Spectral features of perturbed black holes from hidden symmetries

DIFA-UNIBO seminar

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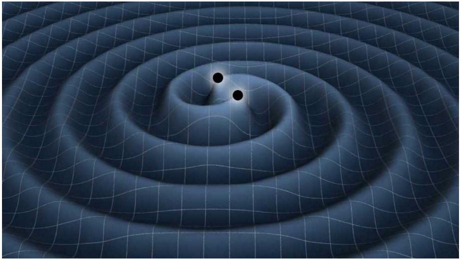

Mainly in collaboration with C.F. Sopuerta, J.L. Jaramillo, A. Montava Agudo, C. Vitel, G. Mena Marugan, A. Minguez Sanchez

- M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 084053 (2021)
- M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021)
- M. Lenzi and C. F. Sopuerta, Phys. Rev. D **107**, 044010 (2023)
- M. Lenzi and C. F. Sopuerta, Phys. Rev. D **107**, 084039 (2023)
- M. Lenzi and C. F. Sopuerta, Phys. Rev. D **109**, 084030 (2024)
- J. L. Jaramillo, M. Lenzi, and C. F. Sopuerta, Phys. Rev. D **110**, 104049 (2024)
- M. Lenzi, A. M. Agudo, and C. F. Sopuerta, JCAP **09**, 021 (2025)
- M. Lenzi, G. M. Marugan, A. M. Sanchez, and C. F. Sopuerta, arXiv:25XX.XXXXX

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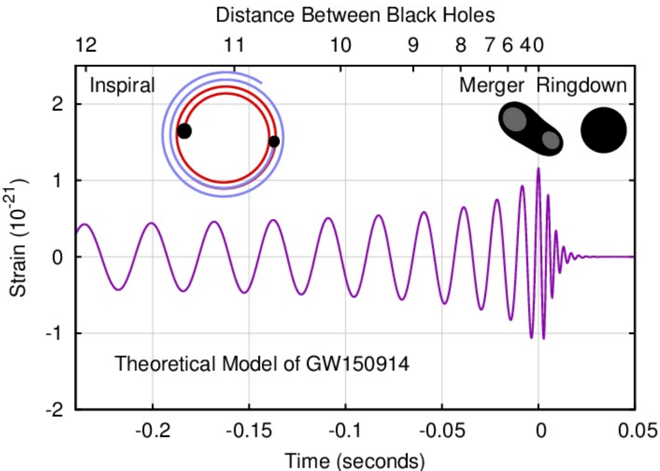
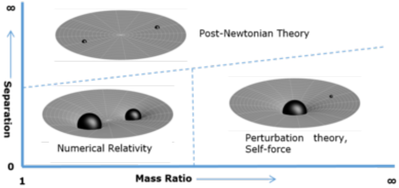
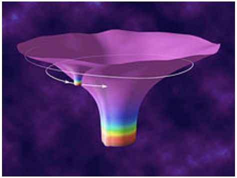


BH perturbation theory

- Scattering of particles and waves through Black Holes (Hawking radiation)
- Binary ringdown signal and Quasi-Normal modes (Black Hole spectroscopy)
- GW with perturbative sources (e.g. EMRI's, environments)





BH perturbation theory

- Scattering of particles and waves through Black Holes (Hawking radiation)
- Binary ringdown signal and Quasi-Normal modes (Black Hole spectroscopy)
- GW with perturbative sources (e.g. EMRI's, environments)
- BH tidal deformations (Love numbers)

Decoupling Einstein equations

- Perturbed Einstein equations at linear order

$$g_{\mu\nu} = \widehat{g}_{\mu\nu} + h_{\mu\nu} \quad \longrightarrow \quad \widehat{G}_{\mu\nu} = 0\,, \quad \delta G_{\mu\nu} = 0$$

- Background metric splitting reflecting spherical symmetry

$$\widehat{g}_{\mu\nu} = \begin{pmatrix} g_{ab} & 0 \\ 0 & r^2\Omega_{AB} \end{pmatrix} \quad \longrightarrow \quad \begin{array}{l} g_{ab}\mathrm{d}x^a\mathrm{d}x^b = -f(r)\,\mathrm{d}t^2 + \mathrm{d}r^2/f(r) \\ \Omega_{AB}\mathrm{d}\Theta^A\mathrm{d}\Theta^B = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2 \end{array}$$

- Scalar, vector, tensor harmonics: even $Y^{\ell m}\,, Y^{\ell m}_A\,, T^{\ell m}_{AB}\,, Y^{\ell m}_{AB}$ and odd $X^{\ell m}_A\,, X^{\ell m}_{AB}$ under parity transformation P : $\quad (\theta,\phi) \to (\pi-\theta, \phi+\pi)$

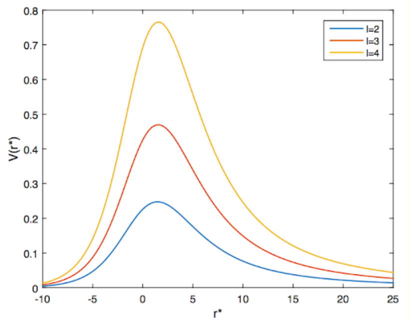
- Harmonic and parity expansion of h $\quad h_{\mu\nu} = \sum_{\ell,m} h^{\ell m,\mathrm{odd}}_{\mu\nu} + h^{\ell m,\mathrm{even}}_{\mu\nu}$

- Decoupled perturbative Einstein equations for each harmonic and parity

Master equations

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-V_\ell^{\mathrm{even/odd}}\right)\Psi^{\ell m}_{\mathrm{even/odd}}=0 \qquad \ell\geq 2$$

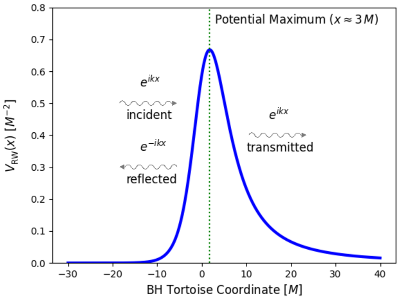
- Master functions $\Psi=\Psi(h,\partial h)$
- Effective potential $V_\ell^{\mathrm{even/odd}}$
- Tortoise coordinate
  $dx/dr=1/f(r)$



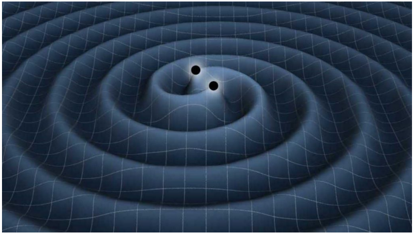
Master equations

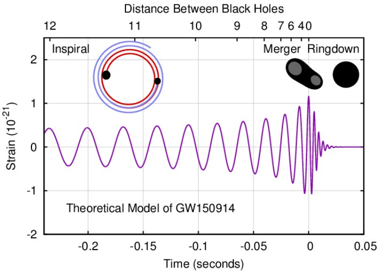
$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-V_\ell^{\text{even/odd}}\right)\Psi^{\ell m}_{\text{even/odd}}=0 \qquad \ell\geq 2$$

- Master functions $\Psi = \Psi(h, \partial h)$
- Effective potential $V_\ell^{\text{even/odd}}$
- Tortoise coordinate $dx/dr = 1/f(r)$
- Quasi-normal modes correspond to vanishing incident wave, $\omega_n \in \mathbb{C}$



BH binary systems





1. GW signal can be written in terms of master functions

$$h_{+/\times} \propto \Psi \Rightarrow h = h_+ - i h_\times \propto \Psi$$

2. Energy and angular momentum fluxes at infinity (Bondi news)

$$\frac{dE}{dt} \propto |\dot{\Psi}|^2, \qquad \frac{dJ}{dt} \propto \Psi \dot{\Psi}$$

K. Martel and E. Poisson, Phys. Rev. D **71**, 104003 (2005)

Known master equations

Odd parity

$$
\begin{aligned}
\Psi_{\mathsf{RW}} &= \frac{r^a}{r}\tilde{h}_a \\
\Psi_{\mathsf{CPM}} &= \frac{2r}{(\ell-1)(\ell+2)}\varepsilon^{ab}\left(\tilde{h}_{b:a} - \frac{2}{r}r_a\tilde{h}_b\right)
\end{aligned}
$$

T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063–1069 (1957), C. T. Cunningham, R. H. Price, and V. Moncrief, Astrophys. J. **224**, 643–667 (1978)

Known master equations

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Even parity

$$\begin{aligned} \Psi_{\mathsf{ZM}} &= \frac{2r}{\ell(\ell+1)}\left\{\tilde{K} + \frac{2}{\lambda}\left(r^a r^b \tilde{h}_{ab} - r r^a \tilde{K}_{:a}\right)\right\} \\ \lambda(r) &= (\ell+2)(\ell-1) - \Lambda r^2 - 3\,(f-1) \end{aligned}$$

F. J. Zerilli, Phys. Rev. D **2**, 2141–2160 (1970), V. Moncrief, Ann. Phys. (N.Y.) **88**, 323 (1974)

Known master equations

Odd parity

$$\Psi_{\mathsf{RW}} = \frac{r^a}{r}\tilde{h}_a$$

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$$\lambda(r) = (\ell+2)(\ell-1) - \Lambda r^2 - 3\,(f-1)$$

Perturbative gauge invariants

$$\tilde{h}_a = h_a - \frac{1}{2}h_{2:a} + \frac{r_a}{r}h_2\,, \qquad \begin{array}{l} \tilde{h}_{ab} = h_{ab} - \kappa_{a:b} - \kappa_{b:a}\,, \\ \tilde{K} = K + \frac{\ell(\ell+1)}{2}G - 2\frac{r^a}{r}\kappa_a \end{array}$$

Assumptions

What are all the possible master equations that one can obtain for the vacuum perturbations of a Schwarzschild BH?

1. Linear in the metric perturbations and first-order derivatives
2. Time independent coefficients
3. Arbitrary perturbative gauge

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 084053 (2021)

Assumptions

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1. Linear in the metric perturbations and first-order derivatives
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3. Arbitrary perturbative gauge

Infinite pairs of master functions and potentials (V, Ψ)

- Standard branch
- Darboux branch

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 084053 (2021)

The standard branch

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-{}_{\mathrm{S}}V_\ell^{\mathrm{odd/even}}\right){}_{\mathrm{S}}\Psi^{\mathrm{odd/even}}=0$$

- Standard branch potentials

$${}_{\mathrm{S}}V_\ell^{\mathrm{odd/even}}=\begin{cases} V_\ell^{\mathrm{RW}} & \text{odd parity} \\ V_\ell^{\mathrm{Z}} & \text{even parity}\end{cases}$$

- Most general master function

$${}_{\mathrm{S}}\Psi^{\mathrm{odd/even}}=\begin{cases} \mathcal{C}_1\Psi^{\mathrm{CPM}}+\mathcal{C}_2\Psi^{\mathrm{RW}} & \text{odd parity} \\ \mathcal{C}_1\Psi^{\mathrm{ZM}}+\mathcal{C}_2\boxed{\Psi^{\mathrm{NE}}} & \text{even parity}\end{cases}$$

$$\Psi^{\mathrm{NE}}(t,r)=\frac{1}{\lambda(r)}t^a\left(r\tilde{K}_{:a}-\tilde{h}_{ab}r^b\right)$$

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-{}_{\mathrm{D}}V_\ell^{\mathrm{odd/even}}\right){}_{\mathrm{D}}\Psi^{\mathrm{odd/even}}=0$$

- Family of potentials ${}_{\mathrm{D}}V_\ell^{\mathrm{odd/even}}$ satisfying

$$\left(\frac{\delta V_{,x}}{\delta V}\right)_{,x}+2\left(\frac{V_{\ell,x}^{\mathrm{RW/Z}}}{\delta V}\right)_{,x}-\delta V=0\,,$$

  with $\delta V={}_{\mathrm{D}}V_\ell^{\mathrm{odd/even}}-V_\ell^{\mathrm{RW/Z}}$.

- Most general (potential dependent) master function

$${}_{\mathrm{D}}\Psi^{\mathrm{odd/even}}=\begin{cases}\mathcal{C}\left(\Sigma^{\mathrm{odd}}\,\Psi^{\mathrm{CPM}}+\Phi^{\mathrm{odd}}\right) & \text{odd parity}\\ \mathcal{C}\left(\Sigma^{\mathrm{even}}\,\Psi^{\mathrm{ZM}}+\Phi^{\mathrm{even}}\right) & \text{even parity}\end{cases}$$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 084053 (2021), Phys. Rev. D **109**, 084030 (2024)

Darboux symmetry

Darboux transformation

- Darboux transformation between (v, Φ) and (V, Ψ)

$$\left(-\partial_t^2 + \partial_x^2 - v\right)\Phi = 0 \longrightarrow \begin{cases} \Psi = \Phi_{,x} + W\,\Phi \\ V = v + 2\,W_{,x} \\ W_{,x} - W^2 + v = \mathcal{C} \end{cases} \longrightarrow \left(-\partial_t^2 + \partial_x^2 - V\right)\Psi = 0$$

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$$\boxed{\left(\frac{\delta V_{,x}}{\delta V}\right)_{,x} + 2\left(\frac{v_{,x}}{\delta V}\right)_{,x} - \delta V = 0}$$

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$$\left(\frac{\delta V_{,x}}{\delta V}\right)_{,x} + 2\left(\frac{v_{,x}}{\delta V}\right)_{,x} - \delta V = 0$$

- Darboux covariance of perturbations of spherically-symmetric BHs

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\left({}_{\mathrm{S}}\Psi^{\mathrm{odd}}, V_\ell^{\mathrm{RW}}\right) \arrow[r, "g^{\mathrm{RW}\to\mathrm{Z}}", "\text{Standard Branch}"'] \arrow[d, "g^{\mathrm{RW}\to\mathrm{odd}}"', "\text{Odd Parity}"] & \left({}_{\mathrm{S}}\Psi^{\mathrm{even}}, V_\ell^{\mathrm{Z}}\right) \arrow[d, "g^{\mathrm{Z}\to\mathrm{even}}", "\text{Even Parity}"'] \\
\left({}_{\mathrm{D}}\Psi^{\mathrm{odd}}, V_\ell^{\mathrm{odd}}\right) \arrow[r, leftrightarrow, "\text{Darboux Branch}", "g^{\mathrm{odd}\leftrightarrow\mathrm{even}}"'] & \left({}_{\mathrm{D}}\Psi^{\mathrm{even}}, V_\ell^{\mathrm{even}}\right)
\end{tikzcd}$$

M. Lenzi and C. F. Sopuerta,
Phys. Rev. D **104**, 124068
(2021)

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- Darboux transformation between (v, Φ) and (V, Ψ)

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\end{tikzcd}$$

Isospectral symmetry

$$\Psi = e^{ikt}\psi$$

$$\phi_{,xx} - v\phi = -k^2\phi$$
$$\psi_{,xx} - V\psi = -k^2\psi$$

Darboux transformation

- Darboux transformation between (v, Φ) and (V, Ψ)

$$\left(-\partial_t^2 + \partial_x^2 - v\right)\Phi = \boxed{\sigma} \longrightarrow \begin{cases} \Psi = \Phi_{,x} + W\,\Phi \\ V = v + 2\,W_{,x} \\ W_{,x} - W^2 + v = \mathcal{C} \end{cases} \longrightarrow \left(-\partial_t^2 + \partial_x^2 - V\right)\Psi = \boxed{S}$$

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\end{tikzcd}$$

Darboux covariance with perturbative sources

$$\boxed{S = \sigma_{,x} + W\sigma}$$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **109**, 084030 (2024)

Standard view on DTs

DT in frequency domain $\Psi(t,r) = e^{ikt}\psi(x;k)$

$$L_V\psi(x;k) \equiv \left(\partial_x^2 - V\right)\psi(x;k) = -k^2\psi(x;k)$$

$$L_V\psi_0 = -k_0^2\psi_0 \;\longrightarrow\; W(x) = -(\ln\psi_0)_{,x} \;\longrightarrow\; \begin{cases} L_v\phi = -k^2\phi \\ \phi = \mathcal{W}[\psi,\psi_0]/\psi_0 \end{cases}$$

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- Darboux transformation between RW and ZM

$$\begin{aligned} \psi_0 &= \tfrac{\lambda(r)}{2}\mathrm{e}^{-ik_0x} \\ k_0 &= \tfrac{i(\ell+2)!}{6M\,(\ell-1)!} \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} W_0(x) &= \tfrac{6M\,f(r)}{\lambda(r)r^2} + ik_0 \\ V_{\mathrm{RW}}^{\mathrm{Z}} &= \pm W_{0,x} + W_0^2 + k_0^2 \end{aligned}$$

S. Chandrasekhar, Proc. Roy. Soc. Lond. A **369**, 425–433 (1980)

Standard view on DTs

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- Darboux generating function as a superpotential

$$\begin{aligned} V &= W_{,x} + W^2 + \mathcal{C} \\ v &= -W_{,x} + W^2 + \mathcal{C} \end{aligned} \quad\longrightarrow\quad \begin{aligned} \left(\partial_x + W\right)\left(\partial_x - W\right)\psi &= -\hat{k}^2\psi \\ \left(\partial_x - W\right)\left(\partial_x + W\right)\phi &= -\hat{k}^2\phi \end{aligned}$$

S. Chandrasekhar, Proc. Roy. Soc. Lond. A **369**, 425–433 (1980)

Kantowski-Sachs (Schwarzschild interior) and Schwarzschild exterior

$$ds^2 = \frac{\mathsf{d}\chi^2}{f(\chi)} - f(\chi)\mathsf{d}\zeta^2 + \chi^2(\mathsf{d}\theta^2 + \sin^2\theta \mathsf{d}\phi^2) \quad \begin{cases} \chi = t & \text{for} \quad \chi < 2M\ (f<0) \\ \chi = r & \text{for} \quad \chi > 2M\ (f>0) \end{cases}$$

- ADM action at second order in perturbations

$$\triangle_1^2[S_0] = \frac{2}{\kappa}\int d\chi \int d\zeta d\Omega \left(h_{ij,t}p^{ij} - C\triangle[\mathcal{H}] - B^i\triangle[\mathcal{H}_i] - \frac{N}{2}\triangle_1^2[\mathcal{H}] - \frac{N^i}{2}\triangle_1^2[\mathcal{H}_i]\right)$$

- Symmetry (harmonics and parity) reduction of the action

$$\triangle_1^2[S_0] = \frac{2}{\kappa}\int d\chi \sum_{\mathfrak{n}\lambda}\left[\sum_{i=1}^{6}\left(q_i'^{\mathfrak{n}\lambda}p_i^{\mathfrak{n}\lambda}\right) - \kappa\left(\mathcal{C}_0 + \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + H_{\mathrm{ax}}^{\mathfrak{n}\lambda} + H_{\mathrm{po}}^{\mathfrak{n}\lambda}\right)\right]$$

D. Brizuela and J. M. Martin-Garcia, Class. Quant. Grav. **26**, 015003 (2009), G. A. Mena Marugán and A. Mínguez-Sánchez, (2024), G. A. Mena Marugán and A. Mínguez-Sánchez, Phys. Rev. D **111**, 086024 (2025)

Steps to (axial) master Hamiltonians

$$\begin{cases} q(\chi) = \alpha(\chi)Q(\chi) + \beta(\chi)P(\chi) \\ p(\chi) = \gamma(\chi)Q(\chi) + \delta(\chi)P(\chi) \end{cases}, \quad \alpha\delta - \beta\gamma = 1$$

- Select gauge invariant quantities by

$$\mathcal{C}_0 = \sum_{\mathfrak{n},\lambda} h_0^{\mathfrak{n},\lambda} F[q_i^{\mathfrak{n},\lambda}, p_i^{\mathfrak{n},\lambda}] \to P_2 \propto F[h_i^{\mathfrak{n}}, p_i^{\mathfrak{n}}] \to \{Q_1, P_2\} = 0$$

so that

$$\kappa\hat{H}^{\text{ax}} = \sum_{\mathfrak{n},\lambda} \frac{\tilde{N}}{2}\left(A_{\text{ax}}(Q_1^{\mathfrak{n},\lambda})^2 + B_{\text{ax}}(P_1^{\mathfrak{n},\lambda})^2 + C_{\text{ax}}Q_1^{\mathfrak{n},\lambda}P_1^{\mathfrak{n},\lambda}\right)$$

- Master function Hamiltonians: i) C'_{ax}, ii) $B'_{\text{ax}} = const$, iii) $A'_{\text{ax}} = \omega^2 - V_{\text{ax}}$

M. Lenzi, G. M. Marugan, A. M. Sanchez, and C. F. Sopuerta, arXiv:25XX.XXXXX

$$\begin{cases} q(\chi) = \alpha(\chi)Q(\chi) + \beta(\chi)P(\chi) \\ p(\chi) = \gamma(\chi)Q(\chi) + \delta(\chi)P(\chi) \end{cases}, \quad \alpha\delta - \beta\gamma = 1$$

- Select gauge invariant quantities by

$$\mathcal{C}_0 = \sum_{\mathfrak{n},\lambda} h_0^{\mathfrak{n},\lambda} F[q_i^{\mathfrak{n},\lambda}, p_i^{\mathfrak{n},\lambda}] \to P_2 \propto F[h_i^{\mathfrak{n}}, p_i^{\mathfrak{n}}] \to \{Q_1, P_2\} = 0$$

so that

$$\kappa \hat{H}^{\mathsf{ax}} = \sum_{\mathfrak{n},\lambda} \frac{\tilde{N}}{2} \left(A_{\mathsf{ax}} (Q_1^{\mathfrak{n},\lambda})^2 + B_{\mathsf{ax}} (P_1^{\mathfrak{n},\lambda})^2 + C_{\mathsf{ax}} Q_1^{\mathfrak{n},\lambda} P_1^{\mathfrak{n},\lambda} \right)$$

- Master function Hamiltonians: i) C'_{ax}, ii) $B'_{\mathsf{ax}} = const$, iii) $A'_{\mathsf{ax}} = \omega^2 - V_{\mathsf{ax}}$

Select a family of canonically Darboux related Hamiltonians

M. Lenzi, G. M. Marugan, A. M. Sanchez, and C. F. Sopuerta, arXiv:25XX.XXXXX

Integrable structures

Korteweg-de Vries isospectral deformations

$$V_{,\sigma} - 6VV_{,x} + V_{,xxx} = 0$$

- Darboux transformation + inverse scattering solves the KdV equation

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. **19**, 1095–1097 (1967)

Korteweg-de Vries isospectral deformations

$$V_{,\sigma} - 6VV_{,x} + V_{,xxx} = 0$$

- Darboux transformation + inverse scattering solves the KdV equation
- KdV deformations of the frequency domain master equation

$$\begin{cases} V(x) \to V(\sigma, x) \\ \psi(x) \to \psi(\sigma, x) \\ k \to k(\sigma) \end{cases}$$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021)

Korteweg-de Vries isospectral deformations

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- KdV deformations of the frequency domain master equation

$$\begin{cases} V(x) \to V(\sigma, x) \\ \psi(x) \to \psi(\sigma, x) \\ k \to k(\sigma) \end{cases} \implies \frac{\partial \mathcal{L}_V}{\partial \sigma} - [P_V, \mathcal{L}_V] = -\mathrm{KdV}[V] \cdot \mathrm{Id}\,, \quad \partial_\sigma \psi = P_V \psi$$

$$\boxed{(k^2)_{,\sigma} = 0}$$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021)

Korteweg-de Vries isospectral deformations

$$V_{,\sigma} - 6VV_{,x} + V_{,xxx} = 0$$

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- KdV equation as an integrable Hamiltonian system with infinite conserved quantities

$$\partial_\sigma V = \{V, \mathcal{H}\} \quad \longrightarrow \quad \mathcal{H}_n[V] = \int_{-\infty}^{\infty} dx\, \kappa_n(V, V_{,x}, V_{,xx}, \ldots)$$

L. D. Faddeev and V. E. Zakharov, Funct. Anal. Appl. **5**, 280–287 (1971)

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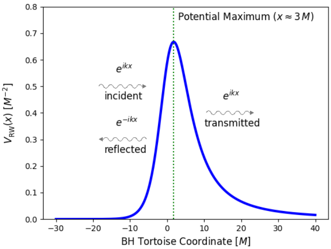
$$\boxed{\mathcal{H}_n[V] = \mathcal{H}_n[V_{\mathrm{RW}}]}$$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021)

BH scattering

$$\psi(x,k,\sigma) = \begin{cases} a(k,\sigma)e^{ikx} + b(k,\sigma)e^{-ikx} \\ e^{ikx} \end{cases}$$



- Bogoliubov coefficients completely determine the physics (greybody factors and QNMs)
  - Greybody factors

$$T(k,\sigma) = \left|\frac{1}{a(k,\sigma)}\right|^2, \quad R(k,\sigma) = \left|\frac{b(k,\sigma)}{a(k,\sigma)}\right|^2$$

  - QNMs: k_i such that $a(k_i,\sigma) = 0$

- Greybody factors and QNMs are conserved by DT and KdV deformations

BH greybody factors from KdV integrals

- Trace identities: a set of integral equations that relate the KdV integrals to the greybody factors

$$\ln a(k,\sigma) = \sum_{n=1}^{\infty} \frac{\mu_n}{k^n} \quad \longrightarrow \quad (-1)^{n+1} \frac{\mathcal{H}_{2n+1}}{2^{2n+1}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, k^{2n} \ln T(k)$$

L. D. Faddeev and V. E. Zakharov, Funct. Anal. Appl. **5**, 280–287 (1971)

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BH moment problem

The greybody factors in BH scattering processes are uniquely determined by the KdV integrals of the BH potential via a (Hamburger) moment problem

$$\mu_{2n} = \int_{-\infty}^{\infty} dk\, k^{2n} p(k)$$

where

$$\mu_{2n} = (-1)^n \frac{\mathcal{H}_{2n+1}}{2^{2n+1}}\,, \quad p(k) = -\frac{\ln T(k)}{2\pi}$$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **107**, 044010 (2023)

Moment problem

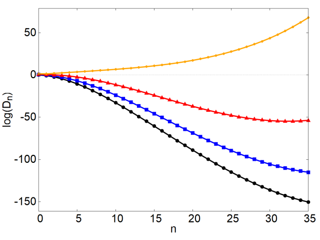
$$\mu_n = \int_{\mathcal{I}} dx\, x^n\, p(x) \quad n = 0, 1, 2, \ldots$$

- Existence: Is there a function $p(x)$ on $\mathcal{I}$ whose moments are given by $\{\mu_n\}$?
- Uniqueness: Do the moments $\{\mu_n\}$ determine uniquely a distribution $p(x)$ on $\mathcal{I}$?
- Solution: How can we construct all such probability distributions?

Moment problem: Existence and Uniqueness

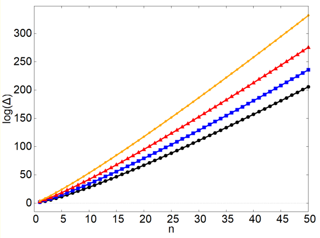
Existence

$$D_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \mu_2 & \mu_3 & \cdots & \mu_{n+2} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} > 0$$



Uniqueness

$$\Delta(n) = C^n (2n)! - \hat{\mu}_{2n} > 0$$



Moment problem: Solution

Solution through Padé approximants

$$T(k) \quad \simeq \quad \exp\left(-2\,\pi\,\sigma\,k\,\sum_{i=1}^{L}\lambda_i e^{-t_i\,\sigma^2 k^2}\right) \quad \lambda_i = \lambda_i\,[\{\mathcal{H}_n\}] \quad t_i = t_i\,[\{\mathcal{H}_n\}]$$

1. Evaluate the first n KdV integrals
2. Obtain the moments from the KdV integrals and construct the MGF

$$M(t) = \int_0^\infty d\xi\, e^{-t\xi}\, \tilde{p}(\xi) = \sum_{n=0}^{\infty} \frac{\tilde{\mu}_n}{n!}(-t)^n$$

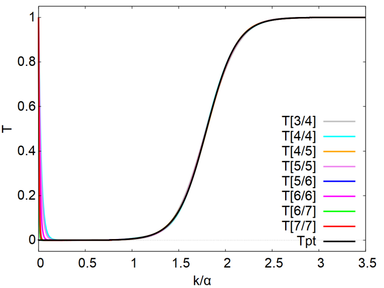
3. Construct Padé approximants of order $[K/L]$, with $K + L < n$
4. Evaluate the poles t_i and residues λ_i of the Padé approximants
5. Apply the Laplace inversion formula

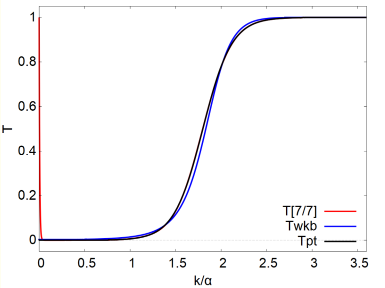
M. Lenzi and C. F. Sopuerta, Phys. Rev. D **107**, 084039 (2023)

Moment problem: Solution

Solution through Padé approximants: Pöschl-Teller

$$T(k) \quad \simeq \quad \exp\left(-2\,\pi\,\sigma\,k\,\sum_{i=1}^{L}\lambda_i e^{-t_i\,\sigma^2 k^2}\right) \qquad \lambda_i = \lambda_i\left[\{\mathcal{H}_n\}\right] \qquad t_i = t_i\left[\{\mathcal{H}_n\}\right]$$

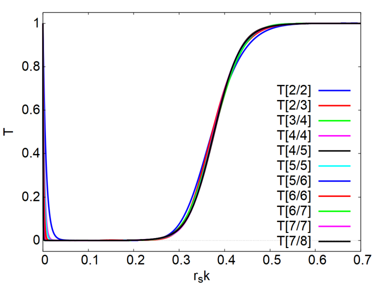


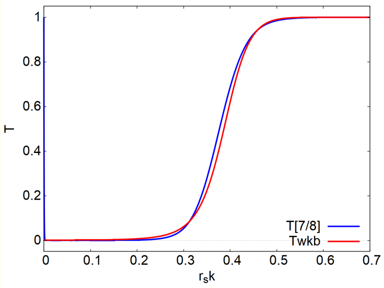


Moment problem: Solution

Solution through Padé approximants: Regge-Wheeler

$$T(k) \quad \simeq \quad \exp\left(-2\,\pi\,\sigma\,k\,\sum_{i=1}^{L}\lambda_i e^{-t_i\,\sigma^2 k^2}\right) \qquad \lambda_i = \lambda_i\,[\{\mathcal{H}_n\}] \qquad t_i = t_i\,[\{\mathcal{H}_n\}]$$





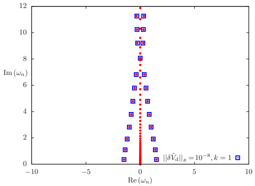
BH spectral (in)stabilities

$$V^{\rm odd}_{\rm even} = V^{\rm RW}_{\rm Z} + \epsilon\, \delta V^{\rm odd}_{\rm even}\,.$$

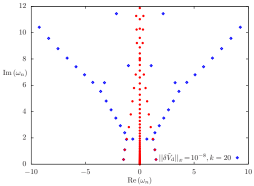
Instability of the QNMs

$$|k_i(\epsilon) - k_i| \leq \epsilon \frac{||w_i|| \, ||v_i||}{|\langle w_i v_i \rangle|}$$

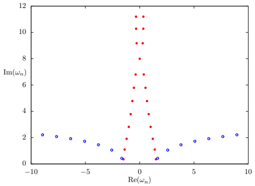
J. L. Jaramillo, R. Panosso Macedo, and L. Al Sheikh, Phys. Rev. X **11**, 031003 (2021)



Low frequency



High frequency



Infrared

Stability of the GFs

$|h_{\ell m}| \simeq C\sqrt{1 - T_{\ell m}(k)}/k^p$ and $T_{\ell m}$ are stable

N. Oshita, Phys. Rev. D **109**, 104028 (2024), N. Oshita, K. Takahashi, and S. Mukohyama, Phys. Rev. D **110**, 084070 (2024), R. F. Rosato, K. Destounis, and P. Pani, arXiv:2406.01692 (2024)

$$\mathcal{H}_{2n+1} = \int_{-\infty}^{\infty} dx\, \kappa_{2n+1}(x) = (-1)^{n+1} 2^{2n+1} \int_{-\infty}^{\infty} dk\, k^{2n} \frac{\ln T(k)}{2\pi}$$

- Qualitative instability similarities with QNMs:
  - ◇ (Mild) Instability of $\mathcal{K}_1$ only under infrared modifications
  - ◇ Strong instabilities in higher KdV integrals under high frequency perturbations
  - ◇ Stability under low frequency perturbations
  - ◇ Simple indicators of parity violations
- Moment problem perspective: solid description of the GFs stability

M. Lenzi, A. M. Agudo, and C. F. Sopuerta, JCAP **09**, 021 (2025)

The KdV-Virasoro-Schwarzian derivative triangle

- Bi-Hamiltonian structure of KdV: Gardner-Zakharov-Faddeev brackets and Magri brackets

$$\partial_\sigma V = \{V, \mathcal{H}_2\}_{\mathrm{GFZ}} = \{V, \mathcal{H}_1\}_{\mathrm{M}} \ ,$$

F. Magri, J. Math. Phys. **19**, 1156–1162 (1978)

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$$\partial_\sigma V = \{V, \mathcal{H}_2\}_{\mathrm{GFZ}} = \{V, \mathcal{H}_1\}_{\mathrm{M}} \ ,$$

- Magri brackets are the classical realization of the Virasoro algebra

$$V(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+1}} \longrightarrow \pi i \{L_n, L_m\}_{\mathrm{M}} = (n-m)L_{n+m} - \frac{n(n^2-1)}{2}\delta_{n+m,0}$$

J.-L. Gervais, Phys. Lett. B **160**, 277–278 (1985), J.-L. Gervais, Physics Letters B **160**, 279–282 (1985)

The KdV-Virasoro-Schwarzian derivative triangle

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- (BH) potentials as a CFT energy-momentum tensor:
  - ◇ Infinitesimal conformal transformation of V: $w(z) = z + \epsilon(z)$

$$\delta_\epsilon V(w) = \{V(w), F_\epsilon\}_{\mathrm{M}} \ , \quad F_\epsilon = -\frac{1}{2}\int dz\, \epsilon(z) V(z) \, , \quad F_\epsilon|_{\epsilon=V} = \mathcal{H}_1$$

  - ◇ Finite conformal transformation of V: Schwarzian derivative

$$V(w) = \left(\frac{dw}{dz}\right)^{-2}\left[V(z) + \frac{1}{2}\mathcal{S}(w(z))\right] \, , \quad \mathcal{S}\left(w(z)\right) \equiv \frac{w_{zzz}}{w_z} - \frac{3}{2}\left(\frac{w_{zz}}{w_z}\right)^2$$

Conformal transformation of the master equation

$$\psi_{,xx} - V\psi = -k^2\psi$$

- Perform the following general transformation

$$\begin{cases} x & \mapsto & x = x(\tilde{x})\,, \\ \psi & \mapsto & \psi(x) = \omega(\tilde{x})\tilde{\psi}(\tilde{x}) \end{cases} \longrightarrow a(\tilde{x})\tilde{\psi}_{,\tilde{x}\tilde{x}} + b(\tilde{x})\tilde{\psi}_{,\tilde{x}} + c(\tilde{x})\tilde{\psi} = -k^2\omega\tilde{\psi}$$

- Cancel first order derivative terms to preserve the operator structure, i.e. $b(\tilde{x}) = 0$ to obtain

$$\tilde{\psi}_{\tilde{x}\tilde{x}} + \left(k^2 x_{\tilde{x}}^2 - \tilde{V}\right)\tilde{\psi} = 0\,, \quad \tilde{V}(\tilde{x}) = \left(\frac{d\tilde{x}}{dx}\right)^{-2}\left[V(x) + \frac{1}{2}\mathcal{S}(\tilde{x}(x))\right]$$

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- Similarity with "Schrödinger symmetry" responsible for the vanishing of the Love numbers (for scalar field perturbations and zero frequency)

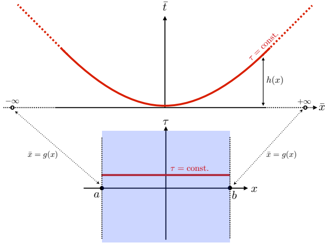
J. Ben Achour, E. R. Livine, S. Mukohyama, and J.-P. Uzan, JHEP **07**, 112 (2022)

Hyperboloidal slicing

$$\left(-\partial_t^2 + \partial_x^2 - V_\ell\right)\phi = 0$$

- Perform the following transformation

$$(t,x) \to (\tau,\xi) \;:\; \begin{cases} t &= \tau - h(\xi) \\ x &= g(\xi) \end{cases}$$



- With $\psi = \partial_\tau \phi$ the master equation becomes

$$\partial_\tau \varphi = i\mathbb{L}\varphi\,, \quad \mathbb{L} = \frac{1}{i}\begin{pmatrix} 0 & 1 \\ \mathcal{L}_1 & \mathcal{L}_2 \end{pmatrix}, \quad \varphi = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

$$\begin{cases} \mathcal{L}_1 = \frac{1}{w(\xi)}\left[\partial_\xi\left(p(\xi)\partial_\xi\right) - q_\ell(\xi)\right] & w(\xi) = \frac{1}{g'}\left(g'^2 - h'^2\right) & q_\ell(\xi) = g' V_\ell \\ \mathcal{L}_2 = \frac{1}{w(\xi)}\left[2\gamma(\xi)\partial_\xi + \partial_\xi \gamma(\xi)\right] & p(\xi) = \frac{1}{g'} & \gamma(\xi) = \frac{h'}{g'} \end{cases}$$

Hyperboloidal slicing

$$\begin{cases} \mathcal{L}_1 = \frac{1}{w(\xi)} \left[\partial_\xi \left(p(\xi)\partial_\xi\right) - q_\ell(\xi)\right] \\ \mathcal{L}_2 = \frac{1}{w(\xi)} \left[2\gamma(\xi)\partial_\xi + \partial_\xi\gamma(\xi)\right] \end{cases} \qquad \begin{array}{ll} \mathcal{L}_1 & \text{bulk/integrable} \\ \mathcal{L}_2 & \text{boundary/dissipative} \end{array}$$

- Define an energy scalar product (crucial to assess QNM instability)

$$\langle \varphi_1, \varphi_2 \rangle = \frac{1}{2}\int_a^b \left(w\partial_\tau\bar{\phi}_1\partial_\tau\phi_2 + p\partial_\xi\bar{\phi}_1\partial_\xi\phi_2 + q_\ell\bar{\phi}_1\phi_2\right) d\xi \quad F = \sum_{\ell m}\gamma\partial_\tau\phi_{\ell m}\partial_\tau\bar{\phi}_{\ell m}$$

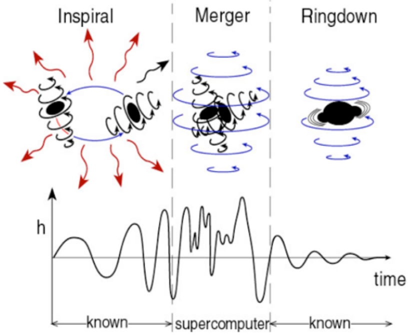
- Non-selfadjointness is due to dissipation at the boundaries

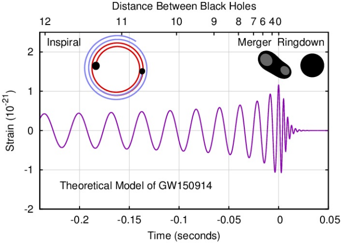
$$\mathbb{L}^\dagger = \mathbb{L} + \mathbb{L}^\partial\,, \quad \mathbb{L}^\partial = \frac{1}{i}\begin{pmatrix} 0 & 1 \\ 0 & \mathcal{L}_2^\partial \end{pmatrix}\,, \quad \mathcal{L}_2^\partial = 2\frac{\gamma}{w}\left[\delta(\xi - a) - \delta(\xi - b)\right]$$

J. L. Jaramillo, R. Panosso Macedo, and L. Al Sheikh, Phys. Rev. X **11**, 031003 (2021), E. Gasperin and J. L. Jaramillo, Class. Quant. Grav. **39**, 115010 (2022)

Universality Conjecture

A Universality Conjecture



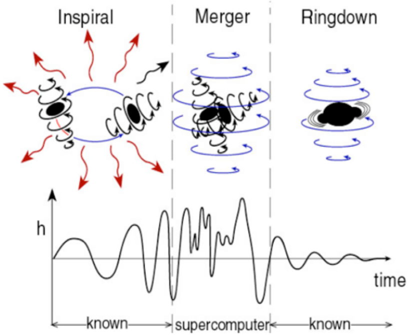


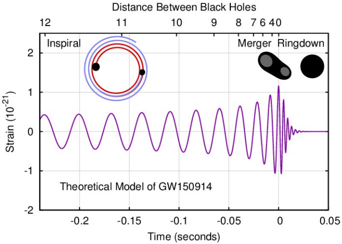
- *Asymptotic reasoning* : filter some DoFs to unveil the underlying (universal) patterns

- *Wave mean flow* : effective separation into "slow" and "fast" DoFs

J. L. Jaramillo, M. Lenzi, and C. F. Sopuerta, Phys. Rev. D **110**, 104049 (2024), J. L. Jaramillo and B. Krishnan, (2022), J. L. Jaramillo, B. Krishnan, and C. F. Sopuerta, Int. J. Mod. Phys. D **32**, 2342005 (2023)

A Universality Conjecture

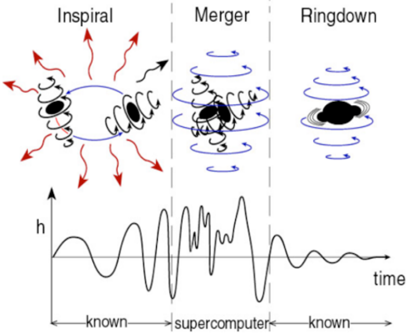


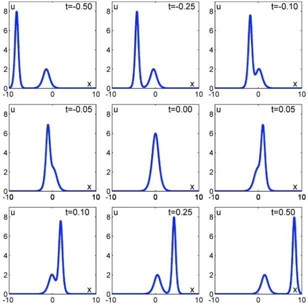


"Even systems which are far from integrable may have an integrable *heart* which tells one much about their behaviour"

N.J. Hitchin, G.B. Segal and R.S. Ward, *Integrable systems: Twistors, loop groups and Riemann surfaces*

A Universality Conjecture





Soliton resolution conjecture

Generic global-in-time nonlinear wave dynamics decouple universally at late times into soliton solutions plus radiation.

T. Tao, Bulletin of the American Mathematical Society **46**, 1–33 (2009)

Conclusions

- Hidden integrable structures in BH physics: analytic results and algebraic structures

- Interplay with asymptotic dynamics and BMS symmetries (resumming the modes and off-shell formulation)

- In GW physics:

  ◇ QNMs

  ◇ Tidal Love numbers

  ◇ BH spectroscopy

  ◇ Extension to Kerr