Distributed-delay epidemiological models and how to inform them with open data

What we learned in 3 years of monitoring COVID-19 in Bologna.

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ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

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$\rightarrow$ We focus on the diffusion of the disease **among individuals** in a population.

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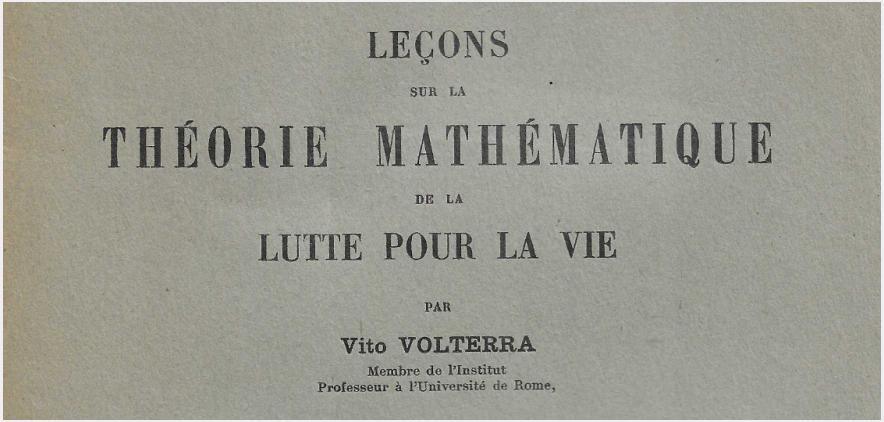

LEÇONS

SUR LA

THÉORIE MATHÉMATIQUE

DE LA

LUTTE POUR LA VIE

PAR

Vito VOLTERRA

Membre de l'Institut
Professeur à l'Université de Rome,

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A Contribution to the Mathematical Theory of Epidemics.

By W. O. Kermack and A. G. McKendrick.

(Communicated by Sir Gilbert Walker, F.R.S.—Received May 13, 1927.)

(From the Laboratory of the Royal College of Physicians, Edinburgh.)

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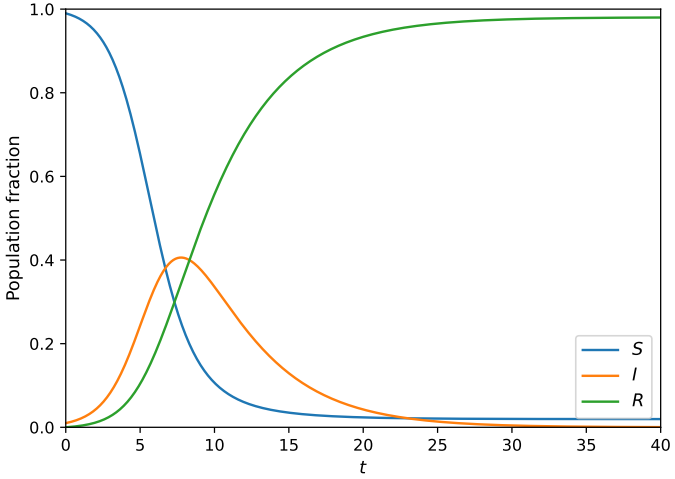
- can be regarded as a mean-field model for a stochastic process.

Example trajectory



- $\beta = 1$,
- $\mu = 1/4$,
- $S_0 = 9.9 \times 10^{-1}$,
- $I_0 = 1 \times 10^{-2}$.

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Identifying $\gamma E(t) = \varphi_{\text{in}}(t)$ *incoming flow*, the ODE for I becomes

$$\dot{I}(t) = \varphi_{\text{in}}(t) - \mu \int_{t_0}^{t} \mathrm{d}\tau e^{-\mu(t-\tau)} \varphi_{\text{in}}(\tau).$$

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- each subject remains in compartment I for a random time τ drawn from $\rho(\tau)$.

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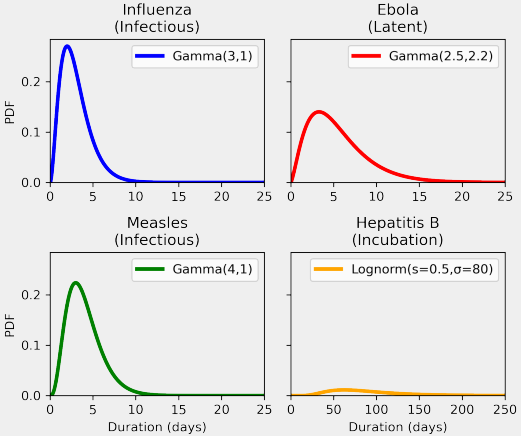

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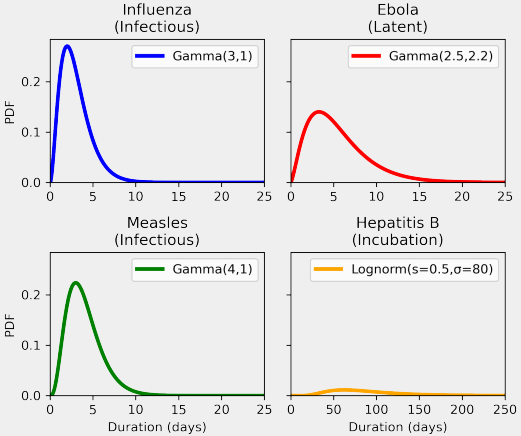

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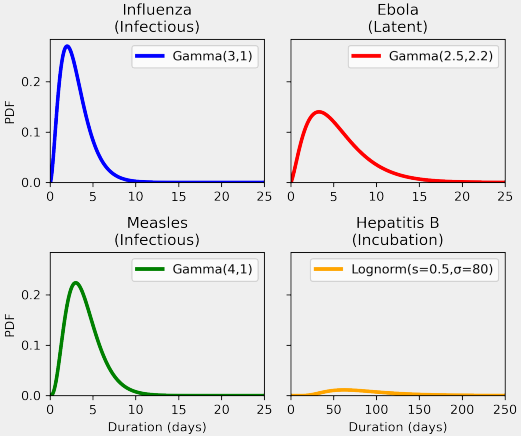
  $\rightarrow$ **Distributed-delay models**.



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Numerical solution	textbook algorithms	computationally expensive & fewer algorithms

DDEs can be more **realistic models** for complex phenomena, but require **specific approaches**.

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Support the **readiness** of local hospitals with an estimate of necessary **ICU** and **ordinary ward** beds.

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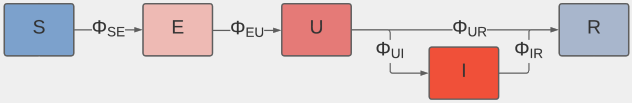

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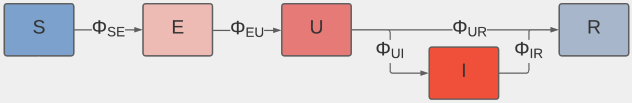

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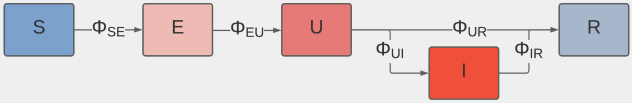

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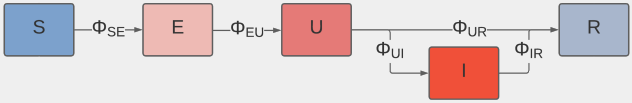

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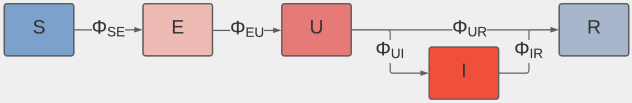

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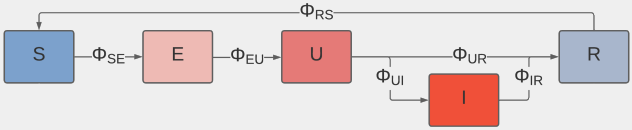

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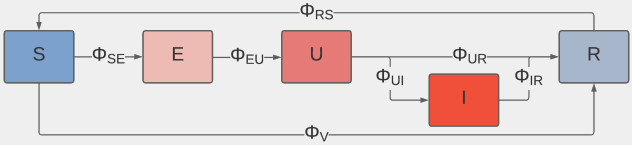

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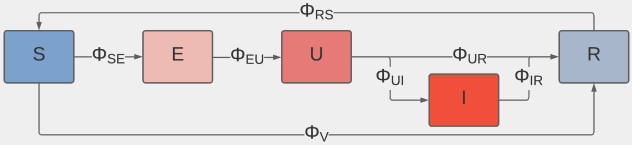

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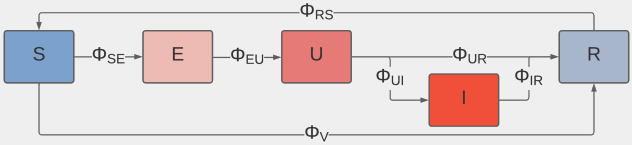

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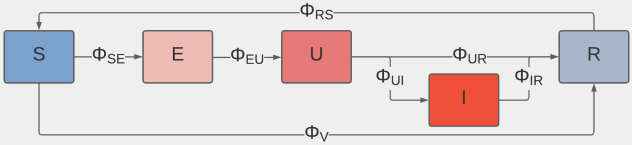

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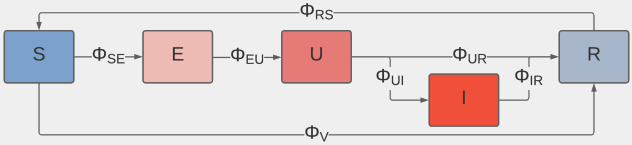

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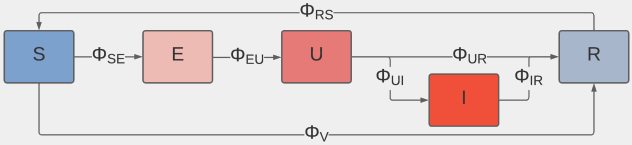

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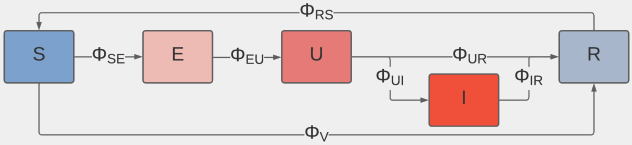

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Intermediate dynamics

$$\dot{E} = \Phi_{SE} - \int_0^\infty ds\Phi_{SE}(t-s)\rho_E(s; T_E, \sigma_E)$$

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$\tau(t)$: relative transmissivity

α: symptomatic fraction

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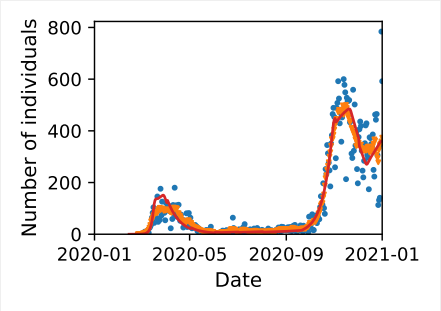

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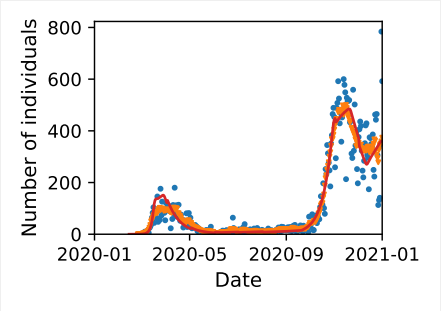

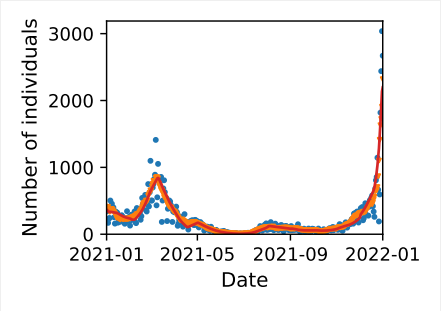

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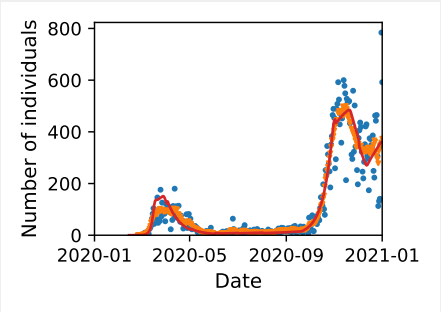

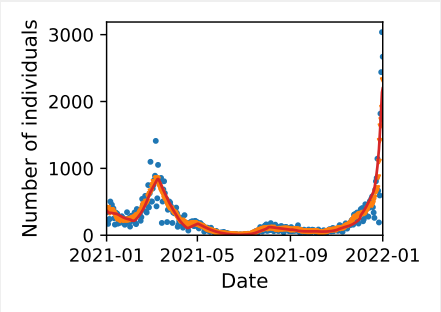

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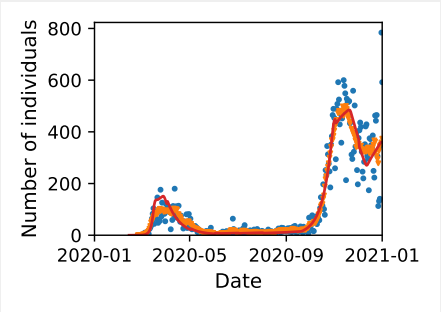

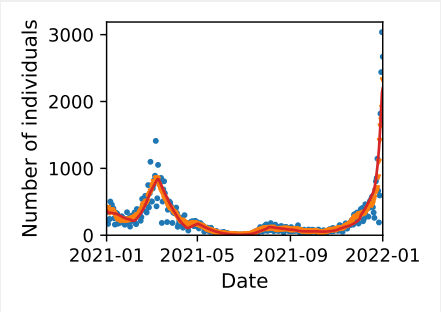

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How sensitive is our prediction to changes in s?

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The U dynamics can be approximated as autonomous

$$\dot{U} = \beta n_o s \left[U(t - T_E) - \int_o^\infty \mathrm{d}\tau \, U(t - T_E - \tau) \rho(\tau; T_U, \sigma_U) \right]$$

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Looking for a solution in the form $U(t) = U(t_o) e^{\lambda(t_o)(t - t_o)}$ we get

$$\frac{\lambda(t_o)}{\beta n_o s} = \left(1 - \frac{b^a}{(\lambda(t_o) + b)^a} \right) e^{-T_E \lambda(t_o)}$$

Passing to $\mu = T_U\lambda$, adimensional LLE we rewrite the equation for λ as

$$\frac{\mu}{T_U s(t_0)\beta n_0} = \left(1 - \frac{a^a}{(\mu + a)^a}\right) \exp\left(-\frac{T_E}{T_U}\mu\right) = F(\mu),$$

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where $r(s) = T_U\beta s n_0$ can be shown to be the *effective reproduction number* $\mathcal{R}_{t_0}$.

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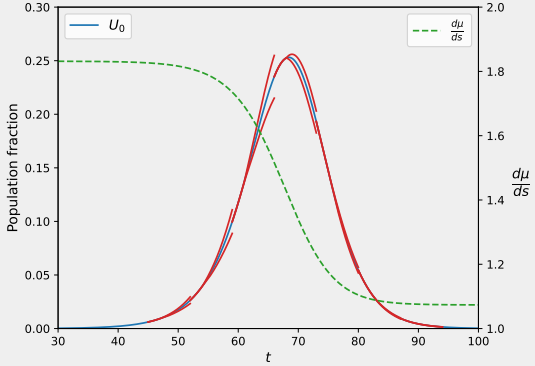

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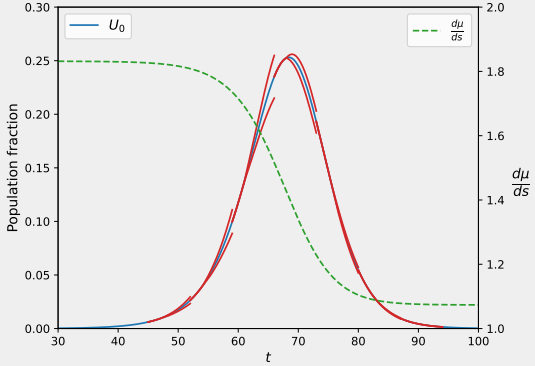

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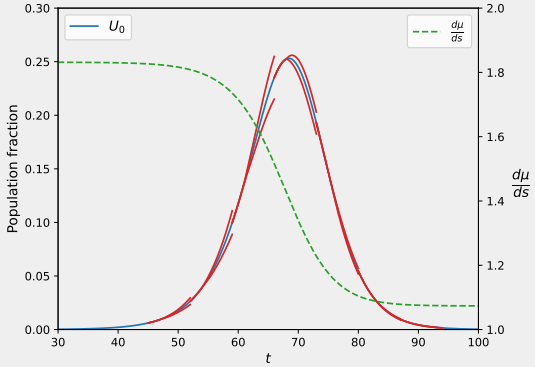

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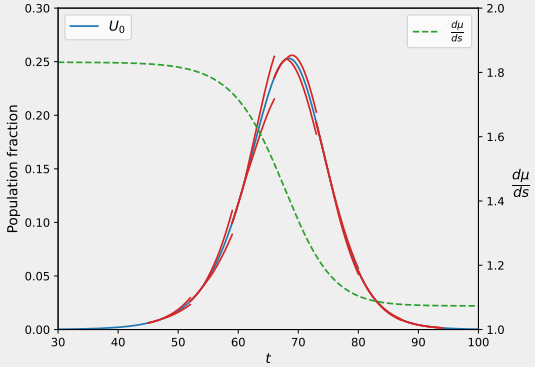

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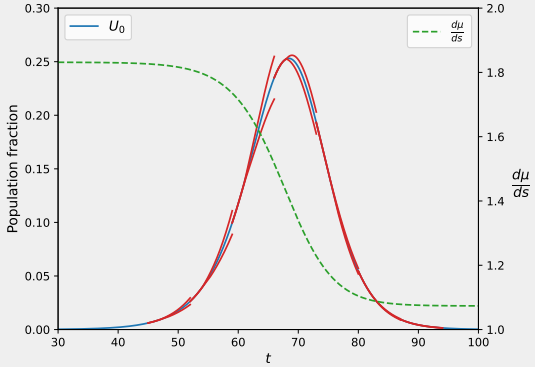
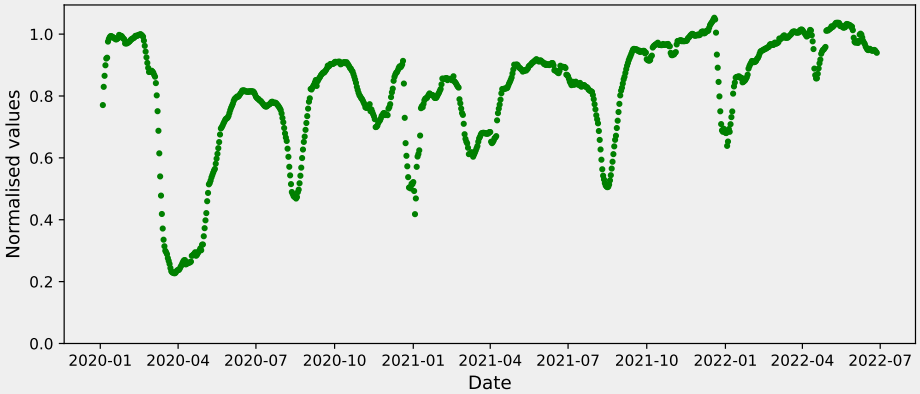


$\to$ need for a reliable way to estimate the sociability , especially in the first phases of the pandemic.

The mobility proxy

Mobility data from magnetic coils.



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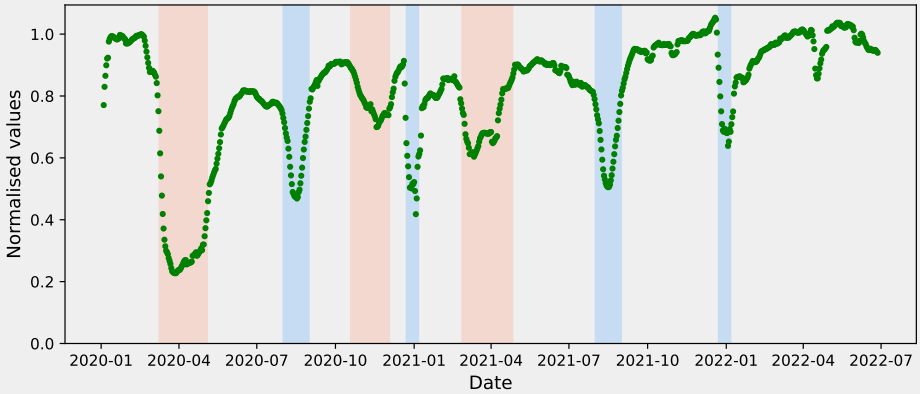

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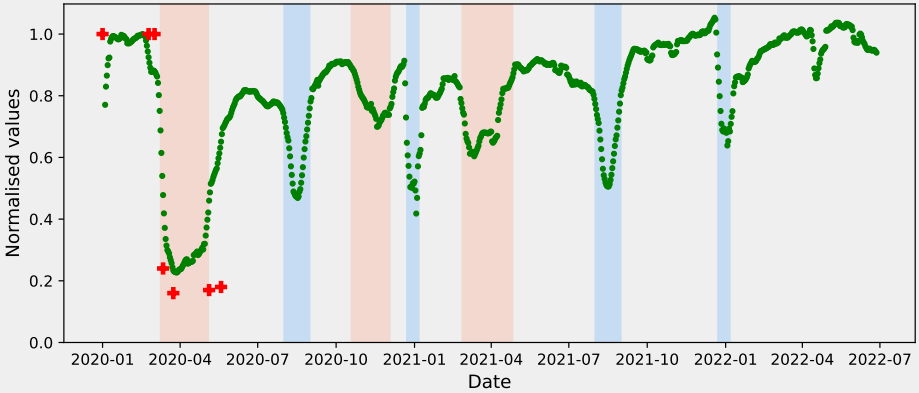

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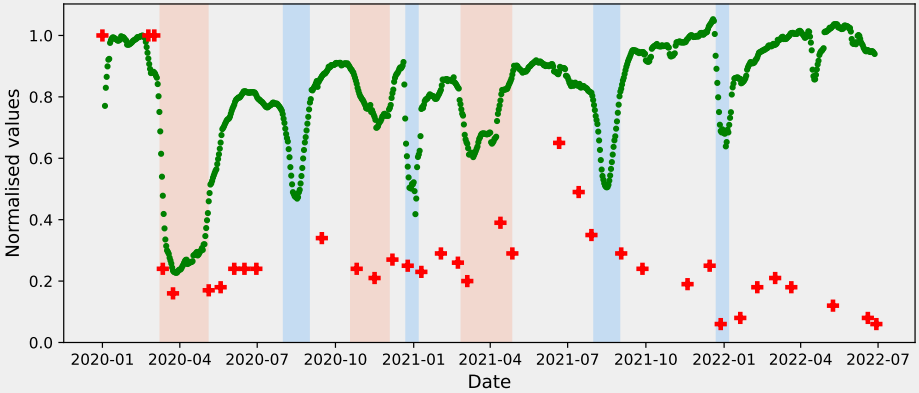

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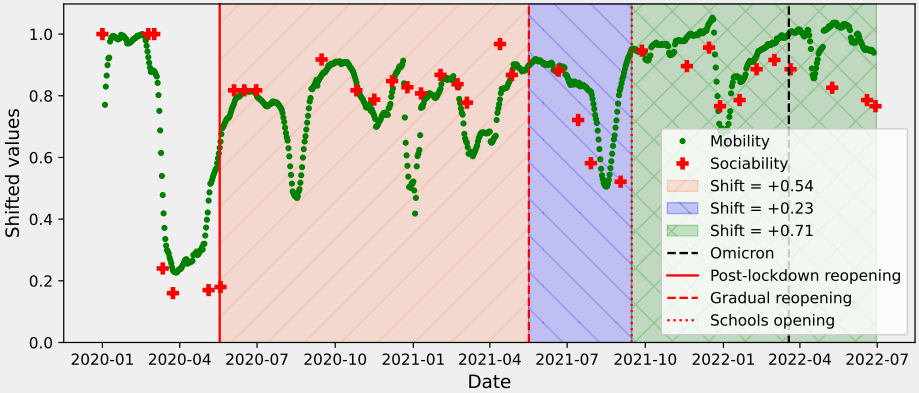

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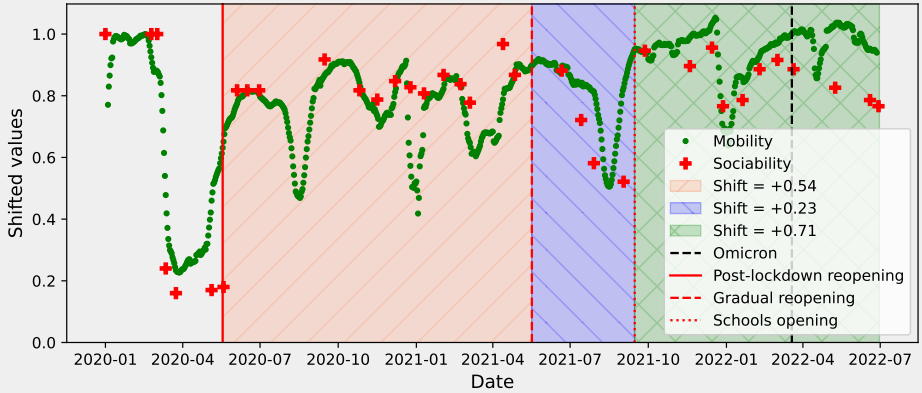

We can realign the time series by performing a few shifts
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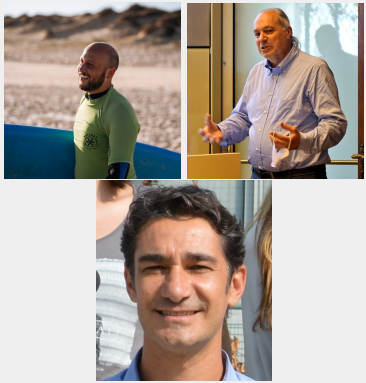
Find our paper here!

Working group

- Dr. Enrico Lunedei
- Dr. Francesco Durazzi
- Prof. Armando Bazzani
- Prof. Daniel Remondini
- Prof. Gastone Castellani
- Bologna MODELS4COVID Group



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- opportunity to **provide ground** for many **useful procedures**, and **automate them**, still quite new for this type of equation.

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- **costs** are of different types: economic, social, workload, infrastructural, ...

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How do people react to the containment measures?

- The government can **face expenses** both due to **measures** and to **lack thereof**,
- individuals can choose between **compliance** with regulations or **refusal to cooperate** depending on their perception of the balancing of costs,
- **costs** are of different types: economic, social, workload, infrastructural, ...
- **game theoretical models** could be coupled to the DDEs.

Thank you!

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Any questions?