An eikonal approach to gravitational scattering and waveforms

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Università di Bologna, December 12th, 2024

Based on:

- 2306.16488: Report on the gravitational eikonal
  Paolo Di Vecchia, CH, Rodolfo Russo, Gabriele Veneziano
- 2312.07452, 2402.06361: Analysis of the NLO waveform
  In collaboration with Alessandro Georgoudis, CH, Rodolfo Russo
- 2406.03937: Angular momentum losses from the NLO waveform
  CH, Rodolfo Russo
- 2407.04128: Logarithmic soft theorems and soft spectra
  Francesco Alessio, Paolo Di Vecchia, CH

Outline

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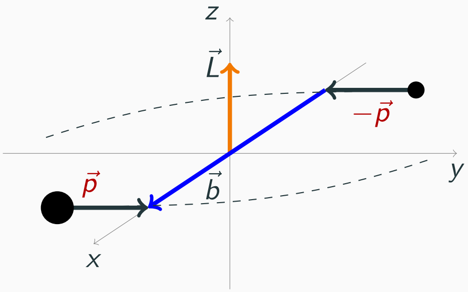
Two-Body Problem: Analytical Approximation Methods

- Post-Newtonian (PN): expansion
  "for small G and small v"

$$\frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \ll 1\,.$$

- Post-Minkowskian (PM): expansion
  "for small G"

$$\frac{Gm}{rc^2} \ll 1\,, \qquad \text{generic } \frac{v^2}{c^2}\,.$$



- Self-Force: expansion
  in the near-probe limit $m_2 \ll m_1$ or

$$m = m_1 + m_2\,, \qquad \nu = \frac{m_1 m_2}{m^2} \ll 1\,.$$

- Soft limit: expansion
  in the limit of small frequencies

$$\omega \ll \frac{v}{r}\,.$$

General Relativity from Scattering Amplitudes

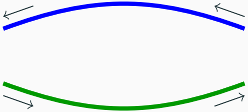
Key Idea: Extract the PM gravitational dynamics from scattering amplitudes.

- Weak-coupling expansion ↔ PM expansion

Weak-coupling:	$\mathcal{A}_0 = \mathcal{O}(G)$	$\mathcal{A}_1 = \mathcal{O}(G^2)$	$\mathcal{A}_2 = \mathcal{O}(G^3)$	$\mathcal{A}_3 = \mathcal{O}(G^4)$
PM:	1PM	2PM	3PM	4PM State of the art: [[Driesse et al. '24; Bern et al. '24] 5PM, 1SF from WQFT]

- Lorentz invariance ↔ generic velocities
- Study scattering events, then export to bound trajectories (V_{eff}, analytic continuation...) [Kälin, Porto '19; Saketh, Steinhoff, Vines, Buonanno '21; Cho, Kälin, Porto '21]

Some Recent Progress



$\mathcal{A}_0^{(4)} = \mathcal{O}(G)$

[Geissler '59]

$\mathcal{A}_1^{(4)} = \mathcal{O}(G^2)$

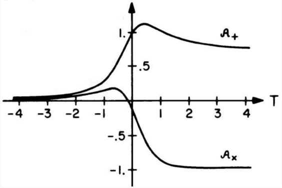
[Westpfahl '85]
[Cheung, Rothstein, Solon '18]
[Collado, Di Vecchia, Russo '19]

$\mathcal{A}_2^{(4)} = \mathcal{O}(G^3)$

[Bern et al. '19]
[Di Vecchia, CH, Russo, Veneziano '20, '21]
[Damgaard et al. '21]
[Brandhuber et al. '21]
[Jakobsen et al. '22]

$\mathcal{A}_3^{(4)} = \mathcal{O}(G^4)$

[Bern et al. '21]
[Dlapa et al. '21,'22]
[Jakobsen et al. '22, '23]
[Damgaard et al. '23]



$\mathcal{A}_0^{\mu\nu} = \mathcal{O}(G^{\frac{3}{2}})$

[Kovacs, Thorne '78]
[Goldberger, Ridgway '16]
[Luna, Nicholson, O'Connell, White '17]
[Jakobsen, Mogull, Plefka, Steinhoff '21]
[Mougiakakos, Riva, Vernizzi '21]
[De Angelis, Gonzo, Novichkov '23]
[Brandhuber et al. '23]
[Aoude, Haddad, CH, Helset '23]

$\mathcal{A}_1^{\mu\nu} = \mathcal{O}(G^{\frac{5}{2}})$

[Brandhuber et al. '23]
[Herderschee, Roiban, Teng '23]
[Elkhidir, O'Connell, Sergola, Vazquez-Holm '23]
[Georgoudis, CH, Vazquez-Holm '23]
[Caron-Huot, Giroux, Hannesdottir, Mizera '23]
[Georgoudis, CH, Russo '23, '24]
[Bini et al. '24]

Ref. [Bini, Damour, Geralico '23] reports **mismatches** with MPM-PN formalism (?)

Classical Soft Theorems

[Sahoo, Sen '18; Saha, Sahoo, Sen '19; Sahoo '20; Sahoo, Sen '21]

- Universal constraints on the soft expansion $\omega \to 0$ of the gravitational waveform:

$$\tilde{w}^{\mu\nu} = -\frac{i}{\omega}\, e^{2iGE\omega \log\omega} \sum_{n=0}^{\infty} \frac{1}{n!} (-i\omega \log\omega)^n\, a_n^{\mu\nu} + \cdots$$

where $\cdots \sim \omega^{n-1}(\log\omega)^m$ and $0 \leq m \leq n-1$, e.g. ω^0 or $\omega\log\omega$.

- Explicitly,

$$a_0^{\mu\nu} = \sum_a \frac{p_a^\mu p_a^\nu}{p_a \cdot n}\,, \quad a_1^{\mu\nu} = G \sum_{a,b} \frac{\tau_{ab}^{(\eta)} p_a^\mu}{p_a \cdot n} n_\rho p^\rho_{[b} p^\nu_{a]}\,, \quad a_2^{\mu\nu} = G^2 \sum_{a,b,c} \frac{\tau_{ab}^{(\eta)} \tau_{ac}^{(\eta)}}{p_a \cdot n} n_\rho p^\rho_{[b} p^\mu_{a]} n_\sigma p^\sigma_{[c} p^\nu_{a]}$$

and $\tau_{ab}^{(\eta)}$ is a function of the **invariants** $\sigma_{ab} = -\eta_a \eta_b p_a \cdot p_b/(m_a m_b)$ (with $\eta_a = +$ if the hard state is outoing, -1 if it is incoming).

$$\tau_{ab}^{(\eta)} = |\eta_a + \eta_b|\, \tau_{ab}\,, \qquad \tau_{ab} = -\frac{\sigma_{ab}(\sigma_{ab}^2 - \frac{3}{2})}{(\sigma_{ab}^2 - 1)^{3/2}} \quad \text{for GR}\,.$$

In general, it is fixed by the IR divergences of the theory.

Outline

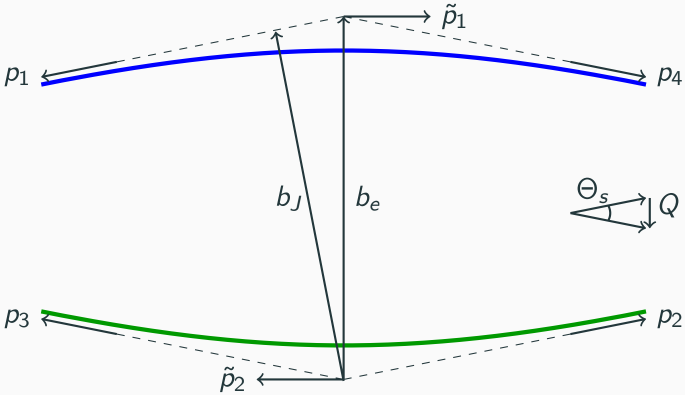
Kinematics of Classical Post-Minkowskian (PM) Scattering

$$\tilde{p}_1^\mu = m_1 \tilde{u}_1^\mu = \frac{1}{2}(p_4^\mu - p_1^\mu)$$

$$\tilde{p}_2^\mu = m_2 \tilde{u}_2^\mu = \frac{1}{2}(p_3^\mu - p_2^\mu)$$

$$Q^\mu = p_1^\mu + p_4^\mu = -p_2^\mu - p_3^\mu$$

$$b_e^\mu = b_J^\mu - \left(\frac{\check{v}_1^\mu}{2m_1} - \frac{\check{v}_2^\mu}{2m_2}\right) Qb$$

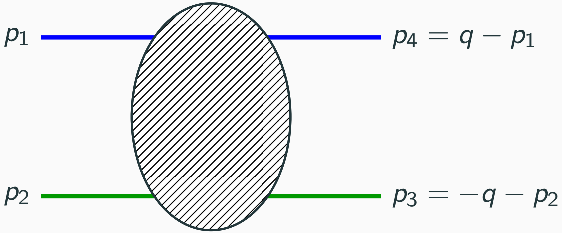


In this way, $v_1 \cdot b_J = v_2 \cdot b_J = 0$ and $\tilde{u}_1 \cdot b_e = \tilde{u}_2 \cdot b_e = 0$. Classical PM regime:

$$\frac{Gm^2}{\hbar} \underset{CL}{\gg} 1\,, \qquad \frac{Gm}{b} \underset{\text{PM}}{\ll} 1\,, \qquad \boxed{\frac{\hbar}{m} \ll Gm \ll b} \qquad \sigma = \frac{1}{\sqrt{1-v^2}} \geq 1 \text{ (generic).}$$

Kinematics of the Elastic $2 \to 2$ Amplitude

$$\bar{p}_1^\mu = \frac{1}{2}(p_4^\mu - p_1^\mu)$$
$$\bar{p}_2^\mu = \frac{1}{2}(p_3^\mu - p_2^\mu)$$
$$\boxed{q^\mu} = p_1^\mu + p_4^\mu = -p_2^\mu - p_3^\mu$$



Defining velocities by $p_1^\mu = -m_1 v_1^\mu$, $p_2^\mu = -m_2 v_2^\mu$

$$\boxed{\sigma} = -v_1 \cdot v_2 = \frac{1}{\sqrt{1 - v^2}}$$

with v the speed of either object as measured by the other one.

Dual velocities: $v_1^\mu = \sigma \check{v}_2^\mu + \check{v}_1^\mu$, $v_2^\mu = \sigma \check{v}_1^\mu + \check{v}_2^\mu$ obey $\check{v}_i \cdot v_j = -\delta_{ij}$.

The Elastic Eikonal

- From q to b: Fourier transform $[q \sim \mathcal{O}(\frac{\hbar}{b})]$

$$\tilde{\mathcal{A}}^{(4)}(b) = \frac{1}{4Ep}\int \frac{d^{D-2}q}{(2\pi)^{D-2}}\, e^{ib\cdot q}\mathcal{A}^{(4)}(q)\,, \qquad \boxed{1 + i\tilde{\mathcal{A}}^{(4)}(b) = e^{2i\delta(b)}}$$

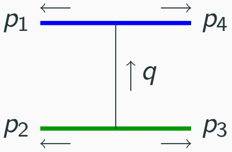
with $2\delta = 2\delta_0 + 2\delta_1 + 2\delta_2 + \cdots \sim \frac{Gm^2}{\hbar}\Big(\log b + \frac{Gm}{b} + \big(\frac{Gm}{b}\big)^2 + \cdots\Big)$

- From b to Q: stationary-phase approximation $[Q \sim \mathcal{O}(p \cdot \frac{Gm}{b})]$

$$\int d^{D-2}b\, e^{-ib\cdot Q} e^{i2\delta(b)} \implies Q_\mu = \frac{\partial\, \mathrm{Re}\, 2\delta}{\partial b_e^\mu}$$

Tree-Level Amplitude and 1PM Impulse

- Tree-level amplitude in $D = 4 - 2\epsilon$ dimensions



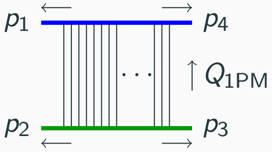
$$\mathcal{A}_0^{(4)}(q) = \frac{32\pi G m_1^2 m_2^2 (\sigma^2 - \frac{1}{2-2\epsilon})}{q^2} + \cdots$$

$$\tilde{\mathcal{A}}_0^{(4)}(b) = \frac{4Gm_1 m_2(\sigma^2 - \frac{1}{2-2\epsilon})}{2\sqrt{\sigma^2-1}} \frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}} .$$

- Matching to the eikonal exponentiation [Kabat, Ortiz '92; Bjerrum-Bohr et al. '18]

$$e^{2i\delta_0} \underset{\text{“small } G\text{”}}{\longrightarrow} 1 + i\tilde{\mathcal{A}}_0^{(4)} \implies 2\delta_0 = \tilde{\mathcal{A}}_0^{(4)} .$$

- From $2\delta_0$, we obtain the leading-order deflection



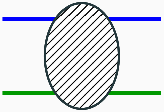
$$Q_{1\mathrm{PM}} = -\frac{\partial 2\delta_0}{\partial b} = \frac{4Gm_1 m_2 \left(\sigma^2 - \frac{1}{2}\right)}{b\sqrt{\sigma^2-1}}$$

$$\Theta_{1\mathrm{PM}} = \frac{4GE\left(\sigma^2 - \frac{1}{2}\right)}{b(\sigma^2-1)} .$$

Elastic $2 \to 2$ Amplitude up to One Loop

[Cheung, Rothstein, Solon '18; Collado, Di Vecchia Russo '19]

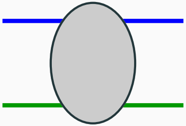
$$\mathcal{A}^{(4)} =$$



$$= \mathcal{A}_0^{(4)} + \mathcal{A}_1^{(4)} + \cdots$$

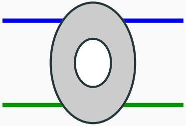
with

$$\mathcal{A}_0^{(4)} =$$

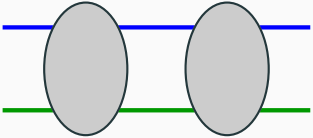


$$= \frac{32\pi G m_1^2 m_2^2 \left(\sigma^2 - \frac{1}{2-2\epsilon}\right)}{q^2} + \mathcal{O}(q^0)$$

$$\mathcal{A}_1^{(4)} =$$



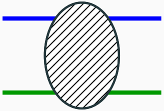
$$= \mathrm{Re}\,\mathcal{A}_1^{(4)} + \frac{i}{2}$$



$$\mathrm{Re}\,\mathcal{A}_1^{(4)} = 2\pi G m_1^2 m_2^2 (m_1 + m_2) \frac{3\pi(5\sigma^2 - 1)}{q} + \mathcal{O}(\log(q))$$

Impulse from the Eikonal Phase up to One Loop

[Collado, Di Vecchia, Russo '19]

$1 + i\,\mathrm{FT}$  $\sim e^{2i\delta}\,,\qquad 2\delta = 2\delta_0 + 2\delta_1 + \cdots \qquad Q_\mu = \frac{\partial 2\delta}{\partial b_e^\mu}$

- Tree level: $i\tilde{\mathcal{A}}_0 = 2i\delta_0$, so

$$2\delta_0 = \tilde{\mathcal{A}}_0^{(4)} = \frac{2Gm^2\nu(\sigma^2 - \frac{1}{2-2\epsilon})}{\sqrt{\sigma^2-1}}\frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}}\,, \qquad Q^\mu_{\mathrm{1PM}} = -\frac{4Gm^2\nu(\sigma^2-\frac{1}{2})}{b\sqrt{\sigma^2-1}}\frac{b_e^\mu}{b}\,.$$

- One loop: By the exponentiation $i\tilde{\mathcal{A}}_1 - \frac{1}{2!}(2i\delta_0)^2 = i\,\mathrm{Re}\,\tilde{\mathcal{A}}_1 = 2i\delta_1$, so

$$2\delta_1 = \mathrm{Re}\,\tilde{\mathcal{A}}_1^{(4)} = \frac{3\pi G^2 m^3\nu\left(5\sigma^2-1\right)}{4b\sqrt{\sigma^2-1}}\,, \qquad Q^\mu_{\mathrm{2PM}} = -\frac{3\pi G^2 m^3\nu\left(5\sigma^2-1\right)}{4b^2\sqrt{\sigma^2-1}}\frac{b_e^\mu}{b}\,.$$

The 3PM Eikonal in General Relativity [Di Vecchia, CH, Russo, Veneziano '20, '21]

[Related work at 3PM: Bern, Cheung, Roiban, Shen, Solon, Zeng '19; Damour '20; Herrmann, Parra-Martinez, Ruf, Zeng '21, Bjerrum-Bohr, Damgaard, Planté, Vanhove '21; Brandhuber, Chen, Travaglini, Wen '21]

- Eikonal phase:

$$\mathrm{Re}\, 2\delta_2 = \frac{4G^3 m_1^2 m_2^2}{b^2}\left[\frac{s\left(12\sigma^4-10\sigma^2+1\right)}{2m_1 m_2\left(\sigma^2-1\right)^{\frac{3}{2}}} - \frac{\sigma\left(14\sigma^2+25\right)}{3\sqrt{\sigma^2-1}} - \frac{4\sigma^4-12\sigma^2-3}{\sigma^2-1}\,\mathrm{arccosh}\sigma\right] + \mathrm{Re}\, 2\delta_2^{\mathrm{RR}}$$

with

$$\mathrm{Re}\, 2\delta_2^{\mathrm{RR}} = \frac{G}{2} Q_{1\mathrm{PM}}^2 \mathcal{I}(\sigma)\,, \quad \mathcal{I}(\sigma) \equiv \frac{8-5\sigma^2}{3(\sigma^2-1)} + \frac{\sigma\left(2\sigma^2-3\right)}{(\sigma^2-1)^{3/2}}\,\mathrm{arccosh}\,\sigma\,.$$

- Infrared divergent exponential suppression:

$$\mathrm{Im}\, 2\delta_2 = \frac{1}{\pi}\left[-\frac{1}{\epsilon} + \log(\sigma^2-1)\right] \mathrm{Re}\, 2\delta_2^{\mathrm{RR}} + \cdots$$

Smoothness and Universality of $\mathrm{Re}\, 2\delta_2$ at High Energy [Di Vecchia, CH, Russo, Veneziano '20, '21]

At high energy, as $\sigma \to \infty$ and $s \sim 2m_1 m_2 \sigma$, i.e. in the massless limit:

- The *complete* eikonal phase is smooth, although the conservative and radiation-reaction parts separately diverge like $\log \sigma$
- Its expression is the same in $\mathcal{N} = 8$ supergravity and in GR,

$$\mathrm{Re}\, 2\delta_2 \sim Gs \frac{\Theta_s^2}{4}\,, \qquad \Theta_s \sim \frac{4G\sqrt{s}}{b}$$

in agreement with [Amati, Ciafaloni, Veneziano '90].

Outline

Kinematics of the $2 \to 3$ Amplitude

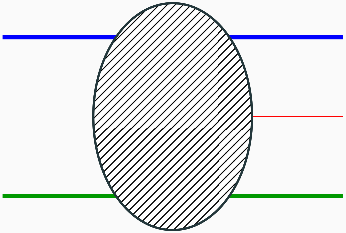
$$\bar{p}_1^\mu = \frac{1}{2}(p_4^\mu - p_1^\mu)$$

$$\bar{p}_2^\mu = \frac{1}{2}(p_3^\mu - p_2^\mu)$$

$$\boxed{q_1^\mu} = p_1^\mu + p_4^\mu$$

$$\boxed{q_2^\mu} = p_2^\mu + p_3^\mu$$

$$0 = q_1^\mu + q_2^\mu + k^\mu$$



More invariants, besides q_1^2, q_2^2, also

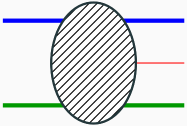
$$\boxed{\sigma} = -v_1 \cdot v_2 \,, \qquad \boxed{\omega_1} = -v_1 \cdot k \,, \qquad \boxed{\omega_2} = -v_2 \cdot k \,.$$

We denote by E, ω the total energy and the graviton frequency in the CoM frame,

$$E = \sqrt{-(p_1 + p_2)^2} \,, \qquad \omega = \frac{1}{E}(p_1 + p_2) \cdot k = \frac{1}{E}(m_1\omega_1 + m_2\omega_2) \,, \qquad \alpha_{1,2} = \frac{\omega_{1,2}}{\omega} \,.$$

$2 \to 3$ Amplitude up to One Loop

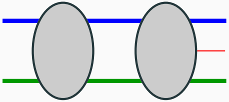
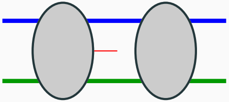
[Brandhuber et al. '23; Herderschee, Roiban, Teng 23; Elkhidir, O'Connell, Sergola, Vazquez-Holm '23] [Georgoudis, CH, Vazquez-Holm '23]

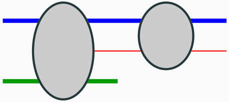
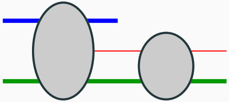
$\mathcal{A} =$  $= \mathcal{A}_0 + \mathcal{A}_1 + \cdots$

with $\mathcal{A}_0$ the tree-level amplitude, and

$$\mathcal{A}_1 = \mathcal{B}_1 + \frac{i}{2}(s + s') + \frac{i}{2}(c_1 + c_2)\,.$$

where $\mathcal{B}_1 = \operatorname{Re}\mathcal{A}_1$ and the unitarity cuts can be depicted as follows,

$s =$  $s' =$ 

$c_1 =$  $c_2 =$ 

Inelastic Final State [Di Vecchia, CH, Russo, Veneziano '22]

[cf. Kosower, Maybee, O'Connell '18; Damgaard, Planté, Vanhove '21; Cristofoli et al. '21]

Eikonal Exponentiation of Graviton Exchanges + Coherent Radiation:

$$e^{2i\hat{\delta}(b_1,b_2)} = e^{i\,\mathrm{Re}\,2\delta(b)} e^{i\int_k \left[\tilde{W}(k)a^\dagger(k)+\tilde{W}^*(k)a(k)\right]} .$$

- Final state, schematically:

$$|\mathrm{out}\rangle = e^{2i\hat{\delta}(b_1,b_2)}|\mathrm{in}\rangle$$

- Unitarity:

$$\langle\mathrm{out}|\mathrm{out}\rangle = \langle\mathrm{in}|\mathrm{in}\rangle = 1$$

- The asymptotic metric fluctuation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ sourced by the scattering (the waveform) is expressed formally as

$$h_{\mu\nu}(x) = \sqrt{32\pi G}\,\langle\mathrm{out}|\hat{H}_{\mu\nu}(x)|\mathrm{out}\rangle \sim \frac{4G}{\kappa r}\int_0^\infty e^{-i\omega U}\,\tilde{W}_{\mu\nu}(\omega n)\,\frac{d\omega}{2\pi} + (\mathrm{c.c.})$$

  where $\kappa = \sqrt{8\pi G}$, r is the distance from the observer and U the retarded time. Normalization $\tilde{W}^{\mu\nu} = \kappa\,\tilde{w}^{\mu\nu}$.

The Waveform Kernel

[See e.g. Herderschee, Roiban, Teng '23]

- Working with "eikonal" variables, we can use the following radiation kernel,

$$W = \mathcal{A}_0 + \left[\mathcal{B}_1 + \frac{i}{2}\left(c_1 + c_2\right)\right].$$

- Tree level: $\mathcal{A}_0$ is a relatively simple rational function
- One loop: We isolate the even and odd parts of $\mathcal{B}_1$ under $\omega_{1,2} \mapsto -\omega_{1,2}$,

$$\mathcal{B}_1 = \mathcal{B}_{1O} + \mathcal{B}_{1E}\,,$$

  and $\mathcal{B}_{1O}$ is fixed in terms of the tree-level amplitude,

$$\mathcal{B}_{1O} = \left[1 - \frac{\sigma\left(\sigma^2 - \frac{3}{2}\right)}{(\sigma^2-1)^{3/2}}\right]\pi G E \omega\,\mathcal{A}_0$$

  while

$$\mathcal{B}_{1E} = \left[\frac{A^R_{\omega_1}}{\omega_1^2(q_2^2+\omega_1^2)^{7/2}} + \frac{A^R_{q_1}}{\omega_2^2 q_1}\right]\frac{m_1^3 m_2^2}{q_2^2 \mathcal{Q}_1^4} + (1 \leftrightarrow 2)\,.$$

- Here, A^R_X are polynomials and $\mathcal{Q}_1 = (q_1^2 - q_2^2)^2 - 4q_1^2\omega_1^2$

The imaginary part is determined by the rescattering or Compton cuts, for instance

$$
\begin{aligned}
\frac{i}{2}\, c_1 &= iGm_1\omega_1 \left(-\frac{1}{\epsilon} + \log\frac{\omega_1^2}{\mu_{\text{IR}}^2}\right)[\mathcal{A}_0]_{D=4} + i m_1^3 m_2^2 \mathcal{M}^{m_1^3 m_2^2}\,, \\
\mathcal{M}^{m_1^3 m_2^2} &= \frac{A^I_{\text{rat}}}{q_1^2 q_2^2(\sigma^2-1)\omega_1\omega_2^2(q_2^2+\omega_1^2)^3\mathcal{Q}_1^3\mathcal{P}\mathcal{Q}} \\
&\quad + \frac{A^I_{\omega_1}}{q_2^2\omega_1^2(q_2^2+\omega_1^2)^3\mathcal{P}\mathcal{Q}_1^4}\,\text{arcsinh}\,\frac{\omega_1}{q_2} + \frac{A^I_{q_1}}{q_2^2\omega_1(\sigma^2-1)\mathcal{P}^2\mathcal{Q}^2}\,\frac{\text{arccosh}\,\sigma}{\sqrt{\sigma^2-1}} \\
&\quad + \frac{A^I_{\omega_1\omega_2}}{\omega_1\omega_2^2\mathcal{P}^2\mathcal{Q}^2}\,\log\frac{\omega_1^2}{\omega_2^2} + \frac{A^I_{q_1 q_2}}{q_1^2 q_2^2\mathcal{Q}_1^4\mathcal{P}\mathcal{Q}^2}\,\log\frac{q_1^2}{q_2^2}
\end{aligned}
$$

with

$$
\mathcal{P} = -\omega_1^2 + 2\omega_1\omega_2\sigma - \omega_2^2\,, \qquad \mathcal{Q} = (q_1^2)^2\omega_1^2 - 2q_1^2 q_2^2\omega_1\omega_2\sigma + (q_2^2)^2\omega_2^2\,.
$$

Infrared Divergences Revisited

[Georgoudis, CH, Russo '24]

- Infrared divergences exponentiate in momentum space,

$$W = e^{-\frac{i}{\epsilon}\,GE\omega}\left[\mathcal{A}_0 + \mathcal{B}_1 + \frac{i}{2}\,(c_1 + c_2)^{\text{reg}}\right] = e^{-\frac{i}{\epsilon}\,GE\omega}\,W^{\text{reg}}\,,$$

  where

$$\frac{i}{2}\,c_1^{\text{reg}} = \frac{i}{2}\,c_1 + \frac{i}{\epsilon}\,Gm_1\omega_1\mathcal{A}_0$$

- This also modifies the finite part by $\frac{i}{\epsilon}\,Gm_1\omega_1$ times the $\mathcal{O}(\epsilon)$ part of $\mathcal{A}_0$.
- After this step, the divergence can be canceled by redefining the origin of retarded time, arriving at the following well defined expression

$$h_{\mu\nu}(x) \sim \frac{4G}{\kappa r}\int_0^{\infty} e^{-i\omega U}\,\tilde{W}^{\text{reg}}_{\mu\nu}(\omega n)\,\frac{d\omega}{2\pi} + (\text{c.c.})\,,$$

See also [Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, Teng '24]

Universal Terms ω^{-1}, $\log\omega$, $\omega(\log\omega)^2$

- Leading $1/\omega$ soft term (memory effect in time domain) [matches Weinberg '64; Sahoo, Sen '18; '21]

$$\tilde{W}^{[\omega^{-1}]} = \frac{i\kappa Q}{b\omega\tilde{\alpha}_1^2\tilde{\alpha}_2^2}(\tilde{\alpha}_1\tilde{u}_2\cdot\varepsilon - \tilde{\alpha}_2\tilde{u}_1\cdot\varepsilon)(2\tilde{\alpha}_1\tilde{\alpha}_2 b_e\cdot\varepsilon + b_e\cdot n(\tilde{\alpha}_1\tilde{u}_2\cdot\varepsilon + \tilde{\alpha}_2\tilde{u}_1\cdot\varepsilon))$$

- Subleading $\log\omega$ soft term [matches Sahoo, Sen '18; '21]

$$\tilde{W}^{[\log\omega]} = \kappa\frac{2Gm_1m_2\sigma(2\sigma^2-3)}{\tilde{\alpha}_1\tilde{\alpha}_2(\sigma^2-1)^{3/2}}(\tilde{\alpha}_1\tilde{u}_2\cdot\varepsilon - \tilde{\alpha}_2\tilde{u}_1\cdot\varepsilon)^2\log\left(\frac{\omega b\,e^{\gamma}}{2\sqrt{\sigma^2-1}}\right) + 2iGE\omega\,\tilde{W}_0^{[\omega^{-1}]}\log\omega$$

- Sub-subleading $\omega(\log\omega)^2$ soft term [matches Sahoo, Sen '18; '21]

$$\tilde{W}^{[\omega(\log\omega)^2]} = 2iGE\omega\tilde{W}_0^{[\log\omega]}\log\omega$$

A Conjecture for $\omega^{\ell-1}(\log\omega)^\ell$ for any $\ell \geq 0$ [Alessio, CH, Di Vecchia '24]

- Considering an elastic $2 \to 2$ hard process, let us define

$$E = (p_1{+}p_2){\cdot}n\,, \quad B^{\mu\nu}(p_1,p_2) = (p_1{+}p_2){\cdot}n\left(\frac{p_1^\mu p_1^\nu}{p_1\cdot n} + \frac{p_2^\mu p_2^\nu}{p_2\cdot n}\right) - (p_1^\mu{+}p_2^\mu)(p_1^\nu{+}p_2^\nu)\,.$$

- Then, the known soft theorems [Sahoo, Sen '18: '21] for $\ell = 0,1,2$ reduce to (define $h(\sigma) = \sigma(2\sigma^2-3)/(\sigma^2-1)^{3/2}$)

$$a_\ell^{\mu\nu} = \frac{1}{E}(-GEh(\sigma))^\ell\left[B^{\mu\nu}(p_1,p_2) - (-1)^\ell B^{\mu\nu}(p_3,p_4)\right]$$

- We **conjecture** that this expression generalizes to any $\ell \geq 0$.

- Frequency-domain resummation

$$\tilde{w}^{\mu\nu} = -\frac{i}{E\omega}\,\omega^{2iGE\omega}\left[\omega^{iGE\omega h(\sigma)}B^{\mu\nu}(p_1,p_2) - \omega^{-iGE\omega h(\sigma)}B^{\mu\nu}(p_3,p_4)\right] + \cdots$$

- Cross-checks: Newtonian quadrupole as $p_\infty \to 0$ to all orders in G (for generic GM/bp_∞^2); 2PN approximation up to $\mathcal{O}(G^3)$ [Bini, Damour, Geralico '24]; near-probe limit $\nu \to 0$ [Fucito, Morales, Russo '24].

- Non-universal ω^0 piece of the tree-level result,

$$\tilde{W}_0^{[\omega^0]} = \kappa(\tilde{\alpha}_1 \tilde{u}_2 \cdot \varepsilon - \tilde{\alpha}_2 \tilde{u}_1 \cdot \varepsilon)^2 \left[\frac{G m_1 m_2 \sigma (2\sigma^2 - 3)}{\tilde{\alpha}_1 \tilde{\alpha}_2 (\sigma^2 - 1)^{3/2}} \log(\tilde{\alpha}_1 \tilde{\alpha}_2) - \frac{2 G m_1 m_2 (2\sigma^2 - 1)}{\tilde{\mathcal{P}} \sqrt{\sigma^2 - 1}} \right]$$
$$+ \frac{4 G m_1 m_2}{\tilde{\mathcal{P}}} \left[\frac{(\tilde{\alpha}_1 \tilde{u}_2 \cdot \varepsilon - \tilde{\alpha}_2 \tilde{u}_1 \cdot \varepsilon)^2}{\tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\mathcal{P}}} \left(g_3 \operatorname{arccosh} \sigma + g_2 \log \frac{\tilde{\alpha}_1}{\tilde{\alpha}_2} \right) \right.$$
$$\left. + \frac{2\sigma^2 - 1}{2 b^2 \tilde{\alpha}_1^2 \sqrt{\sigma^2 - 1}} g_1 \right] + i b_2 \cdot n\, \omega \tilde{W}_0^{[\omega^{-1}]} .$$

- For this one, both regions are needed!

$\omega \log \omega$

[Georgoudis, CH, Russo '23]

- Tree-level $\omega \log \omega$ piece [matches Ghosh, Sahoo '21]

$$\tilde{W}_0^{[\omega \log \omega]} = \kappa \frac{2iGm_1 m_2 \sigma(2\sigma^2 - 3)}{\tilde{\alpha}_1 \tilde{\alpha}_2 (\sigma^2 - 1)^{3/2}} (\tilde{\alpha}_1\, \tilde{u}_2 \cdot \varepsilon - \tilde{\alpha}_2\, \tilde{u}_1 \cdot \varepsilon)$$
$$\times \left[\tilde{\alpha}_1 \tilde{\alpha}_2\, b_e \cdot \varepsilon + \tilde{\alpha}_2 (b_1 \cdot n)(\tilde{u}_1 \cdot \varepsilon) - \tilde{\alpha}_1 (b_2 \cdot n)(\tilde{u}_2 \cdot \varepsilon)\right] \omega \log \omega$$

- Non-universal one-loop $\omega \log \omega$ piece. $\mathcal{B}_{1O}$ contributes in the obvious way, while $\mathcal{B}_{1E}$ does not contribute. Finally,

$$\frac{i}{2} (\tilde{c}_1 + \tilde{c}_2)^{[\omega \log \omega]} = iGE \left[-\frac{1}{\epsilon} + \log \frac{\alpha_1 \alpha_2}{\mu_{\mathrm{IR}}^2}\right] \omega \tilde{W}_0^{[\log \omega]} + 2iGE\omega \log \omega\, \tilde{W}_0^{[\omega^0]} + i \tilde{\mathcal{M}}_1^{[\omega \log \omega]}$$

with

$$i\tilde{\mathcal{M}}_1^{[\omega \log \omega]} = i\kappa\omega \log \omega\, G^2\, m_1^2 m_2 \frac{2\sigma(\alpha_1\, u_2 \cdot \varepsilon - \alpha_2\, u_1 \cdot \varepsilon)^2}{(\sigma^2 - 1)^{3/2} \tilde{\mathcal{P}}}$$
$$\times \left[\frac{2\sigma^2 - 3}{\tilde{\mathcal{P}}} \left(f_3 \frac{\operatorname{arccosh} \sigma}{(\sigma^2 - 1)^{3/2}} + f_2 \frac{1}{\alpha_2} \log \frac{\alpha_1}{\alpha_2}\right) - \frac{f_1}{\alpha_2 (\sigma^2 - 1)}\right] + (1 \leftrightarrow 2)\,.$$

Comparison with Predictions from MPM Formalism

[Georgoudis, CH, Russo '23]

- The result for the $\omega \log \omega$ term was given explicitly in the PN expansion using the Multipolar post-Minkowskian (MPM) formalism in [Bini, Damour, Geralico '23], where a mismatch was found when comparing with the amplitude-based result starting at 2.5PN ($\sim 1/c^5$)
- We find that agreement is restored after performing the following supertranslation [Veneziano, Vilkovisky '22]

$$U \mapsto U - T(n)\,, \qquad T(n) = 2G(m_1\alpha_1 \log \alpha_1 + m_2\alpha_2 \log \alpha_2)$$

or more precisely

$$\delta_T h_{AB} = -T(n)\,\partial_u h_{AB} + r\,[2D_A D_B - \gamma_{AB}\Delta]\,T(n)$$

where only the first term on the RHS (the non-static one) matters.
Here, $n^\mu = (1, \hat{n})$, $e^\mu_A = \partial_A n^\mu$, $h_{AB} = r^2 e^\mu_A e^\nu_B h_{\mu\nu}$, $\gamma_{AB} = e_A \cdot e_B$, D_A is the associated covariant derivative, $\Delta = D_A D^A$.

The PN Limit

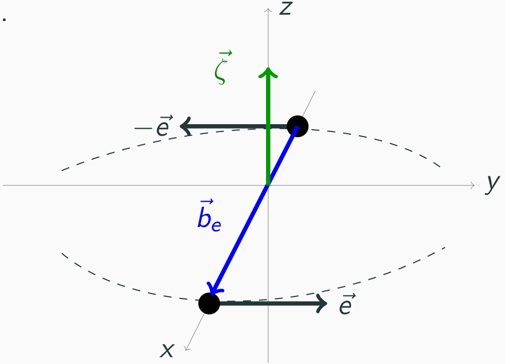
- The PN expansion is defined by the limit

$$p_\infty = \sqrt{\sigma^2 - 1} = \mathcal{O}(\lambda)\,, \qquad \omega = \mathcal{O}(\lambda) \qquad \text{as } \lambda \to 0$$

- Each instance of the Newton constant G increases the PN order by one unit.
- Each power of λ increases it by half a unit.

Reference vectors in the CoM frame:

$$t^\alpha = (1,0,0,0)$$
$$b_e^\alpha = (0,b,0,0)$$
$$e^\alpha = (0,0,1,0)$$
$$\zeta^\mu = (0,0,0,1)$$



Multipolar Decomposition

[Blanchet, Damour '85; Blanchet '13]

- We define the dimensionless frequency
$$u = \frac{\omega b}{p_\infty}\,,$$
which does not scale in the PN limit.
- It is convenient to express the waveform in terms of "multipoles", i.e. symmetric trace-free (STF) tensors $\mathrm{U}_L(u)$, $\mathrm{V}_L(u)$,
$$h_{ij}^{\mathrm{TT}} = \frac{4G}{r}\sum_{\ell=2}^{\infty}\frac{1}{\ell!}\left[n_{L-2}\,\mathrm{U}_{ijL-2}(u) - \frac{2\ell}{\ell+1}\,n_{cL-2}\epsilon_{cd(i}\,\mathrm{V}_{j)dL-2}(u)\right]^{\mathrm{TT}}$$
(decomposition into symmetric, traceless tensors with definite Δ-eigenvalue)
- Order by order in the PN expansion, only the first few U_L, V_L show up.
- **We computed all building blocks of the kernel to NNNLO** in the small λ limit and extracted the associated multipoles.

Examples

[Georgoudis, CH, Russo '24]

Newtonian quadrupole at tree level,

$$\mathrm{U}_{11}^{\mathrm{LO}} = -\frac{4Gm^2\nu}{3p_\infty}(K_0(u) + 3uK_1(u)),$$

$$\mathrm{U}_{12}^{\mathrm{LO}} = -\frac{4iGm^2\nu}{p_\infty}(uK_0(u) + K_1(u)),$$

$$\mathrm{U}_{22}^{\mathrm{LO}} = \frac{4Gm^2\nu}{3p_\infty}(2K_0(u) + 3uK_1(u)),$$

$$\mathrm{U}_{33}^{\mathrm{LO}} = -\frac{4Gm^2\nu K_0(u)}{3p_\infty}$$

1PN correction to the quadrupole due to $\mathcal{B}_{1E}$,

$$\mathrm{U}_{E11} = -\mathrm{U}_{E22} = -\frac{6\pi G^2 m^3\nu}{bp_\infty}(1+u)\,e^{-u},$$

$$\mathrm{U}_{E12} = -\frac{6i\pi G^2 m^3\nu}{bp_\infty}\left(\frac{1}{u} + 1 + u\right)e^{-u},$$

while e.g. one component at 2PN is

$$\mathrm{U}_{E33}^{\mathrm{NLO}} = -\frac{\pi G^2 m^3 \nu p_\infty}{b}(2\nu - 5)(u+1)\,e^{-u}.$$

Comparison with MPM Formalism

[Georgoudis, CH, Russo '24]

- Integer PN terms arise from various corrections to the trajectories.
- Half-odd PN: **Tail formula**

$$\mathrm{U}_L^{\text{tail}} = \frac{2GE}{c^3} i\omega \mathrm{U}_L^{\text{tree}} \left(\log \frac{\omega}{\mu_{\text{IR}}} - \kappa_\ell - \frac{i\pi}{2} \right)$$

(similarly for $\mathrm{V}_L(u)$ with π_ℓ)

- Half-odd PN: **Nonlinear** effects, e.g.

$$U_{ij}^{QQ} = \frac{G}{c^5} \left[\frac{1}{7} I^{(5)}_{a\langle i} I_{j\rangle a} - \frac{5}{7} I^{(4)}_{a\langle i} I^{(1)}_{j\rangle a} - \frac{2}{7} I^{(3)}_{a\langle i} I^{(2)}_{j\rangle a} \right]$$

- Half-odd PN: **Radiation-reaction**

$$x^\mu_{RR} = \frac{8G^2 m^2 p_\infty \nu}{5b^2 r} \left(b^2 e^\mu - (r + p_\infty t) b_e^\mu \right) \qquad U_{ij}^{RR} = 2m\nu \frac{d^2}{dt^2} \left(x_{\langle i}\, x^{RR}_{j\rangle} \right)$$

We checked that C^{reg} completely agrees with the MPM prediction given by tail+nonlinear+radiation-reaction up to and including 2.5PN.

See also [Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, Teng '24]

Outline

Emitted Energy-Momentum

[Georgoudis, CH, Russo '24]

- The operator insertion $\langle \text{out}|\hat{P}^\alpha|\text{out}\rangle = P^\alpha$ leads to leads to

$$P^\alpha = \int_k k^\alpha \rho(k)\,, \qquad \int_k = \int 2\pi\theta(k^0)\,\delta(k^2)\,\frac{d^D k}{(2\pi)^D}$$

where the spectral emission rate ρ is given by

$$\rho = \tilde{w}^{\text{TT}*}_{\mu\nu}\,\tilde{w}^{\text{TT}\mu\nu} = \tilde{W}^*_{\mu\nu}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \tfrac{1}{D-2}\,\eta^{\mu\nu}\eta^{\rho\sigma}\right)\tilde{W}_{\rho\sigma}$$

Note the equivalence between the two expressions, with

$$\tilde{w}^{TT}_{\mu\nu} = \Pi^{\text{TT}}_{\mu\nu\rho\sigma}\tilde{W}^{\rho\sigma}\,, \qquad k_\mu \tilde{W}^{\mu\nu}(k) = 0\,.$$

- We can choose the TT projector to be space-like in the CoM frame, so that

$$\kappa^2 P^0 \equiv \kappa^2 E_{\text{rad}} = G\int_0^\infty \frac{d\omega}{\pi}\oint \frac{d\Omega}{2\pi}\,\omega^2\,\tilde{w}^{\text{TT}*}_{ab}\,\tilde{w}^{\text{TT}}_{ab}\,,$$

$$\kappa^2 P^i = G\int_0^\infty \frac{d\omega}{\pi}\oint \frac{d\Omega}{2\pi}\,\omega^2 n^i\,\tilde{w}^{\text{TT}*}_{ab}\,\tilde{w}^{\text{TT}}_{ab}\,,$$

Emitted Energy-Momentum

[Georgoudis, CH, Russo '24]

Using the explicit waveforms obtained in the PN limit, we get

$$
\begin{aligned}
&E_{\rm rad}/(m\nu^2)\\
&= \frac{G^3m^3}{b^3}\,\pi p_\infty\left[\frac{37}{15}+\left(\frac{1357}{840}-\frac{37\nu}{30}\right)p_\infty^2\right]\\
&+\frac{G^4m^4}{b^4p_\infty}\left[\frac{1568}{45}+\left(\frac{18608}{525}-\frac{1136}{45}\nu\right)p_\infty^2\right]\\
&+\frac{G^4m^4}{b^4}p_\infty^2\left[\frac{3136}{45}+\left(\frac{1216}{105}-\frac{2272}{45}\nu\right)p_\infty^2\right]\\
&+\cdots
\end{aligned}
$$

$$
\begin{aligned}
&P^i/(m\nu^2\sqrt{1-4\nu})\\
&=\frac{G^3m^3}{b^3}\pi\left[-\frac{37}{30}p_\infty^2+\left(\frac{37}{60}\nu-\frac{839}{1680}\right)p_\infty^4\right]e^i\\
&+\frac{G^4m^4}{b^4}\left[-\frac{64}{3}+\left(\frac{32}{3}\nu-\frac{1664}{175}\right)p_\infty^2\right]e^i\\
&+\frac{G^4m^4}{b^4}p_\infty^3\left[\left(\frac{1491}{400}-\frac{26757}{5600}p_\infty^2\right)\pi\,\frac{b_e^i}{b}\right.\\
&\left.+\left(-\frac{128}{3}+\left(\frac{64}{3}\nu-\frac{192}{75}\right)p_\infty^2\right)e^i\right]\\
&+\cdots
\end{aligned}
$$

The component along b_e^μ of $P^\mu_{\rm rad}$ is sensitive to $C^{\rm reg}$ and the ϵ/ϵ!

Perfect agreement with [Bini, Damour, Geralico '21; '22; Dlapa, Kälin, Liu, Neef, Porto '22]

Soft spectra from soft theorems

[Alessio, Di Vecchia, CH '24]

- The resummed waveform in the soft limit gives universal results for the "leading logs" (LL) of the type $(\omega \log \omega)^n$ in the energy emission spectrum $dE/d\omega$.
- In the CoM frame we find, expanding for small deflections $Q \to 0$,

$$\left(\frac{dE}{d\omega}\right)_{\text{LL}} = \left[1 - \cos\left(2GEh(\sigma)\omega \log \omega\right)\right] \frac{2G}{\pi} \mathcal{H}(m_1, m_2, \sigma) + \cos\left(2GEh(\sigma)\omega \log \omega\right) \frac{GQ^2}{\pi} \mathcal{I}(\sigma) + \cdots$$

fixing $G^{2n+1}(\omega \log \omega)^{2n}$ for $n = 1, 2, \ldots$ and $G^{2n+3}(\omega \log \omega)^{2n}$ for $n = 0, 1, 2, \ldots$ (see the additional material for the functions $\mathcal{H}(m_1, m_2, \sigma)$ and $\mathcal{I}(\sigma)$).

- In the ultrarelativistic limit instead

$$\left(\frac{dE}{d\omega}\right)_{\text{LL}} = \frac{4G}{\pi} \left[\sin(2G\sqrt{s}\,\omega \log \omega)\right]^2 s + \frac{4G}{\pi} \cos(4G\sqrt{s}\,\omega \log \omega) \left[Q^2 \log\left(\frac{s}{Q^2} - 1\right) - s \log\left(1 - \frac{Q^2}{s}\right)\right] + \cdots$$

Emitted Angular Momentum

- $\langle \text{out}|\hat{J}_{\alpha\beta}|\text{out}\rangle = \boldsymbol{J}_{\alpha\beta} + \mathcal{J}^{\text{tot}}_{\alpha\beta}$
- Radiative contribution

$$i\boldsymbol{J}_{\alpha\beta} = \int_k \left[\tfrac{1}{2} \left(\tilde{w}^{\text{TT}*}_{\mu\nu} k_{[\alpha} \frac{\partial \tilde{w}^{\text{TT}\mu\nu}}{\partial k^{\beta]}} - \tilde{w}^{\text{TT}\mu\nu} k_{[\alpha} \frac{\partial \tilde{w}^{\text{TT}*}_{\mu\nu}}{\partial k^{\beta]}} \right) + 2\tilde{w}^{\text{TT}*}_{\mu[\alpha} \tilde{w}^{\text{TT}\mu}_{\beta]} \right]$$

or equivalently [Manohar, Ridgway, Shen '22] [Di Vecchia, CH, Russo '22]

$$i\boldsymbol{J}_{\alpha\beta} = \int_k \left[\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \tfrac{1}{D-2}\, \eta^{\mu\nu}\eta^{\rho\sigma} \right) \tilde{W}^*_{\mu\nu} k_{[\alpha} \frac{\overset{\leftrightarrow}{\partial} \tilde{W}_{\rho\sigma}}{\partial k^{\beta]}} + 2\eta^{\mu\nu} \tilde{W}^*_{\mu[\alpha} \tilde{W}_{\beta]\nu} \right].$$

- In particular, for the spatial components in the CoM frame [Compère, Oliveri, Seraj '19],

$$\kappa^2 \boldsymbol{J}^{ij} = G \int_0^\infty \frac{d\omega}{i\pi} \oint \frac{d\Omega}{2\pi} \tilde{w}^*_{ab} \partial_A \tilde{w}_{ab}\, \omega\, \gamma^{AB} n^{[i} \partial_B n^{j]} + 2G \int_0^\infty \frac{d\omega}{i\pi} \oint \frac{d\Omega}{2\pi} \tilde{w}^{*a[i} \tilde{w}^{j]a}\, \omega\,.$$

Emitted Mass Dipole

- The emitted **mass-dipole** or boost charge J_{i0} is related to the initial position of the center of mass of the system (times the energy) Z_i by

$$\dot{J}_{i0} = -\dot{L}_{i0} = \dot{Z}_i \,.$$

- The mass-dipole (space/time) components inherit a time-translation ambiguity from the infrared divergences ("drift") in the waveform.
- We can subtract this off by defining

$$\boldsymbol{M}_i = \boldsymbol{J}_{i0} - \int t\dot{P}_i \, dt$$

- Explicitly [Compère, Oliveri, Seraj '19]

$$\kappa^2 \boldsymbol{J}_{i0} = G \int_0^\infty \frac{d\omega}{i\pi} \oint \frac{d\Omega}{4\pi} \left(\tilde{w}_{ab}^{\mathrm{TT}*} \omega \partial_\omega \tilde{w}_{ab}^{\mathrm{TT}} - \tilde{w}_{ab}^{\mathrm{TT}} \omega \partial_\omega \tilde{w}_{ab}^{\mathrm{TT}*} \right) \omega \, n_i + \kappa^2 \boldsymbol{M}_i \,,$$

$$\kappa^2 \boldsymbol{M}_i = G \int_0^\infty \frac{d\omega}{i\pi} \oint \frac{d\Omega}{4\pi} \left(\tilde{w}_{ab}^{\mathrm{TT}*} \partial_A \tilde{w}_{ab}^{\mathrm{TT}} - \tilde{w}_{ab}^{\mathrm{TT}} \partial_A \tilde{w}_{ab}^{\mathrm{TT}*} \right) \omega \, \gamma^{AB} \partial_B n_i \,.$$

Exponential dressing of the eikonal operator

We can include static/Coulombic modes by letting $e^{2i\hat{\delta}(b_1,b_2)} \mapsto S_{s.r.} e^{2i\hat{\delta}(b_1,b_2)}$ with

$$S_{s.r.} = e^{\int_k^* \left[F^{\mu\nu}(k) a^\dagger_{\mu\nu}(k) - F^{*\mu\nu}(k) a_{\mu\nu}(k)\right]}$$

where [Weinberg '64,'65]

$$F^{\mu\nu}(k) = \sum_a \frac{\sqrt{8\pi G}\, p_a^\mu p_a^\nu}{p_a \cdot k - i0}\,, \qquad n_\mu F^{\mu\nu}(k) = i\pi\sqrt{8\pi G} \sum_{a\in\text{in}} p_a^\mu \delta(\omega) \neq 0$$

and $\int_k^* = \int_k \theta(\omega^* - k^0)$, with ω^* a cutoff (to be sent to zero).

Angular Momentum of the Static Gravitational Field $\mathcal{J}_{\alpha\beta}$

[Di Vecchia, CH, Russo '22] [see also: Veneziano, Vilkovisky '22; Javadinezhad, Porrati '22, '23; Riva, Vernizzi, Wong '23]

This leads to

$$i\mathcal{J}_{\alpha\beta} = \int_k \left[\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \tfrac{1}{D-2}\,\eta^{\mu\nu}\eta^{\rho\sigma} \right) F^*_{\mu\nu} k_{[\alpha} \frac{\overset{\leftrightarrow}{\partial} F_{\rho\sigma}}{\partial k^{\beta]}} + 2\eta^{\mu\nu} F^*_{\mu[\alpha} F_{\beta]\nu} \right] .$$

Angular momentum loss due to static modes

$$\mathcal{J}^{\alpha\beta} = \frac{G}{2} \sum_{a,b} c(\sigma_{ab})\,(\eta_a - \eta_b)\, p_a^{[\alpha} p_b^{\beta]} , \quad c(\sigma_{ab}) = - \left[\left(\frac{\sigma_{ab}^2 - \frac{3}{2}}{\sigma_{ab}^2 - 1} \right) \frac{\sigma_{ab}\,\mathrm{arccosh}\,\sigma_{ab}}{\sqrt{\sigma_{ab}^2 - 1}} + \frac{\sigma_{ab}^2 - \frac{1}{2}}{\sigma_{ab}^2 - 1} \right]$$

- Match with [Damour '20; Manohar, Ridgway, Shen '22; Bini, Damour '22] up to $\mathcal{O}(G^3)$ upon expanding

$$\mathcal{J}^{\alpha\beta} = -\frac{G}{2}\,(p_1 - p_2)^{[\alpha} Q^{\beta]} \mathcal{I}(\sigma) + \mathcal{O}(G^4) , \qquad Q^\mu = Q^\mu_{1\mathrm{PM}} + Q^\mu_{2\mathrm{PM}} + \mathcal{O}(G^3)$$

- **Easy** to include tidal [CH '22] and spin [Alessio, Di Vecchia '22] [CH '23] effects, via Q^α.

Nonlinear memory contribution?

[Heissenberg, Russo '24]

- Take outgoing gravitons into account by

$$\sum_a \mapsto \sum_{a_m} + \int_k \rho(k)$$

where a_m runs over massive states, $\rho(k)$ is the distribution of emitted gravitons.

- This is the operation that gives the nonlinear memory effect,

$$a_0^{\mu\nu} \mapsto a_0^{\mu\nu} + \delta a_0^{\mu\nu}\,, \qquad \delta a_0^{\mu\nu} = \int_k \rho(k)\, \frac{k^\mu k^\nu}{k \cdot n}\,.$$

- For the static contribution to the angular momentum, it gives

$$\mathcal{J}_{\alpha\beta} \mapsto \mathcal{J}^{\text{tot}}_{\alpha\beta} = \mathcal{J}_{\alpha\beta} + \delta\mathcal{J}_{\alpha\beta} + \mathcal{O}(G^7)\,, \qquad \delta\mathcal{J}^{\alpha\beta} = 2G \int_k \rho(k) \sum_{a\in\text{in}} p_a^{[\alpha} k^{\beta]} \log \frac{p_a \cdot k}{m_a \Lambda}$$

and Λ is an energy scale introduced to regulate the collinear divergence. It amounts to a time-translation ambiguity in the mass-dipole components.

- **However**, $\delta\mathcal{J}^{ij} = \mathcal{O}(G^5)$ in the CoM frame! We do not need it to go to $\mathcal{O}(G^4)$.

Angular momentum loss in the PN expansion

Combining everything, $J_{xy} = \boldsymbol{J}_{xy} + \mathcal{J}_{xy} + \mathcal{O}(G^5)$ with

$$\begin{aligned} J_{xy} = & \frac{G^2 m^3}{b} p_\infty^2 \nu^2 \left[\frac{16}{5} + \left(\frac{176}{35} - \frac{8}{5}\nu\right) p_\infty^2 + \mathcal{O}(p_\infty^4)\right] \\ & + \frac{G^3 \pi m^4}{b^2} \nu^2 \left[\frac{28}{5} + \left(\frac{739}{84} - \frac{79}{15}\nu\right) p_\infty^2 + \mathcal{O}(p_\infty^4)\right] \\ & + \frac{G^4 m^5}{b^3 p_\infty^2} \nu^2 \left[\frac{176}{5} + \left(\frac{8144}{105} - \frac{2984}{45}\nu\right) p_\infty^2 + \mathcal{O}(p_\infty^4)\right] \\ & + \frac{G^4 m^5}{b^3} p_\infty \nu^2 \left[\frac{448}{5} + \left(\frac{1184}{21} - \frac{220256}{1575}\nu\right) p_\infty^2 + \mathcal{O}(p_\infty^4)\right] + \mathcal{O}(G^5). \end{aligned}$$

- The first two lines reproduce the small-velocity expansion of the $\mathcal{O}(G^2)$ [Damour '20] and $\mathcal{O}(G^3)$ [Manohar, Ridgway, Shen '22] [Di Vecchia, CH, Russo, Veneziano '22]
- The last two lines are in perfect agreement with the 0PN, 1PN, 1.5PN and 2.5PN contributions at $\mathcal{O}(G^4)$ [Bini, Damour, Geralico '21; '22].

Summary and Outlook

- The **eikonal approach** provides a framework to calculate scattering observables, including the **impulse**, the **waveform**, emitted **energy and angular momentum**.
- The comparison with the **PN** results is interesting both technically and conceptually. There is full agreement up to and including 2.5PN once the amplitudes and the MPM results are written in the same BMS frame

For the future:

- Is the choice of a BMS frame relevant in other comparisons (PN versus NR, PN versus NRGR-EFT, theory vs experiment)? Is it relevant for bound orbits?
- Generalizing/proving the resummation of leading logs?
- When does the naive eikonal exponentiation break down? (If it does)
- Analytic continuation? [Damour, Deruelle '81; Adamo, Gonzo, Ilderton '24]
- Analytic results beyond soft/PN limit? [Brunello, De Angelis '24]
- NNLO waveform? Starting from the soft limit (nonlinear memory)?

ADDITIONAL MATERIAL

The Initial State

- We model the initial state by $|\mathrm{in}\rangle = |1\rangle \otimes |2\rangle$, with

$$|1\rangle = \int_{-p_1} \varphi_1(-p_1)\, e^{ib_1\cdot p_1} | -p_1\rangle$$
$$|2\rangle = \int_{-p_2} \varphi_2(-p_2)\, e^{ib_2\cdot p_2} | -p_2\rangle$$

  and $\int_{-p_i} = \int 2\pi\delta(p_i^2 + m_i^2)\theta(-p_i^0)\frac{d^D p_i}{(2\pi)^D}$ the LIPS measure.

- Wavepackets $\varphi_i(-p_i)$ peaked around the classical incoming momenta.
- Impact parameter $b^\mu = b_1^\mu - b_2^\mu$ lies in the transverse plane $b \cdot p_1 = 0 = p_2 \cdot b$.

Elastic and Inelastic Fourier Transforms

- Elastic Fourier transform:

$$\begin{aligned}\mathrm{FT}\,\mathcal{A}^{(4)} &= \int \frac{d^D q}{(2\pi)^D}\, 2\pi\delta(2m_1 v_1\cdot q)\, 2\pi\delta(2m_2 v_2\cdot q) e^{ib\cdot q}\mathcal{A}^{(4)}(q) \\ &= \frac{1}{4Ep}\int \frac{d^{D-2}q}{(2\pi)^{D-2}}\, e^{ib\cdot q}\mathcal{A}(s,q) = \tilde{\mathcal{A}}^{(4)}\,.\end{aligned}$$

- Inelastic Fourier transform:

$$\begin{aligned}\mathrm{FT}\,\mathcal{A}^{(5)} = \int & \frac{d^D q_1}{(2\pi)^D}\frac{d^D q_2}{(2\pi)^D}\,(2\pi)^D\delta^{(D)}(q_1+q_2+k) \\ & \times 2\pi\delta(2m_1 v_1\cdot q_1) 2\pi\delta(2m_2 v_2\cdot q_2) e^{ib_1\cdot q_1 + ib_2\cdot q_2}\mathcal{A}^{(5)}(q_1,q_2,k) \\ & = \tilde{\mathcal{A}}^{(5)}(k)\,.\end{aligned}$$

Analytic Continuation to the Bound Case

[Saketh, Vines, Steinhoff, Buonanno '21; Cho, Kälin, Porto '21] [CH '23]

- The results discussed so far hold for the **scattering kinematics**, in which the total center-of-mass energy is

$$E = \sqrt{m_1^2 + 2m_1 m_2\,\sigma + m_2^2} \geq m_1 + m_2\,, \qquad \sigma \geq 1\,.$$

- To analytically continue $J(L = pb, a_1, \sigma)$ to the **bound-state kinematics**, $\sigma < 1$, one can sum the two branch choices $\sqrt{\sigma^2 - 1} \to \pm i\sqrt{1-\sigma^2}$

$$J^{\text{bound}}(L, a_1, \sigma) = J(L, a_1, \sigma)_+ + J(L, a_1, \sigma)_-$$

- The $\mathcal{O}(G^3)$ result $J^{\mathcal{O}(G^3)}(L, a_1, \sigma)$ is an analytic function of σ for $\text{Re}\,\sigma > -1$, so

$$J^{\mathcal{O}(G^3)\text{bound}}(L, a_1, \sigma) = 2J^{\mathcal{O}(G^3)}(L, a_1, \sigma)\,.$$

From the Deflection Angle to the Precession Angle

We introduce the effective potential $V(r)$

$$p^2 = p_r^2 + \frac{J^2}{r^2} + V(r)\,, \qquad V(r) = -\left(\frac{G}{r} f_1 + \frac{G^2}{r^2} f_2 + \frac{G^3}{r^3} f_3 + \cdots\right)$$

to extract information about the bound system as well.

- Matching to the **conservative** PM deflection angle, one can fix f_1, f_2, f_3. E.g. in GR, [Bern et al. '19, Damour '20]

$$f_1 = 4m_1^2 m_2^2(\sigma^2 - \tfrac{1}{2})/E\,, \qquad f_2 = \frac{3}{2}(m_1 + m_2)m_1^2 m_2^2(5\sigma^2 - 1)/E\,,$$

- Analytically continuing to $\sigma < 1$ (**bound case**) and working in the Post-Newtonian limit $v_\infty = \sqrt{1-\sigma^2} \to 0$ for fixed $\alpha \equiv G m_1 m_2/(J v_\infty)$ matches the corresponding orders in [Blanchet '13]

$$\Delta\Phi = -2\pi + 2J \int_{r_-}^{r_+} \frac{dr}{r^2\sqrt{p^2 - \frac{J^2}{r^2} - V(r)}} = 3v_\infty^2\alpha^2 - \frac{3}{4}v_\infty^4\alpha^2\left[2\nu - 5 + 5\alpha^2(2\nu - 7)\right]$$

The functions $\mathcal{H}(m_1, m_2, \sigma)$ and $\mathcal{I}(\sigma)$ appearing in the low-frequency spectrum

$$\mathcal{H}(\sigma, m_1, m_2) = \left[2(s - m_1 m_2 \sigma) + \frac{m_2^2(2m_1\sigma + m_2)}{m_1\sqrt{\sigma^2-1}}\,\ell_1 + \frac{m_1^2(2m_2\sigma + m_1)}{m_2\sqrt{\sigma^2-1}}\,\ell_2\right],$$

with

$$\ell_1 = \log\left(\frac{x\,(m_1 x + m_2)}{m_2 x + m_1}\right), \qquad \ell_2 = \log\left(\frac{x\,(m_2 x + m_1)}{m_1 x + m_2}\right), \qquad x = \sigma - \sqrt{\sigma^2 - 1}\,,$$

while

$$\mathcal{I}(\sigma) = \frac{2}{\sigma^2 - 1}\left[\frac{8 - 5\sigma^2}{3} + \frac{\sigma(2\sigma^2 - 3)\,\mathrm{arccosh}\,\sigma}{\sqrt{\sigma^2-1}}\right].$$

$\mathcal{J}_{\alpha\beta}$ vs $\mathcal{J}^{\mathrm{TT}}_{\alpha\beta}$

- Since $n_\mu F^{\mu\nu} \neq 0$, the formula used above is *in general* not equivalent to the one obtained by using the TT-projected static field $f_{\mu\nu} = \Pi_{\mu\nu\alpha\beta} F^{\alpha\beta}$,

$$i\mathcal{J}^{\mathrm{TT}}_{\alpha\beta} = \int_k \left[\tfrac{1}{2} \left(f^*_{\mu\nu} k_{[\alpha} \frac{\partial f^{\mu\nu}}{\partial k^{\beta]}} - f^{\mu\nu} k_{[\alpha} \frac{\partial f^*_{\mu\nu}}{\partial k^{\beta]}} \right) + 2 f^*_{\mu[\alpha} f^{\mu}_{\beta]} \right],$$

- In the CoM frame, letting $\sigma_a = \eta_a\, p^0_a / m_a$, we find [CH, Russo '24]

$$\mathcal{J}^{\mathrm{TT}\alpha\beta} = \frac{G}{2} \sum_{a,b} c(\sigma_{ab})\, (\eta_a - \eta_b)\, p_a^{[\alpha} p_b^{\beta]} + 2G \sum_a c(\sigma_a) \sum_{b\in \mathrm{in}} p_b^{[\alpha} p_a^{\beta]} \,.$$

- So, in the CoM frame, $\mathcal{J}^{\mathrm{TT}}_{ij} = \mathcal{J}_{ij}$, but $\mathcal{J}^{\mathrm{TT}}_{0i} \neq \mathcal{J}_{0i}$.

$\mathcal{J}_{\alpha\beta}$ VS $\mathcal{J}_{\alpha\beta}^{\mathsf{TT}}$

- We note that $\mathcal{J}_{0i}^{\mathsf{TT}}$ admits a smooth high-energy limit,

$$\mathcal{J}^{\mathsf{TT}\alpha\beta} = -2G \log\left(\frac{s}{Q^2} - 1\right)(p_1 - p_2)^{[\alpha} Q^{\beta]} \,.$$

- The TT contribution due to nonlinear memory is cutoff-independent

$$\delta\mathcal{J}^{\mathsf{TT}\alpha\beta} = 2G \int_k \rho(k) \sum_{a\in\text{in}} p_a^{[\alpha} k^{\beta]} \log\frac{p_a\cdot n}{m_a}$$

with $k^\mu = \omega\, n^\mu$ as defined in the CoM frame.