Black holes perturbations from Liouville correlators

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Introduction

Goal: solve

$$\Box_{BH}\psi(\vec{x}) = m^2\psi(\vec{x})\,.$$

$$\psi(\vec{x}) = \quad e^{-i\omega t} \quad \underbrace{S(\Omega)}_{\text{Angular part}} \quad \underbrace{\psi(r)}_{\text{Radial ODE}}$$

We then reduce to

$$\left(\partial_r^2 + V(r)\right)\psi(r) = 0$$

We'll take a more mathematical detour on the class of ODEs appearing in this context, and then come back to black holes.

This is part of a program started by [Aminov-Grassi-Hatsuda]: use

- gauge theory [Bianchi, Di Russo, Fucito, Morales, Russo, Sudano, ...]
- CFT + AGT Here!
- integrability [Fioravanti, Gregori, . . .]

methods to solve for black hole perturbations!

We'll consider

$$\left(\partial_z^2 + V(z)\right)\psi(z) = 0\,.$$

When $V(z)$ is rational and

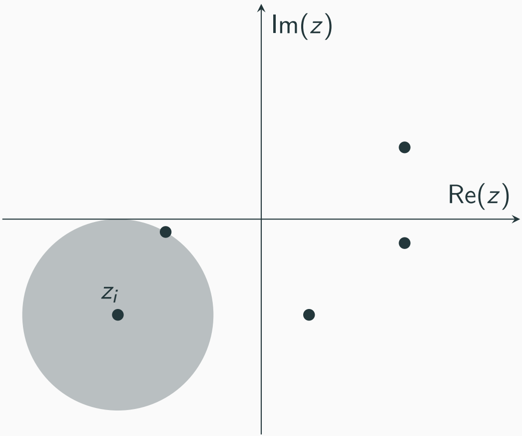
$$V(z) \simeq \frac{\#}{(z-z_i)^2}$$

the ODE is called *Fuchsian* and z_i is a *regular singular point*.

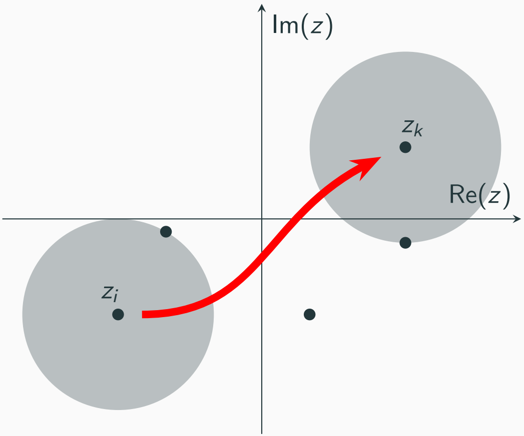
Explicit solutions of Fuchsian ODE are rare, so one considers local solutions (*Frobenius series*)

$$\psi(z) = \mathcal{C}_+ \sum_n c_n^+ (z-z_i)^{\alpha_+ + n} + \mathcal{C}_- \sum_n c_n^- (z-z_i)^{\alpha_- + n}\,.$$

The Frobenius series centered at $z \sim z_i$ only converges up to the next singular point.



In order to study global properties of the solution of the ODE, we need to analytically continue the series out of their domain of convergence.



Consider the local solution

$$\psi(z) = \psi_+(z) = z^{\alpha_+} \sum_n c_n^+ z^n \,.$$

Close to 1 we need to consider a different expansion:

$$\psi_+(z) = \mathcal{A}(1-z)^{\alpha_{1+}} \sum_n t_n^+ (1-z)^n + \mathcal{B}(1-z)^{\alpha_{1-}} \sum_n t_n^- (1-z)^n$$

all the information on the analytic continuation is contained in $\mathcal{A}$ and $\mathcal{B}$.

They are the *connection coefficients* of the ODE, and parametrize most of the complexity of the problem.

Example: 3 reg. singularities (WLOG at $0, 1, \infty$). At $z \sim 0$

$$\psi(z) = z^{\#_\pm}(1-z)^{\#}{}_2F_1(a_\pm, b_\pm; c_\pm, z)\,,$$

$${}_2F_1(a_\pm, b_\pm; c_\pm, z) = \sum_n \frac{(a_\pm)_n (b_\pm)_n}{(c_\pm)_n n!} z^n\,, \quad (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}\,.$$

$\mathcal{A}$, $\mathcal{B}$ for the hypergeometrics have been found by Gauss

$${}_2F_1(a, b; c, z) = \underbrace{\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}}_{\mathcal{A}}\,{}_2F_1(a, b; a+b+1-c, 1-z) +$$

$$+\underbrace{\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}}_{\mathcal{B}}(1-z)^{c-a-b}{}_2F_1(c-a, c-b, 1+c-a-b, 1-z)$$

Adding a 4th singularity greatly complicates the problem.

$4 \gg 3$: we can only fix 3 points on a sphere at $(0, 1, \infty)$. The 4th (t) appears as a new modulus!

The Fuchsian ODE with 4 reg. singularities is solved by *Heun functions*.

Heun connection coefficients can't be computed with classical methods (e.g. integral representations) that work for the ${}_2F_1$.

The following discussions will be devoted to the computation of these connection coefficients with a modern tool: Liouville CFT.

Wavy perturbation of a BH:

Black hole

$$\psi = (r - r_h)^{-i\frac{\beta\omega}{4\pi}} \sum_n c_n (r - r_h)^n \quad = \quad \begin{array}{l} \mathcal{A} r^{-1+2iM\omega} e^{i\omega r} \left(1 + \mathcal{O}(r^{-1})\right) + \\ + \mathcal{B} r^{1-2iM\omega} e^{-i\omega r} \left(1 + \mathcal{O}(r^{-1})\right) \end{array}$$

r_h and $r = \infty$ appear as singularities of a Heun equation.

Relevant physical data are encoded in $\mathcal{A}, \mathcal{B}$.

Outline

- Liouville CFT & the Heun connection problem
- Irregular states and confluences
- Applications to black holes
- Conclusions & further directions

Liouville CFT & the Heun connection problem

Notation:

$$c = 1 + 6Q^2\,, \quad Q = b + \frac{1}{b}\,, \quad b \geq 0\,, \quad \Delta = \frac{Q^2}{4} - \alpha^2\,, \quad \alpha \in i\mathbb{R}$$

Liouville CFT is characterized by [Teschner, ...]

- continuous and diagonal (i.e. scalar) spectrum
- structure constants meromorphic in the Liouville momenta α.

$$\langle \Delta_\alpha | \Delta_\beta \rangle = G_\alpha \delta(\alpha - \beta) = \frac{\Upsilon_b(Q + 2\alpha)}{\Upsilon_b(2\alpha)} \delta(\alpha - \beta)$$

$$\Upsilon_b(x + b) = \frac{\Gamma(bx)}{\Gamma(1 - bx)} b^{1-2bx} \Upsilon_b(x)\,.$$

3 point functions are given by ([DO][ZZ] formula)

$$\langle \Delta_1 | V_2(1) | \Delta_3 \rangle = C_{123}$$

and OPE reads

$$V_{\alpha_1}(t) V_{\alpha_2}(0) = \int_{\alpha \in i\mathbb{R}} d\alpha \, C^{\alpha}_{\alpha_1 \alpha_2} V_{\alpha}(0) (t\bar{t})^{\Delta - \Delta_1 - \Delta_2} (1 + \mathcal{O}(t, \bar{t}))$$

This allows us to compute any correlator

$$\langle \Delta_\infty | V_1(1) V_t(t) | \Delta_0 \rangle = \int d\alpha \, C^{\alpha}_{\alpha_t \alpha_0} (t\bar{t})^{\Delta - \Delta_t - \Delta_0} \langle \Delta_\infty | V_1(1) \left(1 + \mathcal{O}(t, \bar{t})\right) | \Delta_\alpha \rangle$$

$$= \int d\alpha \, C^{\alpha}_{\alpha_t \alpha_0} C_{\alpha_\infty \alpha_1 \alpha} \Bigg| \underbrace{t^{\Delta - \Delta_t - \Delta_0} \left(\sum_n c_n(\alpha_i, b) t^n \right)}_{\text{Conformal Block: } \mathfrak{F}(\alpha_{\infty,1}, \alpha_{0,t}, \alpha, t)} \Bigg|^2$$

A crucial property of CFT correlators is *crossing symmetry*:

$$\langle\Delta_\infty|V_1(1)\underbrace{V_t(t)|\Delta_0\rangle}_{\text{OPE}} = \langle\Delta_\infty|\underbrace{V_1(1)V_t(t)}_{\text{OPE}}|\Delta_0\rangle\,,$$

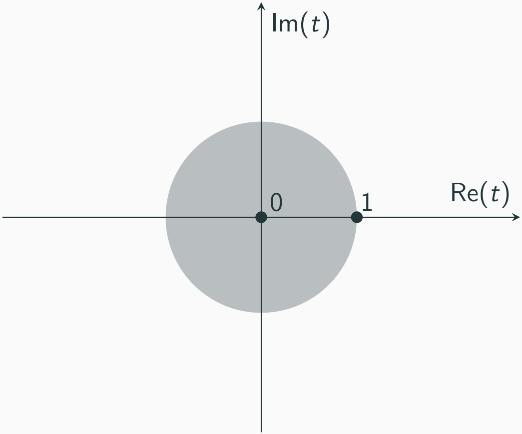
that is

$$\int d\alpha\, C^{\alpha}_{\alpha_t\alpha_0} C_{\alpha_\infty\alpha_1\alpha} \left|\mathfrak{F}(\alpha_{\infty,1},\alpha_{0,t},\alpha,t)\right|^2 = \\ = \int d\alpha\, C^{\alpha}_{\alpha_t\alpha_1} C_{\alpha_\infty\alpha_0\alpha} \left|\mathfrak{F}(\alpha_{\infty,0},\alpha_{1,t},\alpha,1-t)\right|^2$$

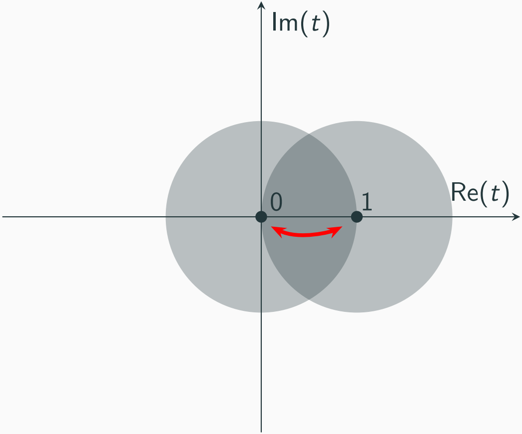
The LHS is a series in $t \sim 0$, while the RHS in $t-1\sim 0$.

It is a nontrivial constraint on the analytic continuation of conformal blocks.

Conformal blocks $\mathfrak{F}(\alpha_{\infty,1}, \alpha_{0,t}, \alpha, t)$ share some similarities with Frobenius series:



Crossing symmetry relates $\mathfrak{F}(\alpha_{\infty,1}, \alpha_{0,t}, \alpha, t)$ and $\mathfrak{F}(\alpha_{\infty,0}, \alpha_{1,t}, \alpha, 1-t)$: we only miss the ODE!



Liouville spectrum can be continued to contain degenerate dimensions

$$\Delta_{2,1} = \frac{Q^2}{4} - \Big(\underbrace{-\frac{Q}{2} - \frac{b}{2}}_{\alpha_{2,1} \notin i\mathbb{R}}\Big)^2 = -\frac{1}{2} - \frac{3}{4}b^2 \,.$$

Conformal families with degenerate dimensions contain null states:

$$|\chi\rangle = \left(b^{-2}L_{-1}^2 + L_{-2}\right)|\Delta_{2,1}\rangle \quad \text{s.t.} \quad \langle\chi|\chi\rangle = 0\,.$$

$|\chi\rangle$ is a primary orthogonal to all states:

$$\langle V_1(z_1)V_2(z_2)\ldots\chi(z)\ldots V_n(z_n)\rangle = 0$$

$$\left(b^{-2}L_{-1}^2 + L_{-2}\right)\cdot\phi_{2,1}(z)$$

L_n act as diff. operators: diff. eqn. ([BPZ]) for $\langle V_1(z_1)\ldots\phi_{2,1}(z)\ldots\rangle$!

Since $\chi(z)$ is a second level descendant, BPZ is a 2nd order PDE

$$\left(\partial_z^2 + f(z, z_i)\partial_z + \underbrace{V(z, z_i; \partial_{z_i})}_{\sim \frac{\Delta_i}{(z-z_i)^2} \text{ as } z\sim z_i}\right)\langle \dots \rangle = 0\,.$$

In principle the PDE has n variables.

$$\text{CFT Ward identities:} \quad n \longrightarrow n-3$$

Example: degenerate 4 pt function

$$\langle \Delta_\infty | V_1(1)\phi_{2,1}(z)|\Delta_0\rangle = \sum_{\pm} (\text{DOZZ}_\pm) \times \left| z^{\frac{bQ}{2}\pm b\alpha_0}{}_2F_1(\dots, z)\right|^2 .$$

Let us study the crossing relation

$$\langle\Delta_\infty|V_1(1)\underbrace{\phi_{2,1}(z)|\Delta_0\rangle}_{\text{OPE}}=\langle\Delta_\infty|\underbrace{V_1(1)\phi_{2,1}(z)}_{\text{OPE}}|\Delta_0\rangle\,.$$

that is

$$\sum_{\pm}C_{\alpha_\infty\alpha_1\alpha_{0\pm}}C^{\alpha_{0\pm}}_{\alpha_{2,1}\alpha_0}\Bigg|\underbrace{z^{\frac{bQ}{2}\pm\alpha_0}{}_2F_1(\ldots,z)}_{\sim\mathcal{A}_2F_1(\ldots,1-z)+\mathcal{B}_2F_1(\ldots,1-z)}\Bigg|^2=$$

$$=\sum_{\pm}C_{\alpha_\infty\alpha_0\alpha_{1\pm}}C^{\alpha_{1\pm}}_{\alpha_{2,1}\alpha_1}\left|(1-z)^{\frac{bQ}{2}\pm\alpha_1}{}_2F_1(\ldots,1-z)\right|^2.$$

Crossing symmetry tells us that

$$(\text{DOZZ})\longrightarrow(\mathcal{A},\mathcal{B})$$

Can we play this game with more complicated correlators?

$$\Psi(z,t) = \langle \Delta_\infty | V_1(1) V_t(t) \phi_{2,1}(z) | \Delta_0 \rangle \,.$$

It satisfies

$$\left(\partial_z^2 + \frac{b^2 \Delta_1}{(z-1)^2} - b^2 \frac{\Delta_1 + t\partial_t + \Delta_t + z\partial_z + \Delta_{2,1} + \Delta_0 - \Delta_\infty}{z(z-1)} + \right.$$
$$\left. + \frac{b^2 \Delta_t}{(z-t)^2} + \frac{b^2 t\partial_t}{z(z-t)} - \frac{b^2}{z}\partial_z + \frac{b^2 \Delta_0}{z^2} \right) \Psi(z,t) = 0 \,.$$

Orange terms make it into a PDE, but in the limit

$$b \to 0 \,, \quad b\alpha_i = a_i$$

$$b^2 t \partial_t \Psi(z,t) \simeq \underbrace{t\partial_t b^2 \langle \Delta_\infty | V_1(1) V_t(t) | \Delta_0 \rangle}_{\text{z-independent constant } u} \times f(z) \,.$$

To evaluate $t\partial_t\langle\dots\rangle$ we expand in conformal blocks:

$$u := \lim_{b\to 0} b^2 \left(\mathfrak{F}\left(\alpha_{\infty,1},\alpha_{t,0},\alpha;t\right)\right)^{-1} t\partial_t \mathfrak{F}\left(\alpha_{\infty,1},\alpha_{t,0},\alpha;t\right)$$

As $b\to 0$, $c\to\infty$, and $\mathfrak{F}(t)\sim\exp\left(b^{-2}F(t)\right)$, so

$$u = t\partial_t F(t) = -\frac{1}{4} + a_t^2 + a_0^2 - a^2 + \mathcal{O}(t)\,.$$

is finite and appears as a new parameters of the equation: the *accessory parameter*.

As $b\to 0$ the BPZ turns into an ODE with 4 reg. singularities at $(0,1,\infty,t)$ and parameters (a_0,a_1,a_t,a_∞,u): the Heun equation.

Now we can play the same game we played for the ${}_2F_1$:

$$\langle\Delta_\infty|V_1(1)\Pi_{\Delta_\alpha}V_t(t)\underbrace{\phi_{2,1}(z)|\Delta_0\rangle}_{\text{OPE}} = \langle\Delta_\infty|V_1(1)\Pi_{\Delta_\alpha}\underbrace{V_t(t)\phi_{2,1}(z)}_{\text{OPE}}|\Delta_0\rangle\,.$$

This gives

$$\begin{aligned} C_{\alpha_\infty\alpha_1\alpha}\sum_\pm C^{\alpha}_{\alpha_t\alpha_{0\pm}}C^{\alpha_{0\pm}}_{\alpha_{2,1}\alpha_0}\left|\mathfrak{F}^{(0)}_\pm\left(t,\frac{z}{t}\right)\right|^2 &= \\ = C_{\alpha_\infty\alpha_1\alpha}\sum_\pm C^{\alpha}_{\alpha_0\alpha_{t\pm}}C^{\alpha_{t\pm}}_{\alpha_{2,1}\alpha_t}\left|\mathfrak{F}^{(t)}_\pm\left(t,\frac{t-z}{t}\right)\right|^2 &. \end{aligned}$$

Plugging an Ansatz

$$\mathfrak{F}^{(0)}_+\left(t,\frac{z}{t}\right) = \mathcal{A}\,\mathfrak{F}^{(t)}_+\left(t,\frac{t-z}{t}\right) + \mathcal{B}\,\mathfrak{F}^{(t)}_-\left(t,\frac{t-z}{t}\right)$$

we can solve for the Heun connection coefficients from 0 to t!

In the convenient notation

$$\mathfrak{F}_{\theta}^{(0)} = \sum_{\theta'=\pm 1} M_{\theta\theta'}(0 \to t)\mathfrak{F}_{\theta'}^{(t)}$$

$$M_{\theta\theta'} = \frac{\Gamma\left(-\theta' 2a_t\right)\Gamma\left(1+2\theta a_0\right)}{\Gamma\left(\frac{1}{2}+\theta a_0-\theta' a_t + a\right)\Gamma\left(\frac{1}{2}+\theta a_0-\theta' a_t - a\right)}$$

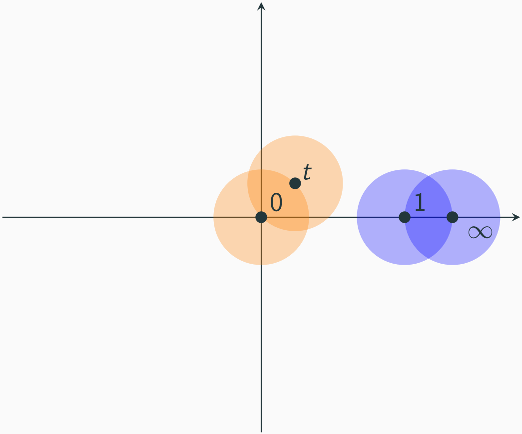
The Heun equation only depended on $(a_0, a_1, a_t, a_\infty, u)$, no a! But recall that

$$u = -\frac{1}{4} + a_t^2 + a_0^2 - a^2 + \mathcal{O}(t)$$

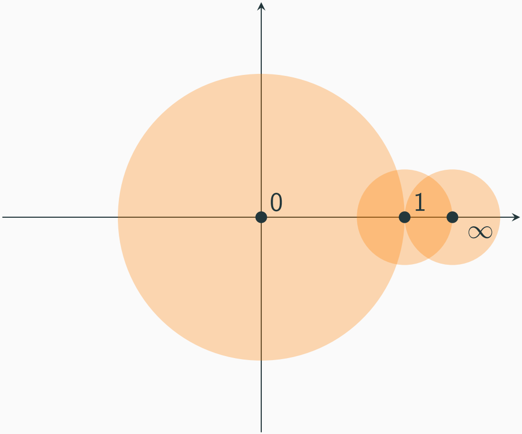
To really compute $M_{\theta\theta'}$ we need to invert the [Matone] relation

$$u = \sum_n c_n(a_i, a)t^n \longrightarrow a = \sum_n k_n(a_i, u)t^n$$

Since $u = \sum_n c_n t^n$, we are assuming $t \ll 1$

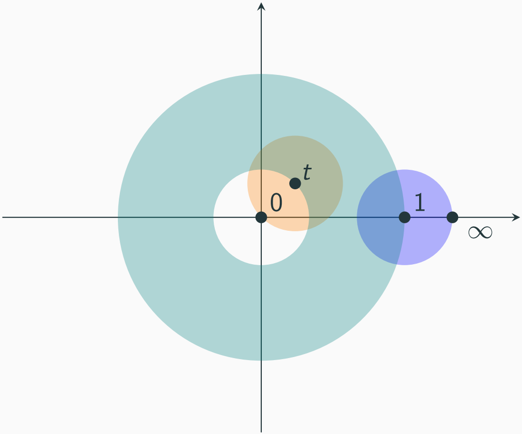


Note that this can't happen for 3 points!



To continue from $t \to 1$ we need to pass through the region

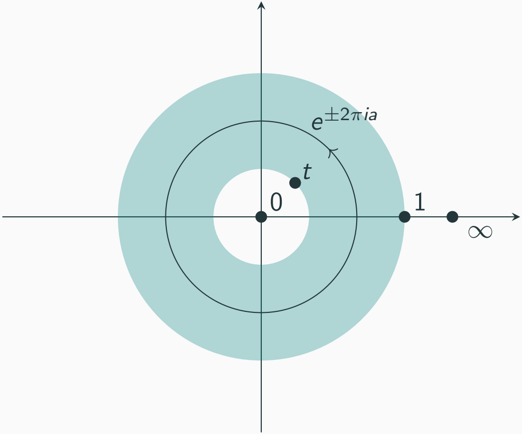
$$\langle \Delta_\infty | V_1(1) \underbrace{\Pi_{\Delta_\alpha} \phi_{2,1}(z)}_{\text{OPE}} V_t(t) | \Delta_0 \rangle$$



Orange $\xrightarrow{M_{\theta\sigma}}$ Teal $\xrightarrow{M_{\sigma\theta'}}$ Blue: $\mathfrak{F}^{(t)}_\theta = \sum_{\sigma=\pm,\theta'=\pm} M_{\theta\sigma} M_{\sigma\theta'} \mathfrak{F}^{(1)}_{\theta'}$

In the teal region

$$\langle \Delta_\infty | V_1(1) \underbrace{\Pi_{\Delta_\alpha} \phi_{2,1}(z)}_{\text{OPE}} V_t(t) | \Delta_0 \rangle \sim z^{\frac{1}{2} \pm a}$$



$a(u) \sim$ monodromy around 2 points

Irregular states and confluences

Irregular states and confluences

Fuchsian ODEs admit confluent limits where reg. singularities collide to produce *irregular* ones.

$$V(z) \supset \lim_{z_1 \to z_2} \frac{\frac{1}{4} - a_1^2}{(z - z_1)^2} + \frac{\frac{1}{4} - a_2^2}{(z - z_2)^2} \sim \frac{L^2}{(z - z_1)^4} + \frac{Lm}{(z - z_1)^3}$$

Rank 0: $\sim (z - z_i)^{-2}$

↓ *confluence*

Rank 1: $\sim (z - z_i)^{-4}$

↓ *reduction*

Rank $\frac{1}{2}$: $\sim (z - z_i)^{-3}$

In CFT singularities are generated by primary states, so confluent limits can be rephrased in CFT language [GT], e.g.

$$\Delta = \frac{Q^2}{4} - \left(\frac{\mu+\eta}{2}\right)^2 , \quad \Delta_t = \frac{Q^2}{4} - \left(\frac{\mu-\eta}{2}\right)^2 , \quad t = \frac{\eta}{\Lambda} ,$$

Rank 1 irreg. state: $\langle \mu, \Lambda | \propto \lim_{\eta\to\infty} t^{\Delta_t - \Delta} \langle \Delta | V_t(t)$.

The resulting state is characterized by

$$\langle \mu, \Lambda | L_0 = \Lambda \partial_\Lambda \langle \mu, \Lambda | , \quad \langle \mu, \Lambda | L_{-1} = \mu \Lambda \langle \mu, \Lambda | , \quad \langle \mu, \Lambda | L_{-2} = -\frac{\Lambda^2}{4} \langle \mu, \Lambda | .$$

Nonvanishing L_{-2} action produces a rank 1 singularity at ∞.

The $b \to 0$ limit of

$$\langle \mu, \Lambda | \Pi_{\Delta_\alpha} V_1(1) \phi_{2,1}(z) | \Delta_0 \rangle$$

solves a ODE with 2 reg. singularities at $0, 1$ and a rank 1 singularity at ∞: the confluent Heun equation.

$$(\text{DOZZ}) \xrightarrow{\text{Crossing sym.}} (\mathcal{A}, \mathcal{B})$$

The collision limit allows us to compute irregular DOZZ factors.

The accessory parameter now reads

$$u = \lim_{b \to 0} b^2 \left(\langle \mu, \Lambda | \Pi_{\Delta_\alpha} V_1(1) | \Delta_0 \rangle \right)^{-1} \Lambda \partial_\Lambda \langle \mu, \Lambda | \Pi_{\Delta_\alpha} V_1(1) | \Delta_0 \rangle =$$
$$= \frac{1}{4} - a^2 + \mathcal{O}(\underbrace{L}_{\lim_{b \to 0} b\Lambda})$$

By considering the *reduction* limit

$$\lim_{\mu\to\infty}\langle\mu,-\frac{\Lambda^2}{4\mu}| = \langle\Lambda^2|\,,$$

we can generate the rank $\frac{1}{2}$ state

$$\langle\Lambda^2|L_0 = \Lambda^2\partial_{\Lambda^2}\langle\Lambda^2|\,,\quad \langle\Lambda^2|L_{-1} = -\frac{\Lambda^2}{4}\langle\Lambda^2|\,.$$

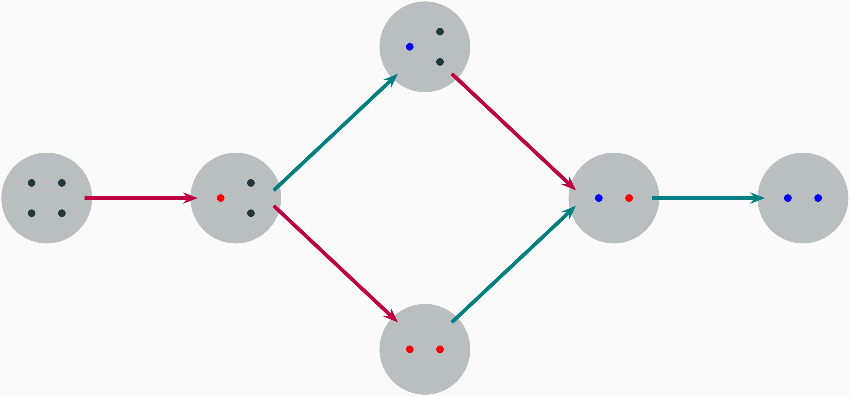
$\langle\Lambda^2|$ excites a rank $\frac{1}{2}$ (i.e. z^{-3}) singularity at ∞,

$$\langle\Lambda^2|V_1(1)\phi_{2,1}(z)|\Delta_0\rangle$$

solves the *reduced confluent* Heun equation, and again

$$(\text{DOZZ}) \xrightarrow{\text{Crossing sym.}} (\mathcal{A},\mathcal{B})$$

Performing successive confluences and reductions we can solve connection problems for Heun functions involving rank 0, 1 and $\frac{1}{2}$ singularities



Remark: we computed connection coefficients in terms of classical blocks

$$u = \lim_{b\to 0} b^2 \left(\mathfrak{F}\left(\alpha_{\infty,1}, \alpha_{t,0}, \alpha; t\right)\right)^{-1} t\partial_t \mathfrak{F}\left(\alpha_{\infty,1}, \alpha_{t,0}, \alpha; t\right) = t\partial_t F(t)\,.$$

AGT duality:

$$\mathfrak{F}\left(\alpha_{\infty,1}, \alpha_{t,0}, \alpha; t\right) = \mathcal{Z}^{SU(2)}_{\mathcal{N}=2}\Bigg(\underbrace{\alpha_1 \pm \alpha_\infty}_{m_1, m_2}, \underbrace{\alpha_0 \pm \alpha_t}_{m_3, m_4}, \alpha, t, \underbrace{\epsilon_1, \epsilon_2}_{1, b^2}\Bigg)$$

Thanks to localization, $\mathcal{Z}^{SU(2)}_{\mathcal{N}=2}$ admits explicit combinatorial formulas!

This representation goes through the confluent diagrams since

$$\text{confluent limits} \sim \text{holomorphic decouplings}\,.$$

Applications to black holes

Applications to black holes

Black hole perturbations in Kerr backgrounds satisfy a separable wave eq.

$$\Box_{BH}\psi(\vec{x}) = m^2\psi(\vec{x})\,.$$

$$\psi(\vec{x}) = \quad e^{im\phi - i\omega t} \quad \underbrace{S_\lambda(\theta, a\omega)}_{\text{Spheroidal harmonics}} \quad \underbrace{\psi(r)}_{\text{Confluent Heun function}}$$

To apply our method we compare the Heun eq. for $\psi(r)$ with the BPZ eq. to get a dictionary

$$u = -\lambda + 8M^2\omega^2 - a^2\omega^2 + 2M\omega^2(r_+ - r_-)\,, \quad a_0 = -i\frac{\omega - m\Omega}{4\pi T_H} + 2iM\omega$$

$$a_1 = -i\frac{\omega - m\Omega}{4\pi T_H}\,, \quad m_3 = 2iM\omega\,, \quad L = -2i\omega(r_+ - r_-)$$

Singularities are at the horizons (r_-, r_+) (reg.) and ∞ (rank 1)

Black hole

$$\psi = (r - r_+)^{-i\frac{\beta\omega}{4\pi}} \sum_n c_n (r - r_+)^n \quad = \quad \begin{array}{l} \mathcal{A} r^{-1+2iM\omega} e^{i\omega r}\left(1 + \mathcal{O}(r^{-1})\right) + \\ + \mathcal{B} r^{1-2iM\omega} e^{-i\omega r}\left(1 + \mathcal{O}(r^{-1})\right) \end{array}$$

$$\mathcal{A} = \sum_{\sigma=\pm} \frac{L^{-\frac{1}{2}-m_3+\sigma a} e^{-\frac{\sigma}{2}-\frac{1}{2}\partial_{m_3} F(L)} \Gamma\left(1 - 2\sigma a\right) \Gamma\left(-2\sigma a\right) \Gamma\left(1 + 2a_1\right)}{\Gamma\left(\frac{1}{2} + a_1 - \sigma a + a_0\right) \Gamma\left(\frac{1}{2} + a_1 - \sigma a - a_0\right) \Gamma\left(\frac{1}{2} - \sigma a - m_3\right)},$$

$$\mathcal{B} = \mathcal{A}\left((L, m_3) \to (-L, -m_3)\right).$$

The dictionary sets $u = u(\text{gravity})$ but a appears in $\mathcal{A}, \mathcal{B}$: inverting

$$a(u) = -\frac{1}{2}\sqrt{1-4u} - \frac{Lm_3(a_0^2 - a_1^2 + u)}{2u\sqrt{1-4u}} + \mathcal{O}(L^2) =$$
$$= -\frac{1}{2}\sqrt{1+4\lambda} - \frac{2\mathrm{a}m\omega}{\sqrt{1+4\lambda}} + \mathcal{O}(\omega^2)$$

- QNMs: $\mathcal{B}(\omega_n) = 0$
- graybody factor: $\sigma = -4\frac{\mathrm{Im}a_1}{\mathrm{Im}L}|\mathcal{A}|^{-2}$
- Love numbers, angular eigenvalue, phase shift ...

Phase shift: $\mathcal{B}/\mathcal{A}$

$$\frac{e^{2i\epsilon(\log(|2\epsilon|)-1/2)}}{|2\omega|^{2s}} e^{\partial_{m_3}F-\frac{L}{2}} e^{i\pi\left(a-\frac{1}{2}\right)} \frac{\Gamma\left(\frac{1}{2}-a-m_3\right)}{\Gamma\left(\frac{1}{2}-a+m_3\right)} \times \frac{1+e^{-i\pi a}\mathcal{K}}{1+e^{i\pi a}\frac{\cos(\pi(m_3-a))}{\cos(\pi(m_3+a))}\mathcal{K}}$$

Perturbative results in $GM\omega$ were known using the Mano-Suzuki-Takasugi method (e.g. [Saketh, Zhou, Ivanov]), so what's new?

Besides efficiency, an insight into the nonperturbative structure of the phase shift: e.g.

symmetries of $F \implies a(a_{BH}) \sim \nu(a_{BH})$ depends polynomially on a_{BH}

Let us consider now perturbations of a 5d AdS Schwarzschild black hole.

$$\Box_{BH}\psi(\vec{x}) = \Delta(\Delta-4)\psi(\vec{x})\,.$$

$$\psi(\vec{x}) = \quad e^{-i\omega t} \quad \underbrace{Y_{\ell m}(\theta,\phi)}_{\text{Spherical harmonics}} \quad \underbrace{\psi(r)}_{\text{Heun function}}$$

Again we compare BPZ Heun with gravity Heun to get a dictionary

$$u = -\frac{\ell(\ell+2)+2(2r_+^2+1)+r_+^2\Delta(\Delta-4)+\frac{r_+^2\omega^2}{2r_+^2+1}}{4(1+r_+^2)}\,, \quad a_1 = \frac{\Delta-2}{2}\,,$$

$$a_t = \frac{i\omega}{2}\frac{r_+}{1+2r_+^2}\,, \quad a_\infty = \frac{\omega}{2}\frac{\sqrt{r_+^2+1}}{1+2r_+^2}\,, \quad a_0 = 0\,, \quad t = \frac{r_+^2}{1+2r_+^2}\,.$$

At $r = \infty$ we have a gravitational wall, accordingly

$$\psi = (r - r_+)^{-i\frac{\beta\omega}{4\pi}} \sum_n c_n (r - r_+)^n = \mathcal{A} r^{\Delta-4}(1 + \mathcal{O}(r^{-1})) + \mathcal{B} r^{-\Delta}(1 + \mathcal{O}(r^{-1}))$$

The main object of interest in this case is

$$G_R(\omega, \ell) = \frac{\mathcal{B}(\omega, \ell)}{\mathcal{A}(\omega, \ell)}$$

According to AdS/CFT, this ratio computes the retarded correlator

$$i\theta(\tau)\langle[\mathcal{O}_\Delta(\tau, \vec{x}), \mathcal{O}_\Delta(0, \vec{x})]\rangle_\beta \propto \int d\omega\, e^{-i\omega t} \sum_\ell (\ell + 1) \frac{\sin(\ell + 1)\theta}{\sin\theta} G_R(\omega, \ell).$$

in the boundary theory.

We find

$$\mathcal{A} = \frac{\Gamma(-2a_1)\Gamma(1+2a_t)\Gamma(-2a)\Gamma(1-2a)}{\prod_{\pm}\Gamma\left(\frac{1}{2}+a_t-a\pm a_0\right)\Gamma\left(\frac{1}{2}-a-a_1\pm a_\infty\right)} t^a e^{-\frac{1}{2}\partial_a F} + \\ + (a \to -a)\,,$$

$$\mathcal{B} = \mathcal{A}(a_1 \to -a_1)\,.$$

where now

$$a(u) = -\frac{1+\ell}{2} + \mathcal{O}(\mu)$$

As $\mu/\ell \ll 1$ [Karlsson-Kulaxizi-Ng-Parnachev-Tadic-Fitzpatrick-Huang-Li-Dodelson-Zhiboedov, ..]

$$G_R(\omega,\ell) \simeq G_R^{pert}(\omega,\ell) = \frac{\Gamma(-2a_1)}{\Gamma(2a_1)}\frac{\prod_{\pm}\Gamma\left(\frac{1}{2}-a+a_1\pm a_\infty\right)}{\prod_{\pm}\Gamma\left(\frac{1}{2}-a-a_1\pm a_\infty\right)}$$

Poles of G_R (i.e. $\mathcal{A}(\omega_n, \ell) = 0$) are QNM of the black holes and resonances of the 2 pt functions.

$$\frac{1}{2} - a_n + a_1 + a_\infty = -n$$

This equation is algebraic and can be solved to very high orders algorithmically

$$\omega_n = \Delta + \ell + 2n - \frac{\mu}{2\ell}(\Delta^2 + \Delta(6n-1) + 6n(n-1)) + \dots$$

Similarly one can compute residues (i.e. holographic structure constants), and consider different expansions.

Conclusions & further directions

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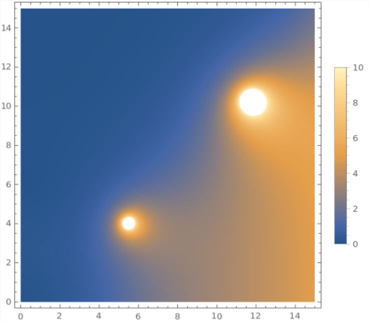
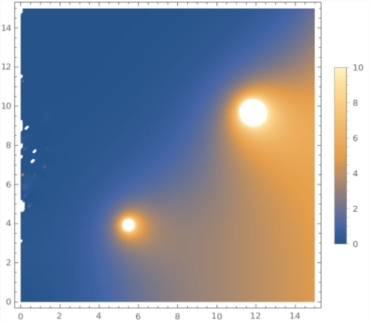
We computed Heun connection coefficients in terms of semiclassical conformal blocks.

They inherit convergence properties of conformal blocks, and are in this sense *exact*.





Conclusions & further directions

- Fuchsian ODE with more/higher order singularities.
- CFT approach is still perturbative in nature: connections to integrability? [Fioravanti-Gregori]
- Non generic choice of parameters (log and polynomial solutions)
- Different gravitational backgrounds (e.g. [Bianchi-Morales-Di Russo-Sudano], [Giusto-CI-Russo])
- ...

Thanks for the attention!