Extracting physical information from the RG in quantum gravity

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based on work in collaboration with Martin Reuter, Kevin Falls and Thomas Thiemann

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Gravity is perturbatively non-renormalizable at two-loops.

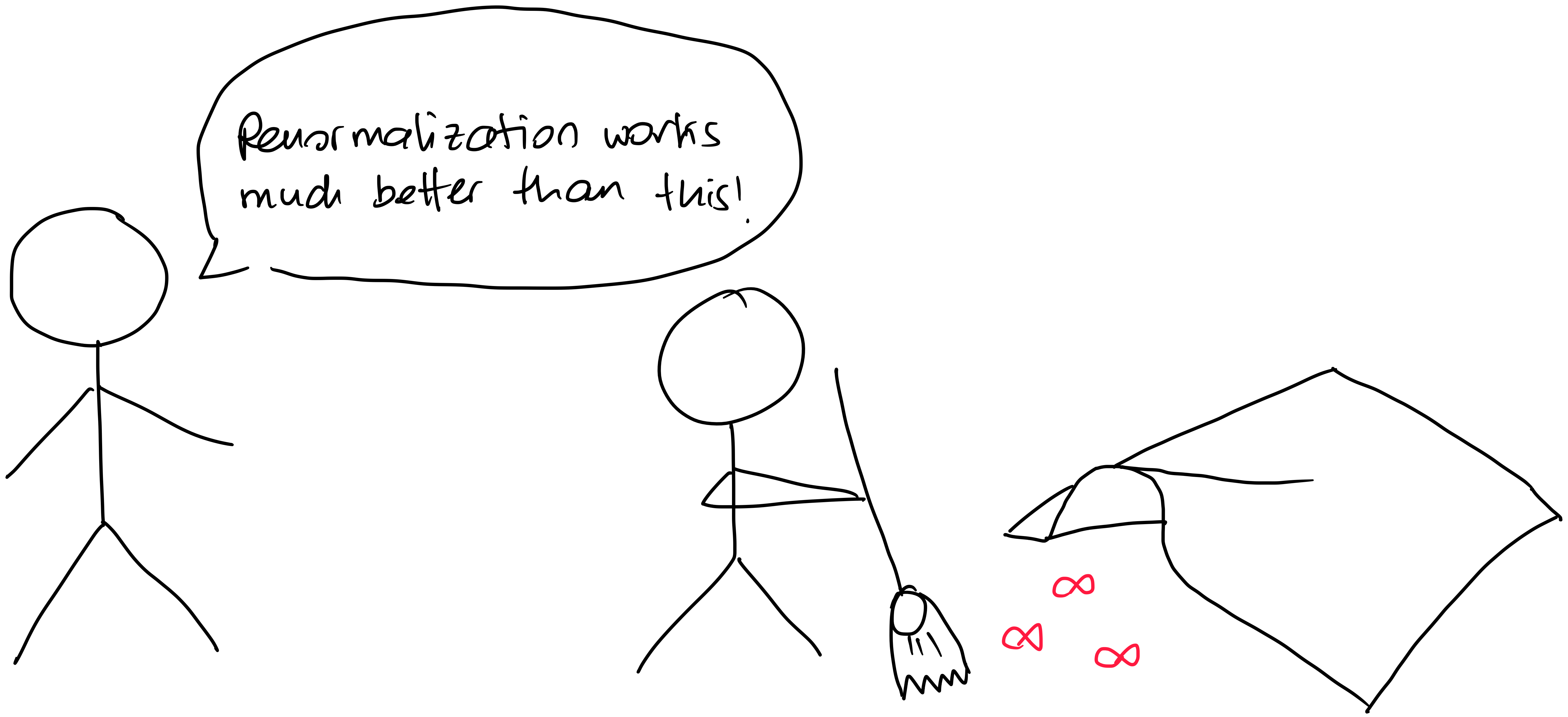
Asymptotic Safety

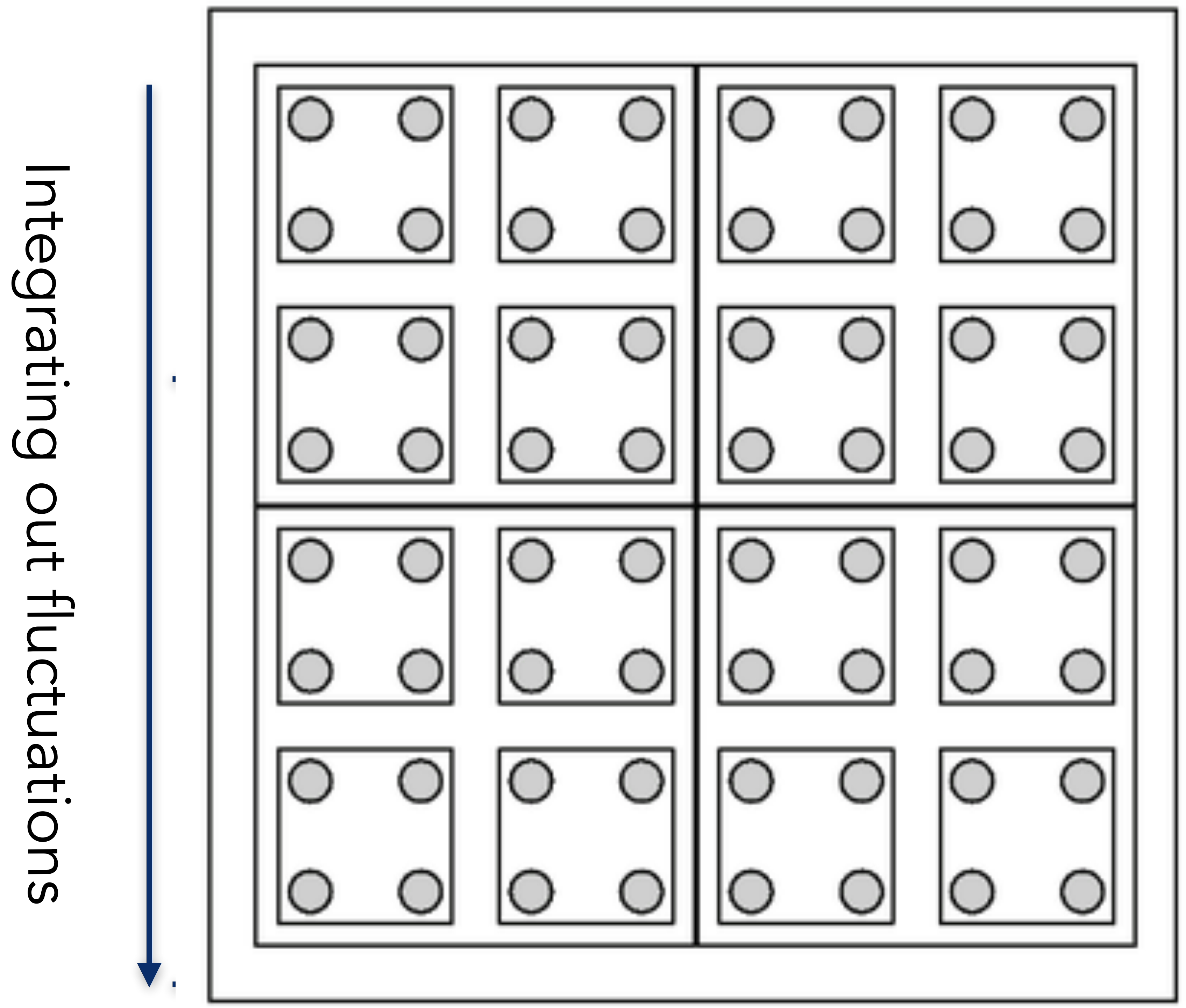
Weinberg's conjecture: There exists a nonperturbative dynamical mechanism which renders physical scattering amplitudes finite and computable at energy scales exceeding the Planck scale: a nontrivial UV fixed point.

[Weinberg (1979)]

Usually technically investigated via the **Functional Renormalization Group**

Dynamical Triangulations





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The RG

Callan-Symanzik equation

RG in perturbation theory

only the finitely many beta functions that are

related to the relevant couplings are considered

[Callan (1970), Symanzik (1970)]

Exact RGE

Wilsonian Exact RG

the quantum fluctuations in the path integral

can be integrated out progressively

[Wilson (1971), Kadanoff (1966)]

Functional RGE

Functional RG

scale-dependent version of the effective action,

the Effective Average Action

[Wetterich (1991)
Reuter and Wetterich (1994)]

FRG in a nutshell

Alternative manipulation of the path integral

- Implement the underlying RG idea already at the level of the EAA, fully independently of the bare action.

Generating functional

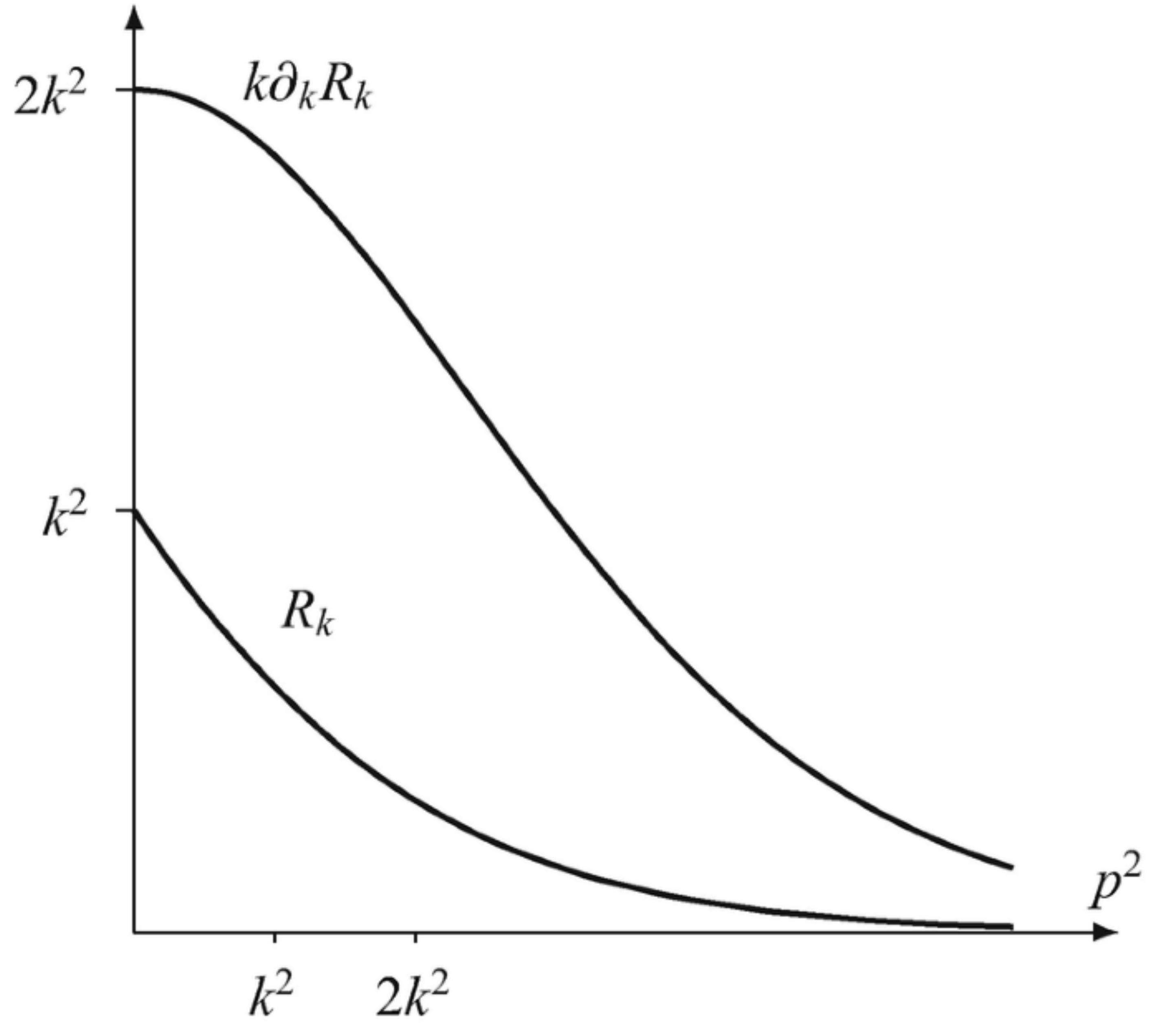
$$W_k[J] = \log \int \mathcal{D}\hat{\phi} \exp\left(-S[\hat{\phi}] - \Delta S_k[\hat{\phi}] + \int \mathrm{d}^d x J(x)\hat{\phi}(x)\right)$$

Smooth cutoff

$$\Delta S_k[\hat{\phi}] = \frac{1}{2}\int \mathrm{d}^d x\ \hat{\phi}(x)\mathcal{R}_k(-\Box)\hat{\phi}(x)$$

RG kernel: mass-like IR regulator

$$\mathcal{R}_k(p^2) \approx \begin{cases} k^2 & \text{for} \quad p^2 \ll k^2 \\ 0 & \text{for} \quad p^2 \gg k^2 \end{cases}$$



Legendre transform

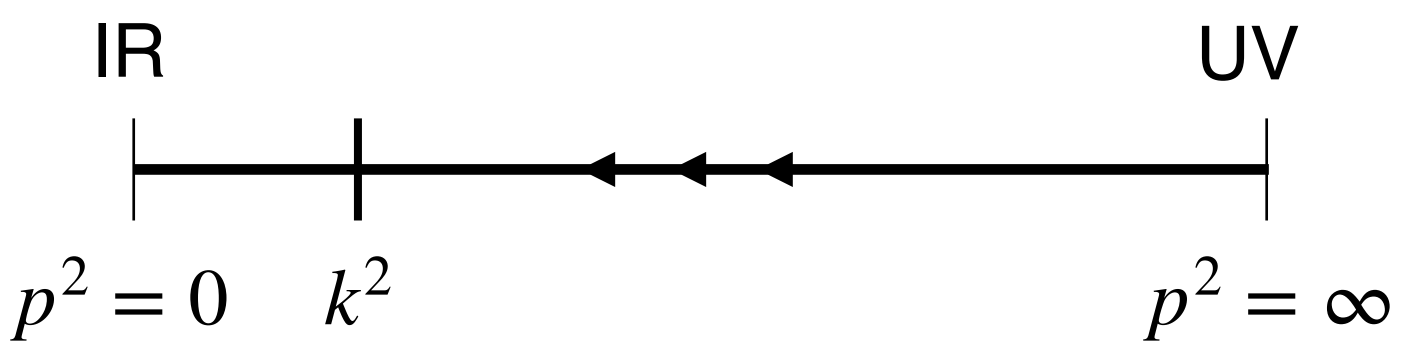
$$\tilde{\Gamma}_k[\phi] = \int \mathrm{d}^d x\ J_k[\phi](x)\phi(x) + W_k[J_k[\phi]] \longrightarrow$$

Effective Average Action

$$\Gamma_k[\phi] = \tilde{\Gamma}_k[\phi] - \Delta S_k[\phi] = \int \mathrm{d}^d x\ J(x)\phi(x) + W_k[\phi] - \Delta S_k[\phi]$$

The Effective Average Action $\Gamma_k[\phi]$

It represents the scale-dependent version of the standard effective action.



$$\lim_{k\to 0} \Gamma_k \to \Gamma \qquad \lim_{k\to\infty} \Gamma_k \to S$$

It satisfies the Functional Renormalization Group Equation:

$$k\partial_k \Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\,\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\partial_k \mathcal{R}_k \right]$$

- UV- and IR finite
- Fully nonperturbative or exact

Predictive solutions do exist in theories that are otherwise perturbatively non-renormalizable.

Asymptotic Safety via FRG: A given trajectory has an acceptable UV limit, if and only if its endpoint in the UV is given by the nontrivial fixed point of the RG flow.

We will use $\Gamma_k[\phi]$ as a constructive object to define our QFT.

Ansatz: $$\Gamma_k[\phi] = \sum_{i=1}^{\infty} U^i(k) P_i[\phi]$$

Fixed points and scaling exponents

Fixed point

$$\beta^i(u_*) = 0$$

Gaussian FP: the scaling exponent agrees with the canonical mass dimension (generally $u^i_* = 0$)

Non-Gaussian FP (interacting, UV FP): at least one of the scaling exponents differs from the canonical mass dimension ($u^i_* \neq 0$)

Linearization & stability matrix

$$k\partial_k u^i(k) = \sum_j \mathcal{B}^i{}_j (u^i(k) - u^i_*), \qquad \mathcal{B}^i{}_j(u_*) \equiv \partial_j \beta^i(u_*)$$

Scaling exponents

$$u^i(k) = u^i_* + \sum_I C_I V^i_I \left(\frac{k_0}{k}\right)^{\theta_I}$$

Anomalous dimension

$$\sum_{l,m} A_{il} \mathcal{B}_{lm}(u_*) A^{-1}_{mj} = -\theta_i \delta_{ij} = -(d_i + \eta_i)\delta_{ij}$$

They encode **physical information** about the **universality class** of the system and its **scaling (observable) behavior**.

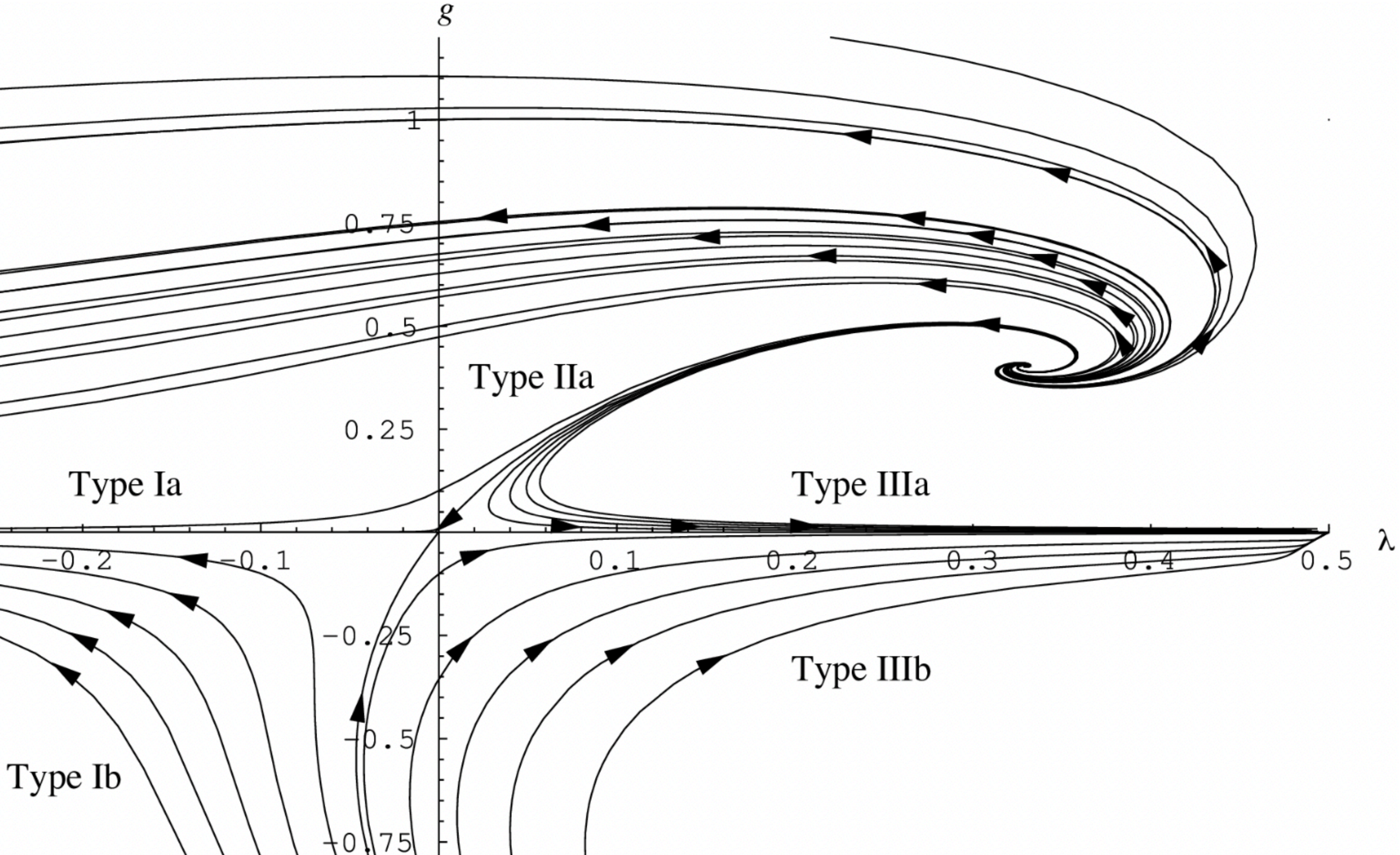
The θ_i s are **universal.**

[Wilson (1971), Wegner (1974)]

Einstein-Hilbert

[Reuter 1996, Reuter-Saueressig and Percacci's book]

$$\Gamma_k[h;\bar{g}] = \frac{1}{16\pi\, G(k)} \int \mathrm{d}^4x\, \sqrt{-g}\, \Big(R(g) - 2\Lambda(k)\Big)\Big|_{g=\bar{g}+h} + \text{gauge fixing} + \text{ghosts}$$



$$g(k) = G(k)k^2 \qquad \lambda(k) = \frac{\Lambda(k)}{k^2}$$

$$\begin{cases} k\partial_k g_k = \beta_g(g,\lambda) \\ k\partial_k \lambda_k = \beta_\lambda(g,\lambda) \end{cases}$$

$$\begin{cases} \beta_g(g_*,\lambda_*) = 0 \\ \beta_\lambda(g_*,\lambda_*) = 0 \end{cases}$$

$g_* = \lambda_* = 0$

Gaussian fixed point

$g_* > 0, \quad \lambda_* > 0$

Non-Gaussian fixed point

ASYMPTOTIC SAFETY

The importance of background independence

$$\mathrm{Tr}[R_k(\bar{g}) + \Gamma_k^{(2)}(\hat{h}, \bar{g})]^{-1}$$

In order to evaluate the traces in a **background independent** way, we can use

Background field method

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$+$$

Heat kernel

$$\Gamma_k^{(2)}(\hat{h}, \bar{g}) \propto \left(\bar{\Box} + \bar{D}^\mu \bar{D}^\nu \bar{g}^{\alpha\beta}\right) \hat{h}_{\mu\nu}\hat{h}_{\alpha\beta} \; + \; \cdots$$

The heat kernel

Based on **Schwinger proper time** representation

$$P_k^{-1} = B^{-1}\,[\bar{\Box} + C_k]^{-1} = -\frac{i}{B}\,[\int_0^\infty dt\; e^{it\bar{\Box} - t\epsilon}]_{\epsilon \to -iC_k}$$

Traces rewritten by the spectral theorem (involving both **minimal and non-minimal** operators)

$$O_k(\bar{\Box}) = \int_{-\infty}^{\infty} dt\; \hat{O}_k(t)\; H_t, \quad H_t := e^{it\bar{\Box}}$$

Heat kernel **on general manifold**:

$$H_t(x,y) = [4\pi|t|]^{-d/2}\; e^{i\frac{\pi}{4}\mathsf{sgn}(t)[2-d]}\; e^{\frac{i}{2t}\,\sigma(x,y)}\; \Omega_t(x,y)$$

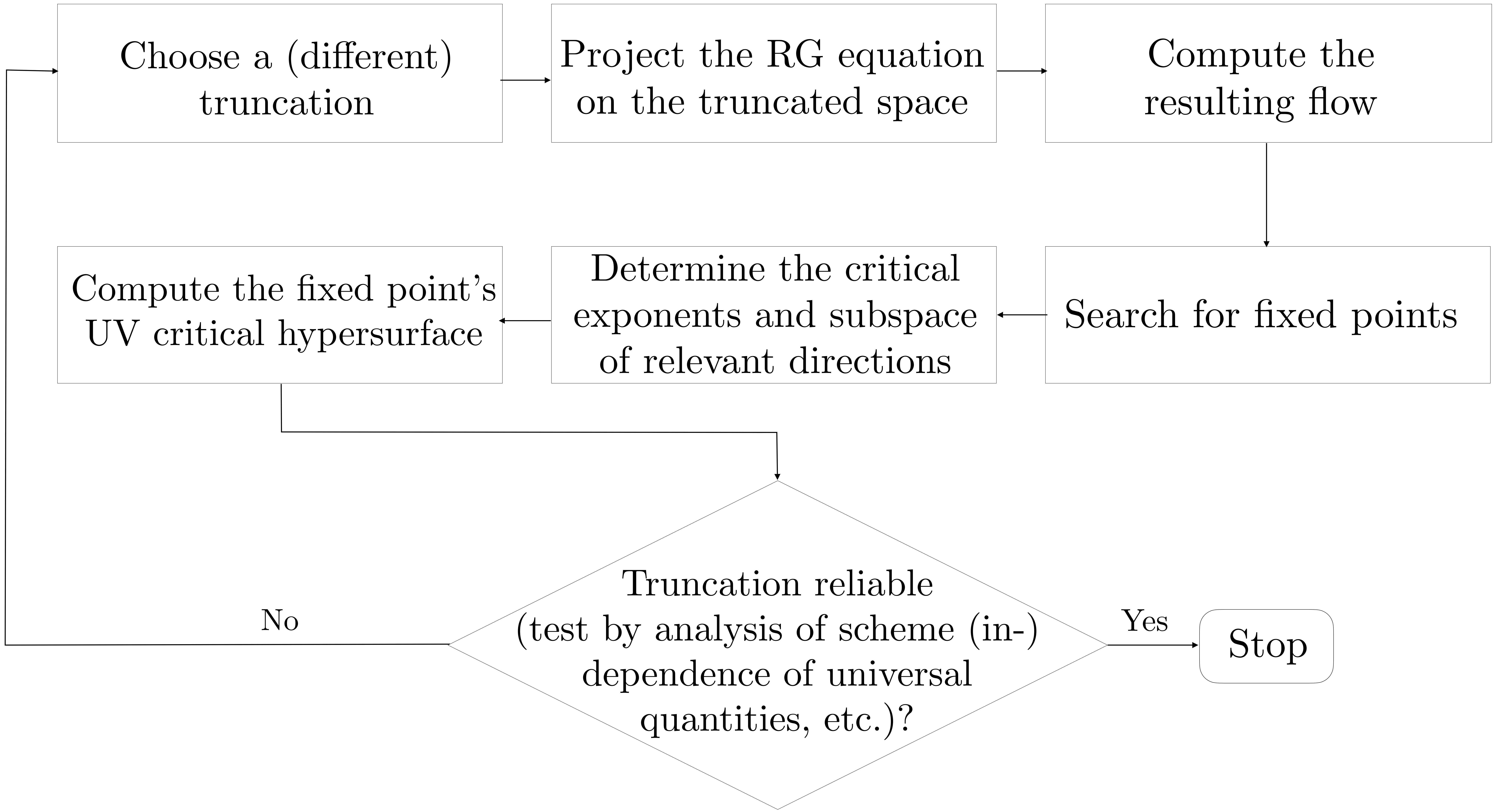
Expansion:

$$\mathrm{Tr}[H_t(\bar{\Box})] = \frac{e^{i\pi/4 \mathrm{sgn}(t)(2-d)}}{(4\pi|t|^{d/2})}\left(1 + \frac{i}{6}\bar{R}t + \cdots\right)$$

[DeWitt (1965), Benedetti Groh, Machado, Saueressig 1012.3081 (2010), Groh, Saueressig, Zanusso 1112.4856 (2011), Christensen (1976), Fulling's book (1989), Moretti (1999), Decanini, Folacci (2006), Parker, Toms' book (2009)]

[RF, Thiemann (2024)]

The program


Choose a (different) truncation
Project the RG equation on the truncated space
Compute the resulting flow
Search for fixed points
Determine the critical exponents and subspace of relevant directions
Compute the fixed point's UV critical hypersurface
Truncation reliable (test by analysis of scheme (in-) dependence of universal quantities, etc.)?
No
Yes
Stop

How to extract physical information from the RG?

in gravity things get
more complicated

What about the **gauge & parametrization** dependence?

Exploit perturbation theory to go on-shell

Part 1 of this talk

Are our operators related to **diffeomorphism invariant observables**?

Relational formalism

Part 2 of this talk

Gauge & parametrization dependence:
On-shell perturbation theory

Disclaimer

I will talk about Einstein gravity.
However, this approach can also be applied to different theories.

Motivation

[Falls, RF (2022) 2411.00938 [hep-th]]

Perturbative renormalization	Non-perturbative renormalization
Dimensional regularization, proper time, …	Asymptotic Safety - Functional Renormalization Group
Gravity is perturbative non-renormalizable	UV-completion via an interacting fixed point
E.g. dimensional regularization in $d = 2 + \epsilon$ The fixed point of order lies beyond perturbation theory. **On-shell effective action**: no unphysical dependencies	Effective action contains **unphysical information**: Field parametrization - gauge dependence 1. Effective action is **off-shell** 2. Regulator **breaks symmetry** (diffeomorpshism invariance)
On-shell effective action results tested in precision measurements in QCD.	Extract physical information from the flow of the effective action is a arduous task.

Motivation

Perturbative renormalization	Non-perturbative renormalization
Perturbative non-renormalizable (−)	(+) Non-perturbative renormalizable
Contains only physical information (+)	(−) Affected by non-physical information

New subtraction scheme + essential renormalization group

Idea

Use perturbative methods to investigate asymptotic safety

E.g. dimensional regularization with non-minimal subtraction scheme

E.g. proper time regularization

Do a fully functional approximation to keep invariants to all orders:
non-minimal subtraction scheme.
RG improvement of one-loop effective action “looks like” non-perurbative RG

Essential RG: RG scheme to keep unphysical dependencies under control

1. One-loop effective action

For **Einstein gravity**:

$$\Gamma = S + \frac{1}{2}\mathrm{Tr}\log\left[K^{-1}(S^{(2)} + S^{(2)}_{\mathrm{gf}})\right] - \mathrm{Tr}[\mathcal{Q}_{\mathrm{FP}}]$$

Faddeev-Popov ghost's term

DeWitt metric

Hessian

gauge fixing: e.g. background covariant harmonic gauge

$$S = \int \mathrm{d}^d x\, \sqrt{g}\left(\frac{\rho}{8\pi} - \frac{R}{16\pi G} + \vartheta\mathfrak{E}(g)\right)$$

Vacuum energy

Euler topological term in d = 4

In this first analysis: investigation of the **parametrization dependence** [Gies, Knorr, Lippoldt 1507.08859]

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + \frac{1}{2}\left(\tau_1 h_{\mu\rho}h^{\rho}_{\nu} + \tau_2 h h_{\mu\nu} + \tau_3 \bar{g}_{\mu\nu} h_{\rho\sigma}h^{\rho\sigma} + \tau_4 \bar{g}_{\mu\nu} h^2\right) + O(h^2)$$

Background field method:

$$S^{(2)\mu\nu,\rho\lambda}(x,y) = \frac{\delta S}{\delta h_{\mu\nu}(x)\delta h_{\rho\lambda}(y)}$$

Use the EoM for S: dependence on the parametrization should disappear.

2. Dimensional regularization

Revisit Weinberg's original conjecture of Asymptotic safety: gravity has a fixed point in $d = 2 + \epsilon$

[Weinberg, Niedermaier, Benedetti, Falls, Jack & Jones, Christensten & Duff, Kawai, Kluth 2409.09252, …]

New subtraction scheme: keep (power law) divergences that appear also in $d_c = 0, 2, 4$

preserves gauge simmetry

Idea: in gravity we should keep track of **two dimensionalities**.
One gets regularized, one is dynamical $d = g^\mu_\mu$ (components of the field)

[Martini, Ugolotti, Zanusso, Vacca, Del Porro, Sauro]

Expand the trace up to second order in curvature, exploiting the EOM

$$\Gamma = \int \mathrm{d}^d x \,\sqrt{g}\left(\frac{a_0(d)}{d}\mu^d + \frac{a_2(d)}{d-2}\mu^{d-2}R + \frac{a_4(d)}{d-4}\mu^{d-4}G\rho R + \frac{a_4'(d)}{d-4}\mu^{d-4}\mathfrak{E}\right) + \text{finite terms}$$

1. We do not need to introduce counter-terms outside the Einstein-Hilbert action.
2. The vacuum energy is only renormalised due to the singular term in $d = 0$ dimensions.

2. Dimensional regularization
Non-minimal subtraction scheme

Compute traces using proper time techniques and evaluate UV singular part $\int_0^\infty \mathrm{d}s\, s^{\frac{d_c-d}{2}-1} \sim -2\frac{\mu^{d-d_c}}{d-d_c}$

Keep $d = g^\mu_\mu$ distinct from d_c, in order not to identify the components of the metric with the regularization parameter.

$$\frac{1}{(4\pi)^{d/2}} \to \frac{1}{(4\pi)^{d_c/2}}$$

The heat kernel coefficients then become

$$\begin{aligned}
\bar{a}_0(d) =& \frac{1}{2}(d-3)d \\
\bar{a}_2(d) =& \frac{1}{(4\pi)}\frac{1}{12}\left(d^2-3d-36\right) \\
\bar{a}_4(d) =& \frac{1}{(4\pi)^2}\frac{d^3+19d^2-566d+1200}{120(d-2)} \\
\bar{a}_4'(d) =& \frac{1}{(4\pi)^2}\frac{1}{360}\left(d^2-33d+540\right)\ ,
\end{aligned}$$

2. Dimensional regularization
Essential renormalization group

[Baldazzi, Falls, Zinati 2105.11482, 2107.00671]

Renormalization scheme which restricts the analysis to the running of the **essential couplings**

Couplings which contribute to the **scaling of physical observables** such as scattering cross sections (scaling exponents)

Inessential couplings associated with redundant operators
$\longrightarrow$ **fixed by renormalization conditions** achieved by a field reparameterisation along the RG flow.

$G \to 0$

Minimal essential scheme: fix inessential couplings to values at Gaussian fixed point (perturbative)

For the free theory (flat space, GFP)

$$S = \int \mathrm{d}^d x \, \sqrt{g} \left(\frac{\rho_k}{8\pi} - \frac{R}{16\pi G_k} + \vartheta \mathfrak{E}(g) \right)$$

$$S_{\mathrm{ct}} = - \int \mathrm{d}^d x \, \sqrt{g} \bar{a}_0(d) \frac{\mu^d}{d}$$

$$\beta_{\tilde{\rho}} = -d\tilde{\rho} + 8\pi \bar{a}_0 \longrightarrow \frac{\tilde{\rho}}{8\pi} = \frac{\bar{a}_0}{d} = \frac{1}{2}(d-3)$$

Renormalization condition

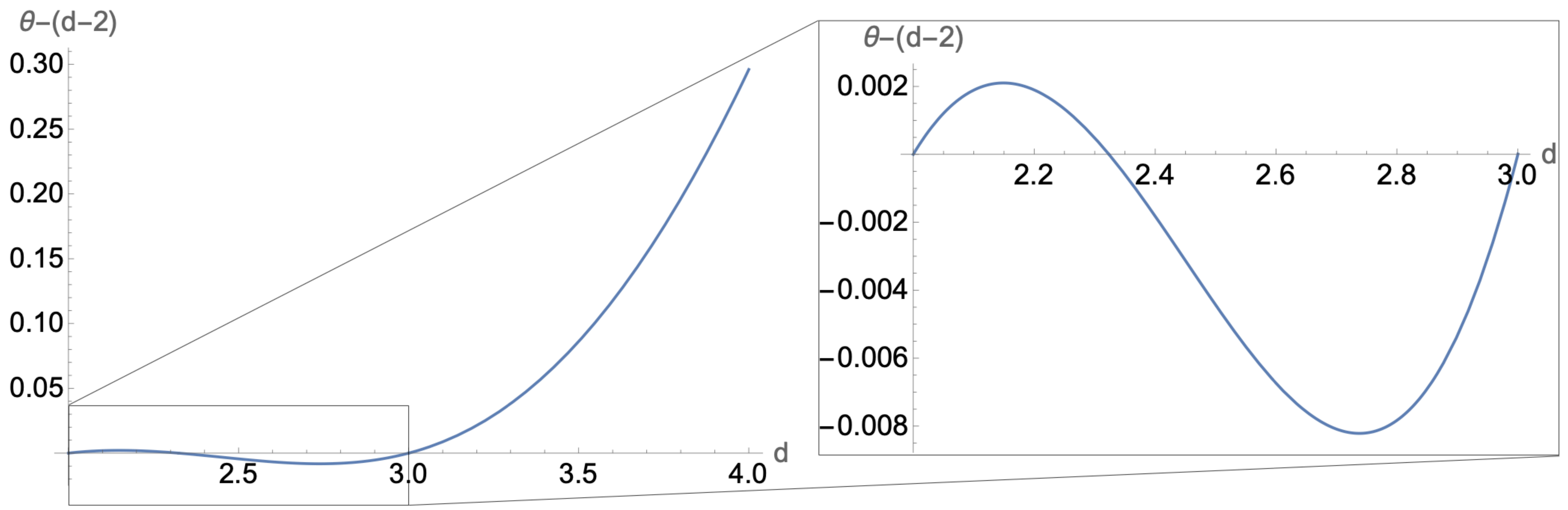
2. Dimensional regularization
Essential renormalization group

Essential coupling (dimensionless - invariant under rescaling of metric)

$$\eta \equiv G\left(\frac{\rho}{4\pi(d-3)}\right)^{\frac{d-2}{d}} \quad \xrightarrow[\text{using ren. condition}]{} \quad g$$

$$\beta_\eta = (d-2)\eta + \frac{1}{3}((d-3)d-36)\eta^2 + \frac{(d-3)(d(d(d+19)-566)+1200)\eta^3}{30(d-2)}$$

Fixed point and critical exponent (d = 4): $\eta_* = 0.16, \qquad \theta = -\left.\frac{\partial\beta_\eta}{\partial\eta}\right|_{\eta=\eta_*} = 2.296$



This was dim. reg.

Can we implement this into a flow equation?

3. Proper time regularization

Generalized proper time flow

One-loop effective action

$$\Gamma = S + \frac{\hbar}{2}\mathrm{Tr}\log K^{-1}S^{(2)}/M^2 + O(\hbar^2)\ .$$

arbitrary scale, take $M \to \infty$

$\downarrow$ evaluate the trace via proper-time parametrization

IR and UV regulated effective action

$$\Gamma_{k,M} = S - \frac{\hbar}{2}\int_{M^{-2}}^{k^{-2}} ds \frac{1}{s}\mathrm{Tr}\left(\exp(-sK^{-1}S^{(2)}) - \exp(-sM^2)\right) + O(\hbar^2)\ .$$

IR-cutoff

$\downarrow$ take k-derivative

Flow equation

$$k\partial_k\Gamma_k = \hbar\ \mathrm{Tr}\ \exp(-K^{-1}S_k^{(2)}k^{-2}) + O(\hbar^2)$$

$\downarrow$ replace classical action and the metric by the effective action and an RG-improved DeWitt metric

One-loop RG-improvement:

$$k\partial_k\Gamma_k = \mathrm{Tr}\ \exp(-K_k^{-1}\Gamma_k^{(2)}[\phi]k^{-2})$$

It suffers from unphysical dependencies, it is off-shell

3. Proper time regularization
Essential renormalization group

Add an extra term: allow the field variable coupling to the source to flow with k

When deriving the effective action, do the Legendre transform:

$$e^{-\Gamma_k[\phi]} = \int d\hat{\chi} e^{-S[\hat{\chi}]+(\hat{\phi}_k[\hat{\chi}]-\phi)\cdot\frac{\delta\Gamma[\phi]}{\delta\phi}}$$

Introduce the RG kernel: $\Psi_k[\phi] := \langle k\partial_k \hat{\phi}_k[\hat{\chi}]\rangle$

Generalized perturbative essential flow equation

$$\boxed{k\partial_k\Gamma_k[\phi] = -\Psi_k[\phi]\cdot\frac{\delta}{\delta\phi}\Gamma_k[\phi] + \mathrm{Tr}\,\exp(-K_k^{-1}\Gamma_k^{(2)}[\phi]\ k^{-2})}$$

absorb off-shell term
into the kernel

proportional to the EoM

Only the essential couplings run.

3. Proper time regularization MES @ order curvature squared

Consider again Einstein-Hilbert truncation $\longrightarrow$ remove curvature squared terms by field redefinitions

$$\left(k\partial_k + \int \mathrm{d}^d x\sqrt{g}\Psi_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\right)\Gamma = \mathrm{Tr}\, e^{-K^{-1}(\Gamma^{(2)}+S^{(2)}_{\mathrm{gf}})k^{-2}} - 2\mathrm{Tr}\, e^{-\mathcal{Q}_{\mathrm{FP}}[\phi]k^{-2}}$$

RG kernel: linear combination of operators up to desired truncation order

$$\Psi^g_{\mu\nu}[g] = \gamma_g g_{\mu\nu} + \gamma_R R g_{\mu\nu} + \gamma_{Ricci} R_{\mu\nu}$$

EOM

$$\frac{\delta\Gamma}{\delta g_{\mu\nu}} = \frac{\sqrt{g}}{2}\frac{\rho}{8\pi}g^{\mu\nu} + \frac{\sqrt{g}}{16\pi G}\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)$$

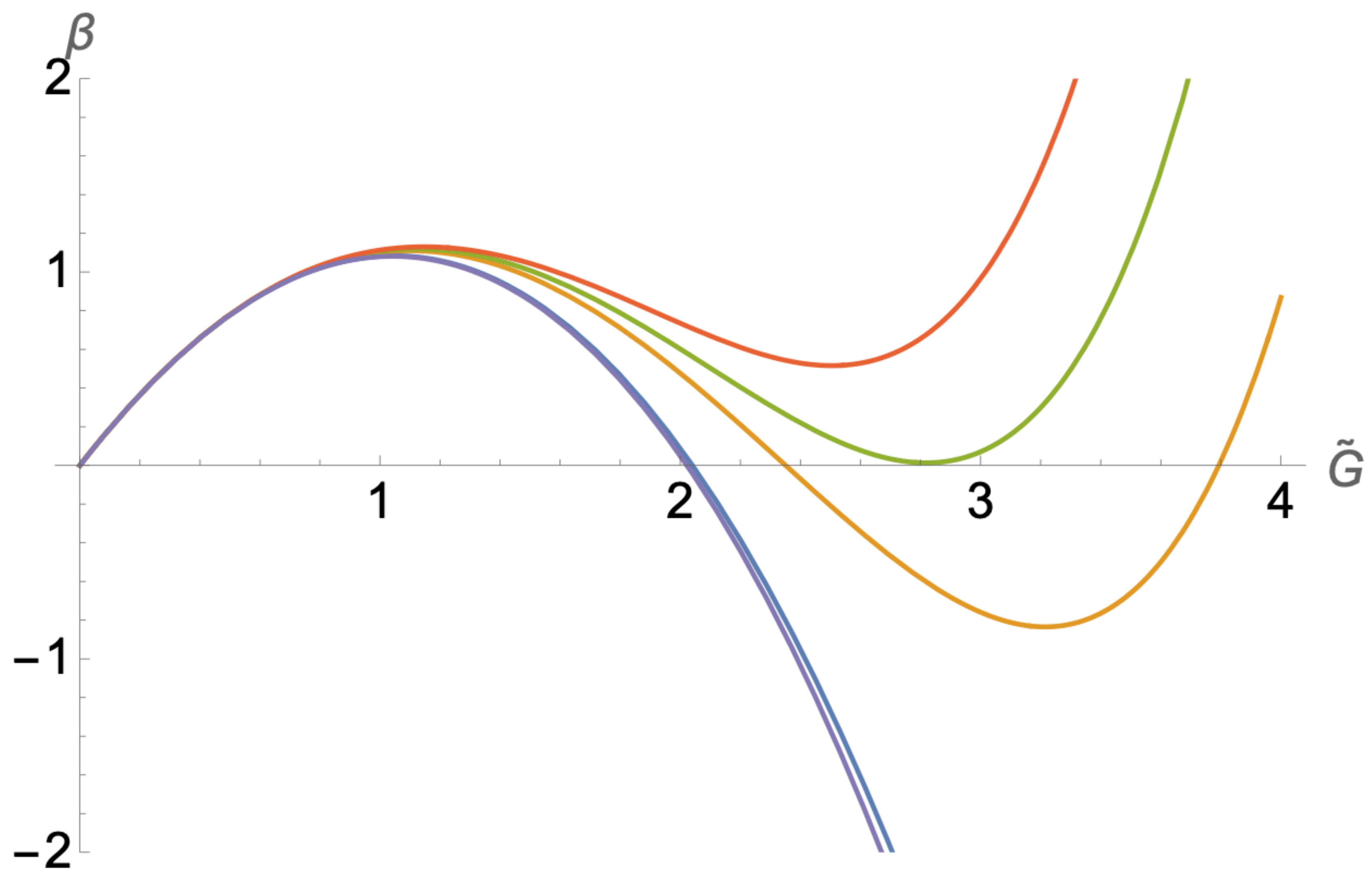
MES: flat space and $\Psi_{\mu\nu} = 0$ $\longrightarrow$ RG condition same as before (up to a constant)

$\frac{\tilde{\rho}}{8\pi} = \frac{\bar{a}_0}{d} = \frac{1}{2}(d-3)$ $\longrightarrow$ Solve for γ_g $\longrightarrow$ Absorb curvature squared terms by solving for γ_R, γ_{Ricci} $\longrightarrow$ Solve for $k\partial_k G_k$

Our non-minimal variant of dimensional regularization coincides with proper time regularization, at least within the early time heat kernel expansion.

3. Proper time regularization
MES @ order curvature squared

Let us use it as a tool to analyze parametrization dependence.



Expand the essential beta function:

$$\beta_{\tilde{G}} = (d-2)\tilde{G} + \frac{1}{3}(-36+(d-3)d)(4\pi)^{1-d/2}\tilde{G}^2 + \frac{(d-3)(1200+d(-566+d(d+19)))(4\pi)^{2-d}}{30(d-2)}\tilde{G}^3 + O(\tilde{G}^4)$$

No dependence on the τ's up to this order.

4. All curvature order on a max. symmetric background

Going to higher orders in curvature, one can lift the dependence on the parameterisation at higher order in $\tilde{G}$.

Expansion up to $O(R^2)$ $\longrightarrow$ No dependence on the parametrization up to $O(\tilde{G}^3)$

Expansion up to $O(R^3)$ $\longrightarrow$ No dependence on the parametrization up to $O(\tilde{G}^4)$

$\vdots$ $\vdots$

Expansion up to $O(R^N)$ $\longrightarrow$ No dependence on the parametrization up to $O(\tilde{G}^{N+1})$

WHY?

$$\left.\begin{array}{l}\text{In EOM terms linear in } R \text{ and } G_k\rho_k \text{ on RHS} \\ \text{Resolving terms } \int \mathrm{d}^d x \sqrt{g} R G_k^{N-1} \text{ requires going } R^N \\ \\ \text{On the LHS we have } \int \mathrm{d}^d x \sqrt{g} \beta_{\tilde{G}} / \tilde{G}^2 \end{array}\right\} \text{up to } O(\tilde{G}^{N+1})$$

4. All curvature order on a max. symmetric background

Technical details:

- **Heat kernel expansion** on a *d*-sphere [Kluth, Litim 1910.00543]
- **Spectral sum** on a *d*-sphere or *d*-hyperboloid
- Evaluation of **non-commuting traces**

RG kernel: $\Psi_{\mu\nu} = \gamma(R) g_{\mu\nu}$

Fixed point and the critical exponent converges rapidly

$O(R^N)$	θ
R	2
R^2	2.296
R^3	2.312
R^4	2.312
R^5	2.311
R^∞	2.311

Diffeomorphism invariance:
relational observables

Problem of defining observables in gravity

Construct a physical coordinate frame
e.g. by adding matter fields

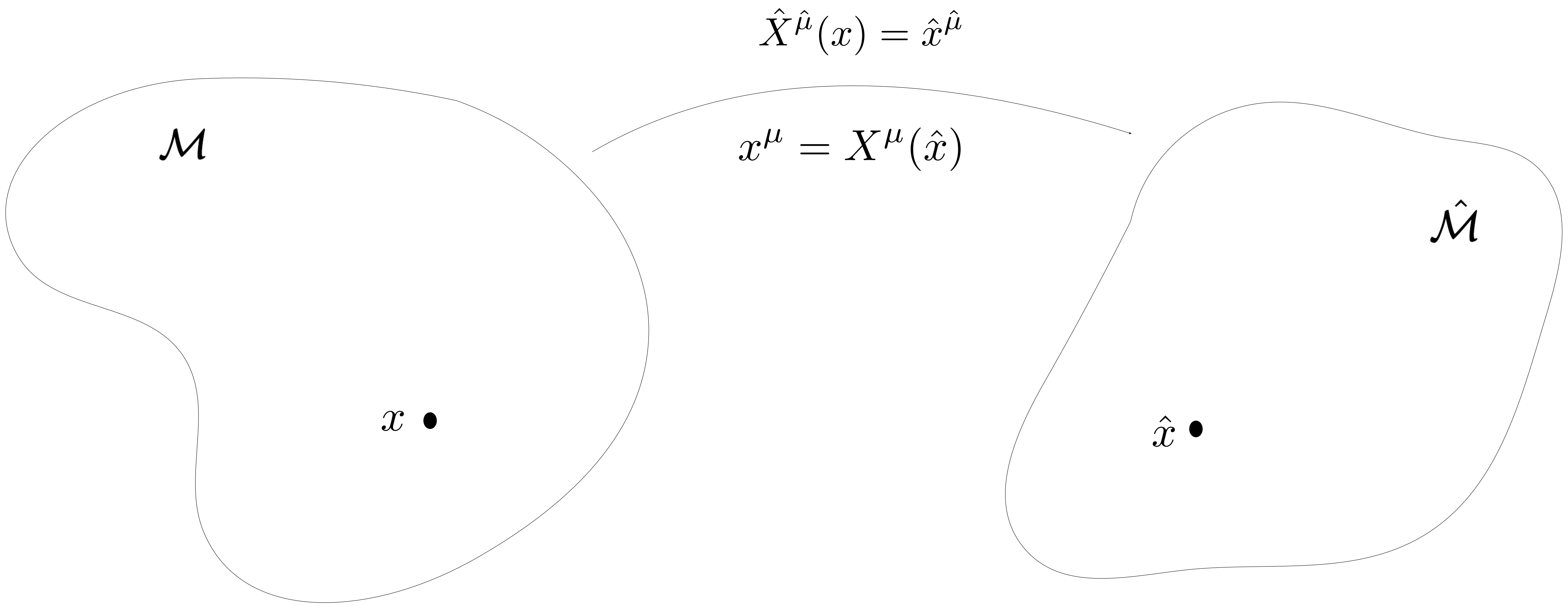
s.t.

perform a diffeomorphism
transformation

transform the tensor
$R(x) \mapsto \varphi * R(x)$

transform the physical frame
$X \mapsto \varphi^{-1}(X)$

composed transformation leaves
the tensor invariant
$R(X) \mapsto \varphi * R(\varphi^{-1}(X)) = R(X)$



Lessons from the canonical theory

The notion of absolute time and space has to be corrected.

GR is a totally constrained theory with vanishing canonical Hamiltonian.

It is the time parameter defined by the Hamiltonian which corresponds to the notion of **time of a physical observer**.

1 By using techniques of group theory: find out orthonormal basis of Hilbert space and define fundamental quantum observables of **area** and **volume** operators.

2 Definition of observables that encode **relations between dynamical fields**.

Analysis of the spectra shows that spacetime at microscopic scale is **discrete**.

For deparameterizable models construct a full set of gauge invariant observables.

[Dittrich, Rovelli, Thiemann,…]

Three options:

1 Physical gauge fixing

via material reference frame

$\sim$ reduced phase space quantisation

phase space path integral or coherent state path integral

[RF, Thiemann (2024) 2404.18224 [hep-th],
RF, Han, Liu (2025) 2502.07696 [gr-qc]]

2 Relational observables as composite operators

after standard gauge-fixing include matter (physical reference frame) in the EAA

[Falls, RF (2021) 2112.02118 [hep-th]]

3 Relational effective action

build in the EA the quantum reference frame and gauge fix on the frame

[WIP with Philipp Höhn & Luca Marchetti]

What about Lorentzian?

Integrating out dofs

Timelike-spacelike: no distinguished ordering of the modes with a standard canonical status is given.

State dependence - Observer dependence

Problem set 1

Obtain RG trajectories on a theory space which is constituted of functionals that are constructed on Lorentzian metrics.

One can work out a Lorentzian **heat kernel proper time regularization.** Price to pay: complex-valued flow.

[RF, Thiemann (2024) 2404.18224 [hep-th]

One can work out a Lorentzian **flow in AQFT (Hadamard propagator).**

[D'Angelo, RF, Fröb (2025) 2502.05135 [hep-th]]

Problem set 2

Analyze the flows of hyperbolic kinetic operators, typically of the d'Alembertian in the background of the running metrics.

[RF, Reuter (2022) 2203.08003 [hep-th]]

1. Physical gauge fixing

Introduce a matter system to fix the gauge:

deparametrize at the classical level.

E.g. four scalar fields or Gaussian dust

[RF, Thiemann (2024) 2404.18224 [hep-th],
RF, Han, Liu (2025) 2502.07696 [gr-qc]]

Write down the "physical" Hamiltonian and quantize it.

On those states construct the path integral.

2. Relational observables as composite operators

Construct observables usually requires information about **operators which usually are not taken into account in a truncated** EAA.

In order to obtain information regarding an arbitrary operator one can **couple it to an external source.**

Composite operators

Compute its **correlation functions** by taking functional derivatives with respect to the source.

Flow of the composite operator

$$\int \mathrm{d}^d x \, \varepsilon \, k\partial_k \mathcal{O}_k = -\frac{1}{2} \mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} \left(\int \mathrm{d}^d x \, \varepsilon \mathcal{O}_k^{(2)} \right) \left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\partial_k \mathcal{R}_k \right]$$

[Pagani, Reuter (2016)]

$$\lim_{k\to\infty} \mathcal{O}_k = \hat{\mathcal{O}}|_{\hat{\phi}\to\phi}$$

$$\mathcal{O}_{k=0} = \langle \hat{\mathcal{O}} \rangle$$

2. Relational observables as composite operators

[Falls, RF (2021) 2112.02118 [hep-th]]

Relational Effective Average Action:

$$\Gamma_k^{\text{rel.}} \equiv \int \mathrm{d}^4x\, \tilde{e}(x)\mathcal{L}_k^{\text{rel.}}(x)$$

Flow of the relational observables:

$$k\partial_k\Gamma_k^{\text{rel.}} = -\frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)}+\mathcal{R}_k\right)^{-1}\left(\Gamma_k^{\text{rel.}(2)}\right)\left(\Gamma_k^{(2)}+\mathcal{R}_k\right)^{-1}k\partial_k\mathcal{R}_k\right]$$

A natural expansion for observables?

Derivative expansion: This means that the frame fields only involve a finite number of derivatives.

Example:

$$\Gamma_k^{\text{rel.}} = \int \mathrm{d}^4\hat{x}\left(\underbrace{a_0(k)}_{\text{relational volume}} + \underbrace{a_R(k)\hat{R}(\hat{x})}_{\text{relational Ricci scalar}} + \underbrace{a_1(k)\delta_{\hat{\mu}\hat{\nu}}\hat{g}^{\hat{\mu}\hat{\nu}}(\hat{x})}_{\text{relational inverse metric}}\right)$$

$$= \int \mathrm{d}^4x\,\tilde{e}\left(a_0(k) + a_R(k)R + a_1(k)\delta_{\hat{\mu}\hat{\nu}}g^{\mu\nu}(\partial_\mu\hat{X}^{\hat{\mu}})(\partial_\nu\hat{X}^{\hat{\nu}})\right)$$

2. Relational observables as composite operators

Find the fixed points

EAA

$$\Gamma_k = \int \mathrm{d}^4x \sqrt{-g} \left(\frac{1}{16\pi G_N(k)} (R - 2\Lambda(k)) + \frac{1}{2} \delta_{\mathrm{AB}} g^{\mu\nu} \partial_\mu \varphi^{\mathrm{A}} \partial_\nu \varphi^{\mathrm{B}} \right) + \dots$$

Identify the relational observables

PHYSICAL REFERENCE SYSTEM

$$\varphi^{\hat{\mu}} = \hat{X}^{\hat{\mu}}$$

Compute the flow of the observables

RELATIONAL EAA

$$\Gamma_k^{\mathrm{rel.}} = \int \mathrm{d}^4x \, \tilde{e} \left(\alpha_0(k) + \alpha_R(k) R + \alpha_1(k) \delta_{\hat{\mu}\hat{\nu}} g^{\mu\nu} (\partial_\mu \hat{X}^{\hat{\mu}})(\partial_\nu \hat{X}^{\hat{\nu}}) \right)$$

Scaling dimension at the fixed point

RESULT

Matter content	θ_0	θ_R	θ_1
SM (type II)	-4	-5.97643	-7.92358
SM (type I)	-4	-5.97467	-7.8177
SM + SF (type II)	-4	-5.97505	-7.80603
SM + 3 ν (type II)	-4	-5.98015	-7.78084
	-4	**-6**	**-8**

3. QRF and relational effective action

Perform the gauge fixing on the QRF but on the quantum level.

E.g. 4 scalar fields + harmonicity condition:

perturbative equivalent as Feynman harmonic gauge fixing

Stay tuned.

Conclusions & Outlook

In view of recent successes of application of RG rechniques to gravity: how to extract physical info?

What about the **gauge & parametrization** dependence?

↓

Exploit perturbation theory to go on-shell

Are our operators related to **diffeomorphism invariant observables**?

↓

Relational formalism

ToDo Analyze also gauge-dependence

Test scheme in other theories - add matter

Go two loop in proper time?

ToDo Derive an explicit relational effective action

What about universality?

What about state dependence?