Positroid varieties

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arXiv: 1111.3660 (w. Knutson, Speyer)
arXiv: 2012.09745 (w. Galashin)

$[n]=\{1,2,\dots n\}$

Grassmannians and Schubert varieties

$$Gr(k,n) = \{k\text{-planes in } \mathbb{C}^n\} \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$$

$$= \{ k\underbrace{[\qquad\qquad]}_{n} \} / \text{row operations}$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

$\Delta_{13} = 3$

$$\mathbb{C}^n \supset V \longmapsto (\Delta_I(V))_{I \in \binom{[n]}{k}}$$

Δ_I = Plücker coord.

$$Gr(k,n) = \coprod_{\binom{[n]}{k}} \Omega_I$$

$$\Omega_{136} = \left\{ \begin{bmatrix} 1 & * & 0 & * & * & 0 & * \\ 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix} \right\} \cong \mathbb{C}^8$$

Schubert cell → Ω_I

$$X_I := \overline{\Omega_I}$$

Schubert variety → X_I

Positroid varieties.

$$\chi : \mathbb{C}^n \longrightarrow \mathbb{C}^n \qquad \text{induces } \chi : Gr(k,n) \longrightarrow Gr(k,n)$$
$$e_i \longmapsto e_{i+1 \bmod n}$$

An open **positroid variety** is a (non-empty) intersection

$$\mathring{\Pi}_{\mathcal{I}} = \Omega_{I_1} \cap \chi(\Omega_{I_2}) \cap \chi^2(\Omega_{I_3}) \cap \dots \cap \chi^{n-1}(\Omega_{I_n})$$

for some $(I_1, I_2, \dots, I_n) \in \binom{[n]}{k}^n$. **Closed positroid variety**:

$$\Pi_{\mathcal{I}} = \overline{\mathring{\Pi}_{\mathcal{I}}} \underset{\substack{\uparrow \\ \text{Th'm}}}{=} X_{I_1} \cap \chi(X_{I_2}) \dots \cap \chi^{n-1}(X_{I_n})$$

irreducible
projective

Juggling patterns

$V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$ $\quad v_i,\ i \in \mathbb{Z} \quad$ by $\quad v_i := v_{i+n}$

$f_V : \mathbb{Z} \to \mathbb{Z} \qquad f(i) := \min\{ j \geq i \mid v_i \in \operatorname{span}(v_{i+1}, v_{i+2}, \ldots v_j) \}$

Example $k = 3 \quad n = 6$

$$\begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 1 & 2 & 4 & 1 \\ 0 & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array} \right] \end{array}$$

$f(1) = 3$

$f(2) = 6$

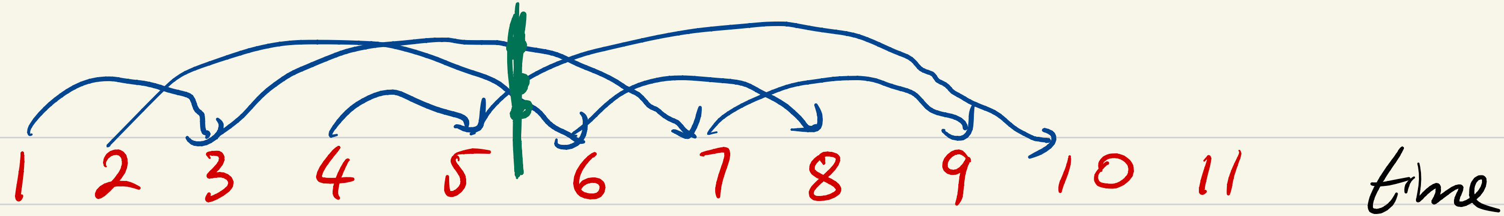
$f(3) = 7$

$f(4) = 5$

$f(5) = 10$

$f(6) = 8$

$f(i+n) = f(i) + n$


1
2
3
4
5
6
7
8
9
10
11
time

$$\left.\begin{array}{l} f : \mathbb{Z} \to \mathbb{Z} \text{ bijection} \\ (1)\ f(i+n) = f(i) + n \\ (2)\ i \leq f(i) \leq i+n \\ (3)\ \displaystyle\sum_{i=1}^{n} (f(i)-i) = kn \end{array}\right\} \text{(k,n)-bounded affine permutation}$$

$$\mathring{\Pi}_f = \{ V \mid f_V = f \} \subset Gr(k,n)$$

<u>Th'm</u> [Knutson-L.-Speyer]

- f_V is a (k,n)-bounded affine permutation.
- The induced stratification

$$Gr(k,n) = \coprod_{f \in Bound(k,n)} \mathring{\Pi}_f$$

is the open positroid stratification.

other ways to index:

Bruhat intervals

positroids

plabic graphs

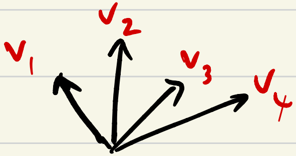
cyclic rank matrices

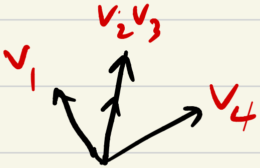
- $\Pi_f := \overline{\mathring{\Pi}_f} = \coprod_{g \geq f} \mathring{\Pi}_g$

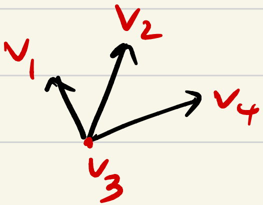
affine Bruhat order

- $\mathring{\Pi}_f$ is smooth, irreducible
- Π_f is normal, Cohen-Macaulay

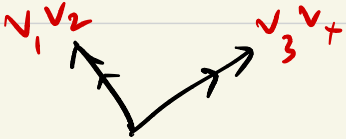
Example Gr(2,4) $id = 3456$ $id_{k,n}(i) = i+k$







$f(3) = 3$



Positroid varieties

Total positivity [Postnikov] [Lusztig]

[Brown-Goodearl-Yakimov] G-Y

Poisson Geometry

Schubert Calculus [KLS]

Frobenius splitting [KLS]

Cluster algebras

Mirror symmetry

Scattering amplitudes

q,t-Catalan [Galashin-L.]

Hodge: $I(X_I) = \langle \Delta_J \mid J \leq I \rangle$ $\quad \begin{bmatrix} 0 & 1 & * & * & 0 & * & 0 & * & * \\ 0 & & & & 1 & * & 0 & * & * \\ 0 & & & & & & 1 & * & * \end{bmatrix}$

Th'm [KLS] The scheme-theoretic intersection

$$X_{I_1} \cap \chi(X_{I_2}) \ldots \cap \chi^{n-1}(X_{I_n})$$

is reduced. Thus the ideal is linearly generated.

$$I(\Pi_f) = \langle \Delta_I \mid I \notin \mathcal{M} \rangle$$

prime!

positroid.

Let $R(k,n) = \bigoplus_{d \geq 0} R(k,n)_d \cong \mathbb{C}[Gr(k,n)] \cong \mathbb{C}[\Delta_I] / \ldots$

Plücker relations

$\Gamma(Gr(k,n), \mathcal{O}(d)) = R(k,n)_d \circlearrowleft GL(n)$ irreducible Weyl modules

[Borel-Weil]

$$\mathbb{C}[X_I] = \bigoplus_{d\geq 0} V(I)_d \circlearrowleft B$$ Demazure modules.

upper-△ matrices

$$\mathbb{C}[\Pi_f] = \bigoplus_{d\geq 0} V(f)_d$$ cyclic Demazure module.

$\Gamma(\Pi_f, \mathcal{O}(d))$

Thm (L.) These spaces have canonical bases (at $q=1$).

$k=2$ $n=4$

$d=2$

$$\wedge \overset{\leq}{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array}} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 4 \\ \hline \end{array}$$

all tableaux

Gr(2,4)

$$X_{13} = \overline{\left\{ \begin{bmatrix} 1 & * & 0 & * \\ & & 1 & * \end{bmatrix} \right\}}$$

columns ≥ 13

SSYT shape $(2,2) = k \times d$ rectangle

filled $[n]$

$f = 2547 \in \mathrm{Bound}(2,4)$ $\quad \mathcal{I} = (13, 23, 13, 41)$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 4 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 4 & 4 \\ \hline \end{array}$$

$$\dim \mathbb{C}[\Pi_f]_d = 9$$

To belong to cyclic Demazure crystal must belong to all promotion-rotated Demazure crystal

Open Problem: what is the character of the cyclic Demazure module?

Cohomology class

$$H^*(Gr(k,n)) \cong \bigoplus_{I} \mathbb{C}.[X_I] \cong \Lambda/I_{k,n} \cong \mathbb{C}[e_1,e_2,\dots,e_k]/\dots$$

$$[X_I] \underset{\text{Giambelli}}{\longmapsto} s_\lambda$$ Schur polynomial

Th'm [KLS] Under $\Lambda/I_{k,n} \cong H^*(Gr(k,n))$,

affine Stanley function $\tilde{F}_f \longmapsto [\Pi_f]$

$$\tilde{F}_f = \sum_{f = v_1 v_2 \dots v_r} x_1^{\ell(v_1)} x_2^{\ell(v_2)} \cdots\cdots x_r^{\ell(v_r)}$$

$\ell(f) = \ell(v_1) + \ell(v_2) + \dots \ell(v_r)$

v_i: cyclically decreasing.

e.g. $v_i = s_7 s_5 s_4 s_3 s_1$

$$[n]_q := \frac{1-q^n}{1-q} = 1+q+q^2+\cdots+q^{n-1}$$

$$[n]_q! := [n]_q\,[n-1]_q \cdots [1]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

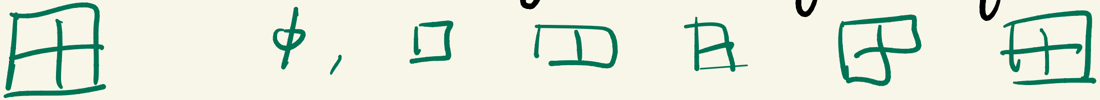
$$\sum_{\lambda \subset \underbrace{\square}_{n-k}\, k} q^{\mathrm{area}(\lambda)}$$

Gaussian polynomial

Theorem $\sum_i \dim H^{2i}(Gr(k,n))\, q^i = \begin{bmatrix} n \\ k \end{bmatrix}_q$

$$\# Gr(k,n)(\mathbb{F}_q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

Example $k=2,\ n=4$ $\quad 1+q+2q^2+q^3+q^4$



$$\mathring{\Pi}_{id} := \{ V \mid \Delta_{123\ldots k}(V) \neq 0, \ldots \Delta_{i\,i+1\ldots i+k-1}(V) \neq 0, \ldots \}$$

$\subset Gr(k,n)$ top open positroid variety

Thm [Galashin-L.] assume $\gcd(k,n) = 1$

$$\left.\begin{aligned} \# \mathring{\Pi}_{id}(\mathbb{F}_q) &= \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q (q-1)^{n-1} \\ \sum \dim H^i(\mathring{\Pi}_{id}) q^i &= (q+1)^{n-1} \sum_{D} q^{area(D)} \end{aligned}\right\} q\text{-Catalan numbers}$$

D: $(k, n-k)$ rational Dyck paths

$\binom{n}{k}$ binomial coefficient

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}$$ Catalan numbers

$$C_{a,b} = \frac{1}{a+b}\boxed{\binom{a+b}{a}}$$ rational Catalan number

gcd(a,b)=1

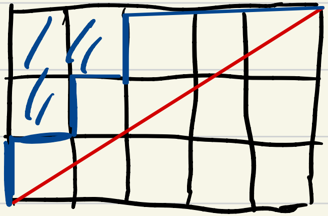
$= \#$ (a,b) rational Dyck paths

Th'm

Grossman,
Bizley

Two q-analogues of $C_{a,b}$!

e.g. $(a,b) = (3,5)$ $\quad \frac{1}{[8]_q}\begin{bmatrix}8\\3\end{bmatrix}_q = 1+q^2+q^3+q^4+q^5+q^6+q^8$



- mixed Hodge polynomial $(\mathring{\Pi}_{id}) \rightsquigarrow q,t$-Catalan

$J_{k,n-k}$ comp. Jacob.

$x^{k} = y^{n-k}$

mixed Hodge-Tate

$H^{k,(p,p)}(\mathring{\Pi}_{id})$

Garsia-Haiman
Haglund
Loehr-Warrington
Gorsky-Mazin

- $H^{*}(\mathring{\Pi}_{id})$ satisfies curious Lefschetz

$\gamma \in H^{2,(2,2)}(\mathring{\Pi}_{id}) \longrightarrow$ unimodality + symmetry of $C_{a,b}(q,t)$

- generalization to $\mathring{\Pi}_{f}$. Point count = (KL R-poly) is given by a coefficient of HOMFLY.