Teacher's choices as the cause of misconceptions in the learning of the concept of angle

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Abstract. *In this research, we highlight that pupil's misconceptions about the concept of angle, extensively treated in literature, depend also on the didactic choices made by the teachers in didactic transposition of knowledge and in the educational design. It is often driven by unique and binding choices which do not take into account that mathematical objects usually have various definitions elaborated in the history of mathematics. Mathematical objects are usually imposed by the teacher, instead of being the result of mediation and negotiation within a community of practices, with the aim of reaching a shared knowledge by the pupils. Another important cause of difficulty, on which this research specifically concentrates, consists in the incoherencies in the intentionality of the teachers deriving from a limited and unaware use of the semiotic means of objectification with respect to the conceptual and cultural aspect of the knowledge they want pupils to reach.*

Abstract. *Nesta pesquisa ressaltamos que as concepções errôneas dos alunos sobre o conceito de ângulo, destacadas da ampla literatura no assunto, também dependem das escolhas didáticas dos professores na transposição didática do saber e na engenharia didática. São escolhas muitas vezes unívocas e vinculantes que não levam em conta que os objetos da matemática tem geralmente variadas definições que a mesma história desenvolveu e que são impostas, ao invés de resultar de mediações e negociaçõesdentro de uma comunidade de práticas, com o objetivo de alcançar um conhecimento compartilhado pelos alunos. Outra importante causa de difficuldades em que no nos concentramos especificamente nesta pesquisa são as incoerências na intencionalidade dos professores resultantes resultantes de uma utilização limitada e inconsciente dos meios semióticos de objetivação respeito ao lado conceitual e cultural do saber ao que deseja-se que os próprios alunos cheguem.*

1. Introduction

There is a term that has been much used for decades in Mathematics Education research: such term is "misconception". This word is interpreted in different ways by

several authors, but, in most cases, it has negative connotations, as a synonym of ‘error’, ‘erroneous judgment’, or ‘wrong idea’; also ‘ambiguous’ or ‘misunderstanding’. For this reason misconceptions are often cited in didactics when reference is made to errors.
Many authors agree on the fact that the first uses of this term, in the sense of ‘error’ or ‘misunderstanding’, came about in the domain of Physics or Economics. Reference is usually made to works of Di Sessa (1983), of Kahneman & Tversky (starting from 1982) with regard to decisional processes, and of Voss et al. (1989).
One of the first documented appearances of the term ‘misconception’ in mathematics happened in the USA in 1981, by Wagner (1981), in a work which dealt with the learning of equations and functions. Also in 1981, there was a famous text of Kieran on the solution of equations. Subsequently, several works appear in 1985 where the term ‘misconception’ is explicitly used: Schoenfeld (1985), Shaughnessy (1985) and Silver (1985), who use it mostly with regard to problem solving, together with the term ‘convictions’.
Silver (1985, pp. 255-256) explicitly says that there is a strong connection between misconceptions and mistaken convictions.
Schoenfeld (1985, p. 368) highlights how students can develop in the correct way incorrect conceptions, especially as regards procedures.
As one can well see, in the first half of the 1980s scholars in Mathematics Education work intensively on this theme.
Therefore, several authors took under examination, in a critical manner, the substantive misconception, for example in the sphere of the French School. In a private letter which the author has kindly authorised us to make public, Colette Laborde affirms:
«The term misconception which had its origin in the United States may not be the most appropriate if one refers to the “incorrect” knowledge of the students. The notion of “correctness” is not absolute and it refers to a given piece of knowledge; the reference knowledge can also evolve. The criteria to identify mathematical rigour have changed considerably over time. Each conception has its domain of validity and functions for that precise domain. If this does not happen, the conception does not survive. Each conception is partly correct and partly incorrect. Therefore, it would seem appropriate to speak of conceptions with respect to a domain of validity and to try to establish to which domain these belong» (quoted in D’Amore & Sbaragli, 2005, p. 12).
Keeping in mind both the researchers’ various positions and the rather different occurrences of this term, we maintain that the attention to misconceptions has been very productive. A consequence is that it has forced scholars to no longer identify the errors with something absolutely negative, to be avoided at all costs, but also with human products owing to evolving situations. The number of researchers focusing on the topic increases over the years. They have outlined a shared meaning of ‘misconceptions’ as causes of errors or better as *reasonable* causes of errors. Such causes are often easily explainable and sometimes even convincing (D’Amore & Sbaragli, 2005, p. 12).
It is undeniable that these studies have forced examining the interpretation of the classroom activities on the part of the subject. Misconceptions are interpretations created on the basis of convictions developed through learning. Therefore, misconceptions are considered as the fruit of a piece of knowledge, not as a lack of knowledge.
From this point of view, another possible approach, not far from the position of Laborde and in agreement with our perspective, is that of preserving such a term, but analysing it in a more constructive way, supplying it with a more elaborate and less negative

interpretation. This interpretation takes into account the current research in Mathematics Education and allows a deeper investigation of the causes for the lack of learning. From this point of view, initially D'Amore (1999, p. 124) and later D'Amore & Sbaragli (2005, p. 19) refer to misconception not only as a completely or certainly negative situation, but also as possible moments of passage. Such transitions are sometimes necessary for the construction of a concept that is in the process of being organised.
Within this perspective, misconceptions are distinguished in two large categories: *unavoidable* and *avoidable* (Sbaragli, 2005, p. 56 and following). The first category refers to misconceptions that directly depend neither on *didactic transposition* carried out by the teacher, nor on *educational design*. On the contrary, they depend on the necessity to say and to show something in order to explain a concept; often relative to a knowledge that is never exhaustive in what is being proposed, also because of the ontogenetic characteristics tied to the pupil. The second misconceptions *depend precisely on the choices that the teacher makes for carrying out the didactic transposition* and choices concerning the *educational design* which can negatively influence the training of the pupils.
In this paper the attention is focused on *avoidable misconceptions*, analysed in a *cultural-semiotic* framework (Radford, 2005a, 2006). We consider the *intentionality* of the teacher as a possible cause of such misconceptions relative to the notion of angle.

2. Theoretical Outline

2.1. Cultural-semiotic frame work

The *cultural semiotic approach* (Radford, 2005a, 2005b) bestows a central role to semiotics within an anthropological perspective toward thinking, mathematical objects, and learning. Both the arising of mathematical objects and their learning require a *reflexive mediated activity*. In particular, learning is seen as a process of *objectification* that allows the pupil, through the reflexive mediated activity, to become aware of the mathematical object.
Referring to the phenomenology of Edmund Husserl (1913), Radford (2006) consider sobjectification, regarded as a meaning-making process through the mediated reflexive activity, to an *intentional act* which places the subject in relationship to the object of knowledge and provides a particular understanding of such object. When considering scientific knowledge, particularly in mathematics, we have to face the issue of the interpersonal and general nature of mathematical objects. The subjective and situated meaning of intentional acts does not fully encompass the generality that characterises scientific knowledge.
In *Ideas: General Introduction to Pure Phenomenology,* Husserl (1913) overcomes this problem by distinguishing the intentional act which determines the way in which the object is presented to consciousness (noesis) from the conceptual contents of individual experience (noema). To each intentional experience of the subject, noesis, there corresponds a special conceptual meaning, the noema: «A tree *ut sic*, the thing in nature, is anything far from this perceived-tree as such, which as sense of the perception belongs inexorably to the single perception. The tree *ut sic* can burn, dissolve into its chemical elements, etc. However, the sense – the sense of this perception, that is

something that necessarily belongs to its essence – cannot burn, does not have chemical elements, forces, or real properties. […] The sense of perception also obviously belongs to the non phenomenologically reduced perception (to the perception in the psychological sense). Therefore, one clearly sees how the phenomenological reduction can acquire, also for the psychologists, the useful methodological function of fixing the noematic meaning in clear distinction with respect to the object *ut sic* and of recognizing it as inseparably belonging to the purely psychological essence of the intentional Erlebnis, conceived in this case as real» (Husserl, 1965, p. 203).

Husserl's phenomenology, to be understood as an epistemology and not as an ontology, attributes centrality to the role of the subject, but presupposes, on the one hand, the existence of a transcendent object that assures consistency and unity to the different intentional acts of the individual and, on the other hand, relegates the intentional experience to a relationship which exclusively involves the subject and the object.

According to the cultural-semiotic approach that we are following, intentional acts in Husserl's phenomenological understanding play an important role in learning processes. Nevertheless we cannot reduce our individual experience to a solitary sensory and cognitive interaction with the world, but the way in which we intentionally enter in contact with reality is intrinsically determined by historical and cultural factors. The mediators of the reflexive activity, the artefacts, the gestures, the symbols, and the words which Radford calls *semiotic means of objectification* (Radford, 2003) are not only tools by which we manipulate the world, but bearers of a historical consciousness built from the cognitive activity of the preceding generations. Such means determine and constitute the socially shared practices in which the processes of objectification develop: «In giving *meaning* to something, we resort to language, to gestures, signs or concrete objects through which we make our intentions apparent. […] Language, signs, and objects are bearers of an embodied intelligence (Pea, 1993) and carry in themselves, in a compressed way, cultural-historical experiences of cognitive activity and artistic and scientific standards of inquiry (Lektorsky, 1984)» (Radford, 2006, p. 52).

It is also advantageous to see the relationship between *noesis* and *noema* also addressing Mason's shifts of attention. Mason (2003, p.12) considers learning as making new distinctions, to discern previously undiscerned aspects of a mathematical object. But discerning is possible only within the interplay of change and a background of invariance: «probing what learners see as possible to vary, sheds some light on what they are attending to and on their to, currently available *example-space*, leading us to the language of *dimensions-of-possible-variation.* Furthermore, within each dimension-of-possible-variation, there is a perceived *range-of-permissible-change*. That is, there may be perceived constraints on the extent and nature of permissible change in any of the dimensions. So prompting learners to construct objects 'which no-one else will think of' reveals perceived dimensions-of-possible-variation, and hence something of the structure of their attention» (Mason, 2003, p.12). The process of objectification can be analysed also looking at the change of the structure of attention prompted by the teacher's reflexive mediated activity.

According to cultural semiotic approach, learning processes develop along a dialectical interaction between two complementary dimensions of meaning: the *personal meaning* which is «linked to the individual's most intimate personal history and experience; it conveys that which makes the individual unique and singular» (Radford, 2006, p. 53); the *cultural meaning* which is «a cultural construct in that, prior to the subjective experience has been endowed with cultural values and theoretical content that are reflected and refracted in the semiotic means to attend to it» (Radford, 2006, p. 53). We refer the reader also to the classical and fundamental researches on this topic proposed by Godino & Batanero (1994). Learning, as a process of objectification requires a dynamic and dialectical alignment between the personal dimension determined by the pupil's intentional acts and the cultural one that involves the historical and cultural aspects. The teacher plays a crucial role in prompting pupil's shifts of attention towards the manifold dimensions of the mathematical object (Mason, 2003, 2010). The construction of such a meaning, in which the unity of the individual with his culture is realised, is possible through the semiotic means of objectification, contribute to the creation of a shared meaning space that brings about the unity between the person and the culture, between personal meaning and cultural meaning, between individual intention and the object to which the intention is addressed.

It is necessary, therefore, to consider the complex network of individual and social practices, customs, beliefs, and convictions within which the teacher must daily orientate himself when he activates the mediators to foster the learning of mathematical knowledge on the part of his pupils. The complexity of such a network can sometimes be the cause of inconsistent behaviours on the part of the teacher.

It is from this point of view that it is possible to interpret the avoidable misconceptions within the cultural semiotic perspective. In fact, such misconceptions depend directly on the choices of the teachers tied to the didactic transposition and the educational design; two factors which, in the light of the cultural semiotic setting, turn out to be determining in the aligning of the personal meaning of the pupil and the cultural one, when the teacher manages the classroom practices.

Therefore, we want to evaluate if the teacher is able to unify his personal meaning and the cultural one using the semiotic means of objectification in an appropriate way; i.e., if the intentional acts of the teacher and the meaning objectified by the semiotic means turns out to be consistent with the cultural meaning of the mathematical object (the angle) which he proposes to the class. The existence of inconsistency from this point of view can create misconceptions in the pupils (of the avoidable category); misconceptions which, from a semiotic point of view, bring with them the pupil's inability to adequately coordinate the different representations when he tries to give meaning to the mathematical object.

2.2. Argumentative activity: Toulmin's and Habermas's models

There is a distinctive reflexive activity that characterizes in general human thinking and in particular mathematical thinking: the need to justify and ground our claims, both from an epistemological and communicative point of view. In a weaker and broader sense, we refer to such an activity with the term *argumentation*, when the validity of a claim is obtained through a reasoning process based on: (1) previous knowledge made up of cultural and philosophical viewpoints, experience, beliefs, convictions etc.; (2) inference rules that cogently link the previous knowledge to the claim. In mathematics there is also a special form of argumentation, institutionally accepted, to sanction the validity of its claims: the *proof*, a strong reasoning process based on a set of axioms, definitions and hypotheses linked to the claim through established deductive rules.

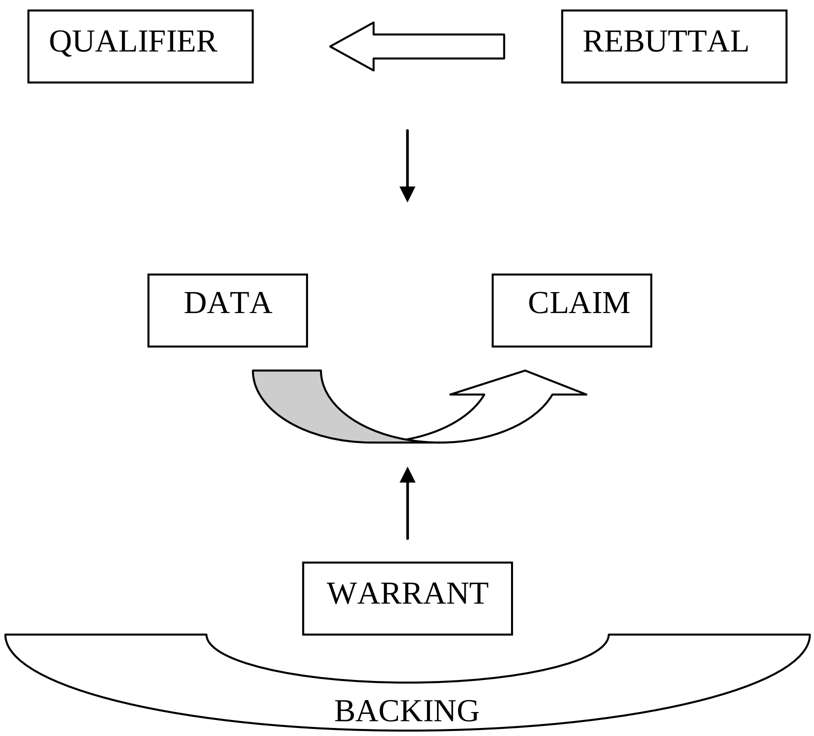
For the purpose of our investigation, we shall focus only on argumentation. In fact, argumentative processes play an important role in the construction and acceptance of a mathematical concept and, therefore, also in the arising of misconceptions. We use an integrated theoretical framework, proposed by Boero, Doudek, Morselli & Pedemonte (2010), that connects Toulmin's model of argumentation and Habermas' model of rational behaviour.

Toulmin's model of argumentation

Toulmin (1974) provides a model for argumentation based on the following elements:

- a *claim* C: the statement we want to justify;
- a set of *data* D that justify the claim;
- a *warrant* W: an inference rule that connects the data D to the claim C;
- a *backing* B: a system of principles, a theory, etc. that support the inference rule;
- a *qualifier* Q that expresses the strength of the argument;
- a *rebuttal* R, one or more exceptions to the rule.

Therefore, according to Toulmin's model, an argumentation is a process that links a set of a data to a statement through an inference rule. This process is embedded in a larger system that supports the argumentation and it has a definite degree of certainty according to the exceptions that are brought against the claim. Toulmin's model is a flexible instrument to frame and understand argumentative activity in mathematics.



Habermas' model for rational behaviour

Habermas proposes a model for rational behaviour that goes beyond the structure of argumentation and proving processes but he encompasses also the individual's activity.

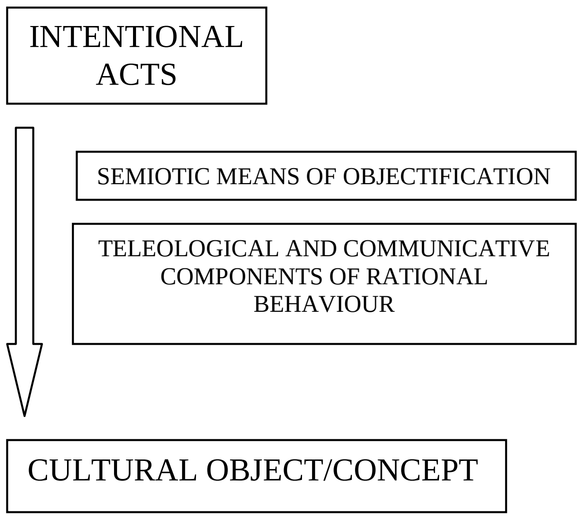
It is a model that can effectively support our analysis of the relation between the semiotic means of objectification and the objective towards which they direct the subject's intentional acts.
Hebermas (2003) highlights three interwoven dimensions that characterize rational behaviour:
• the *epistemic component* that encompasses the control over propositions and chains of propositions that sustain an argument;
• the *teleological component* that encompasses the *conscious* choice of suitable tools in view of the *objective* of the rational and argumentative activity;
• the *communicative component* that encompasses the *conscious* choice of communicative instruments within a social and cultural context.

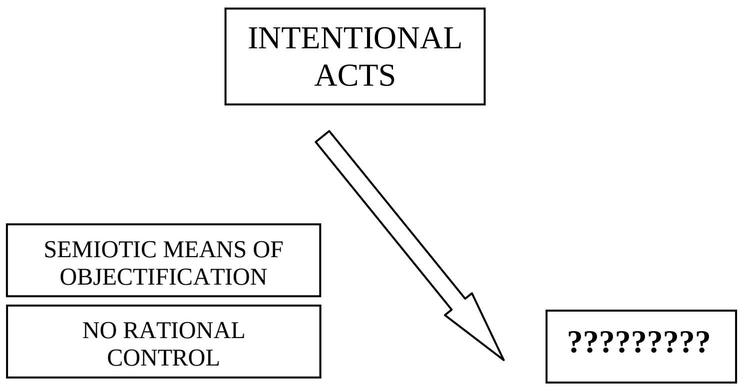
As regards our interest in the inconsistency between the semiotic means of objectification and the objective of the intentional acts as a source of avoidable misconceptions, the teleological and communicative components can provide important insights. The lack of rational control could be a cause of inadequate semiotic choices on the part of the teachers that determine mathematical activities inconsistent with the concept students have to objectify.

Integrating Habermas' and Toulmin's models
Boero, Doudek, Morselli & Pedemonte (2010) propose and integration of Habermas' and Toulmins' that considers the influence of argumentation's constraints (Toulmin's data-claim-warrant chain) on the three dimensions of rational behaviour (Habermas' epistemic, teleological, and communicative components). They propose a twofold structure to analyse the argumentative and proving activities, that take into account teacher and the role of the student separately but intertwined.
As regards the teacher, whose task is to foster pupils argumentative and proving skills, the authors outline a *meta-level of argumentation* characterized by «the awareness of the constraints on the three components of rational behaviour» (Boero et al. 2010, p. 13). At the meta-level, the warrants affect the awareness, control, and nature on the epistemic, teleological, and communicative components of logical behaviour. For example, at the meta-level the teacher is able to evaluate the reliability and grounding of the structure of an argument; the effectiveness of an argumentative strategy in view of the statement that has to be justified; the appropriateness of the communicative elements related to the cultural and social context in which they are adopted.
As regards the pupil, committed to grasp argumentative and proving mathematical activity, the authors outline a *level of argumentation* that requires to handle «the specific nature of the three components» (Boero et al., 2010, p. 13). At this level of argumentation Toulmins model of argumentation mainly controls the structure of the three components of rational behaviour, therefore the warrants mainly drive the epistemic component. Typically the warrants put forth by the students to justify their claims can resort to visual resources, natural language, algebraic formalism etc.
In teaching-learning processes, the aforementioned levels of argumentation play different and interwoven roles. The teachers operates at the meta-level of argumentation to design her teaching activity and guide students learning activity both towards mathematical concepts and argumentative and proving thinking. Under the teacher's direction, the pupils act at the level of the epistemic, teleological, and communicative components in order to justify their claims. The final objective is to foster their

awareness, at the meta-level, of Habermas' components of rational behaviour driven by Toulmin's data-claim-warrant argumentative structure.

We are interested in the role of teachers' rational behaviour in binding the semiotic resources to a given mathematical object. According to what criteria or principles does the teacher identify the most effective semiotic means? The teleological component allows a conscious choice of the most effective semiotic means in view of the mathematical concept to objectify. The communicative component creates a connection between the teacher, the pupil, and the mathematical concept within a system of social and cultural shared practices. Attention to the communicative aspects is important for the teacher to select the semiotic means and for the pupil to use and accept them to objectify the mathematical object. Within the teleological and communicative components the activation of Toulmin's argumentative structure, data-claim-warrant, allows to appropriately connect the semiotic means of objectification to the mathematical concept. There are three possible attitudes on the part of the teacher when proposing a mathematical activity according to the semiotic resources she adopts: her choices are oriented by habits, beliefs, convictions, didactical contract etc; they are oriented by an inconsistent rational behaviour; eventually the expected and desirable situation, the choices are oriented by an argumentative process consistent with the cultural and historical nature of the mathematical object and the social and educational features of the class. The inconsistencies between the semiotic means of objectification and the cultural meaning of the mathematical object, introduced in section 2.1, can be therefore also traced back to an incoherent rational control of the mathematical concept and the justification the supports semiotic choices..





2.3. A mathematical subject: the angle

In this work we have chosen to concentrate our attention on the "angle" as the mathematical topic we focus our investigation on. We refer to some researches present in the international environment; the numerous articles by Mitchelmore on this theme should be remembered, amongst them Mitchelmore & White (2000) who describe the evolution of the conceptualisation of angle through a theory based on three levels of sequential abstraction. The pupils start with physical experiences relative to angle, classifying them in specific situations, then passing on to more and more general contexts, until reaching abstract domains that are obtained from the different elementary mathematical conceptions that the students have about the angle. In these passages, the authors highlight the difficulties in coordinating different aspects of this concept. According to this theory, it is important that the formal definition of a mathematical concept captures the essence of the elementary mathematical conceptions from which it is abstracted.
An application of this theory is found in Prescott, Mitchelmore & White (2002) where, starting from the data supplied by a group of 12 teachers involved in a pilot research project, they show how a didactic unit that uses the teaching paradigm for abstraction

proposed in Mitchelmore & White (2000) and cited previously, brought about good learning. The article also shows areas for further improvement of the didactic unit.
Still from the same authors, we remember the research works: Mitchelmore (1997) and Mitchelmore & White (1998) who have confirmed that children structure different conceptual situations of the angle from the beginning of school, therefore independently of the teaching they received.
D'Amore & Marazzani (2008) show how, over the course of the millennia, mathematics has elaborated various definitions of the object angle. Some of these are profoundly different amongst themselves. Even if in Italian classrooms one currently dominates, it isn't necessarily the case that it is the only correct one (in other countries, other definitions are used). It has been shown that, spontaneously, young pupils prefer to revert to one of the others, even if they have not been used or mentioned in the classroom. Particularly, 8 different definitions of angle are presented and it is shown how, in individual interviews, students of different ages, before and after the presentation of one of these in the classroom, spontaneously refer to others.
The research literature highlights the complexity students have to face in the cognitive construction of the notion of angle. In this sense, Foxman & Ruddock (1984) and Mitchelmore & White (1998) highlight how the students, who should already have conceptualised the mathematical object angle, cannot manage to incorporate rotation as a way to consider this concept. In this last article, other researches which confirm this aspect are cited.
Vadcard (2002) proposes, as meaningful, the notion of angle as inclination that has an historical importance of this definition, used by Euclid in the Ist book of the *Elements*, and the application of such definition used, for example, by topographers. Furthermore, in the article, textbooks are analysed to identify the practices through which the students construct their notions of angle.
There are, moreover, numerous works that report a review of different definitions of angle present in the history of mathematics; in particular, we remember D'Amore (1985) who presents 8 definitions that go from the interpretation of Euclid (-300) to that of Hilbert in the XX century. Mitchelmore (1989), Roels (1985) and Schweiger (1986) classify the different definitions of this concept from the mathematical point of view, concentrating mostly on three particular classes of definitions held to be recurrent: angle as rotation of one half-line with respect to another around a common point, angle as two half-lines with their origin in common, and angle as a region formed of the intersection of two half-planes.

3. The motivations for the research

According to what we developed in the previous sections the learning of the angle entails a dynamic and dialectical relationship, driven by the teacher, between the cultural and the student's personal meaning of the mathematical object. He selects the mathematical knowledge and he is in charge of directing the pupils intentional acts and prompting shifts of attention.
Sometimes, the pieces of knowledge brought into play are even contrasting, for example when the teacher naïvely believes that there is only one possible conceptualisation of

the mathematical object and, as a consequence, only one definition, the one in his possession.
As it has emerged in the research presented in the theoretical outline, it can happen that the definition institutionally proposed in the classroom contrasts with the intuitive image that the student has already constructed for himself, thanks to the contexts of use outside of the school. In proposing a definition, it is necessary, therefore, to sift well the difficulties that the student will have in eliminating or overcoming his own intuitive image, perhaps having already constructed a model, and substitute it with the teacher's proposal. If it is true that the definition of a mathematical object should be the result of mediation and negotiation within a community of practices, it is still necessary for each of the components of the community to bring his personal contribution, according to his own convictions, negotiating the knowledge in the micro-society (the class) and arriving, one hopes, at a shared knowledge.
In this research, we want to demonstrate that the choices relative to the definition of the mathematical objects and to the use of semiotic representations involve only the teachers and exclude the pupils who must disambiguate the representations proposed to them in the didactic practice. Our conjecture is that it only has to do with a mediation done by the teacher who wants to lead his pupils toward that knowledge shared by the adults, by the teachers, and by the mathematicians, belonging to a specific culture, while the subject involved in the learning process is held at a safe distance from such negotiations. Therefore, the pupil's learning is not carried out as an objectification process within a reflexive mediated activity, as suggested by the cultural-semiotic approach we are advocating in this study. Moreover, we want to verify if the definition chosen by the teacher to help his students learn the concept of angle is univocal and if it results even in contrast with the semiotic choices carried out by the teacher for presenting the subject. Besides, we hypothesise that such semiotic choices turn out to be limited and stereotypical, when instead objectification process requires to *synchronically* activate a variety of semiotic means of objectification organized in a *semiotic node* (Radford, 2009; Radford, Demers, Guzmán & Cerulli, 2003). These aspects, univocal choice of the definition and of the semiotic proposals, lack of negotiation on the part of the pupils, and inconsistency in the intentionality of the teacher between conceptual aspect and semiotic proposal, can be some of the causes of the difficulties of the pupils dealing with the notion of angle, that emerge from several researches.
In order to scrutinize the consistency between the teacher's personal meaning and the cultural meaning of the mathematical object – therefore also the consistency between the semiotic means proposed to the students and the concept they have to learn – we focus our attention on the definition of the angle. There are many possible aspects of the angle that we could take into account beyond its definition, for example argumentation and proof, problem solving etc. We decided to draw our attention on the definition as it can be a clear and simple aspect of the angle on which the teacher can draw its intentional acts and thereby also the one of the students. It is a direct way to look at the teacher's intentions in relation and compared with a well outlined object of its intentions.The definition can also be the starting point for other possible activities that can be carried out with the angle in terms of conceptualization, strategic thinking, communication etc. Furthermore, it enables to decouple the role of argumentation in specific and complex argumentative tasks (Habermas' epistemic level), for example

mathematical proving, from its role in the choice of appropriate semiotic means to objectify a mathematical object (Habermas' telelological and communicative levels).

Specifically, our research questions are the following:
Q1 In the didactic transposition of the object angle, do the teachers have in mind a single definition to propose to the students or do they hypothesise working on different interpretations of such a concept that emerge from the pupils and that are present in the history of mathematics?
Q2 On the part of the teachers who want to propose a specific definition of angle to their pupils, is there consistency amongst the semiotic means of objectification chosen to present such a concept and the definition to which they want to arrive?
Q3 Do the semiotic proposals supplied by the teachers to present the subject of angle turn out to be varied, or are they stereotypical and limited?
Q4 What is the role played by the student in his interaction with the teacher and the angle as a mathematical object at personal and cultural level?

4. Research hypothesis

H1 In our opinion, in the carrying out of the didactic transposition of the object angle most of the teachers propose a single definition to their pupils; the most common one usually found in the textbooks.
H2 In our opinion, the semiotic means of objectification the teacher proposes to the pupils, for the learning of the angle, is not always consistent with the definition towards which he intends to direct them. Sometimes, in fact, such inconsistencies can derive from habits and stereotypes that take over in the semiotic choices, from the lack of critical analysis, of rational control on the mathematical concept and the justifications supporting such semiotic choices, and of personal reflection on the situation to propose in the classroom. In particular, it is important a rational control at the meta-level of argumentation of the teleological and communicative components.
H3 In our opinion, the semiotic proposals relative to the object angle turn out to be stereotypical and limited, deriving in an almost exclusive way from the proposal of the textbooks. The consequence is that there is no synchronic use of several semiotic means of objectification, especially when, through the reflexive activity, the students have to access higher levels of generality.
H4 In our opinion the student has to play an active role in the learning process. In fact the definition of angle is often proposed/imposed without negotiating with the students about their own convictions. Therefore, there is no involvement of the students in significative mediated reflexive activities to objectify the angle as a mathematical cultural object. Learning fails to be an objectification process without any aware prompting on the part of the teacher of pupil's shifts of attentions. There is a distance between the cultural meaning of the angle and the personal meaning of the student, between the teachers's learning intention and the object of the pupils intention and attention.

5. Research methodology and setting

Experimental setting

The research involved both primary school teachers and their pupils. The experimental setting was an attempt to mirror the relationship between the teacher's personal meaning, the cultural meaning of the object, and the semiotic means of objectification involved. Our aim was to identify inconsistencies in the teacher and investigate their effect on the student's misconceptions. For this reason we decided to involve in the research a group teachers and for each one of them a group of their students. We tried to involve schools from different Italian cities in order to have a wider range of cultural, social and educational environments.

The teachers we have chosen did not undergo any specific training on the angle both from an historical, epistemological, and didactical point of view. This to avoid any interference between their deeper beliefs and convictions on the angle and a specific training on the subject.

For this first research we did not design any activity that involved the administration of a test or a classroom activity that could have biased both the teachers and the relationship between the teacher and its students. This could have hindered the possibility to scrutinize the relationship between the teacher's inconsistencies and the students' difficulties in objectifying the angle.

Of course, it is necessary to carry out further researches with a more structured design in order to see the strength of the effects of the inconsistencies and possible ways to overcome them.

Methodology

The research developed in two phases: the first based on interviews carried out with primary school teachers relative to the concept of angle and to the semiotic means of objectification chosen to communicate this knowledge to the class, while the second was based on questions regarding the conceptual aspect of angle posed to their pupils in the 5th year of primary school. We then compared the consistency between the teacher's intention, the semiotic means of objectification she has chosen, and the consequent direction of the students' intentional acts and prompting of shifts of attention. Our aim was to highlight the teachers' use of semiotic resources in order to analyse it with the cultural semiotic lens and Haberma's teleological and communicative components of argumentation.

We believe that interviews are the less intrusive research method to single out the nature of teachers' and students' mathematical reflexive activity. The dialogical form allows individuals to express their ideas and their feelings in a positive atmosphere without the negative consequences, that can hit both pupils and teachers, of a test. As regards the teachers we proposed questions regarding the elements involved in the reflexive mediated activity; the mathematical object and its meaning they have in mind; the semiotic arsenal they put into play; how such an arsenal is used; the reasons that guided the choice of the way they present the mathematical concept and the instruments, and the relationship they created between concepts, semiotic means and mathematical discourse. As regards the students, we proposed them describe their personal meaning of the object and thus infer relationship between the personal meaning, the cultural meaning and the semiotic means involved in their objectification process.

The individuals were audiotaped and during the interviews they used pen and paper to express their ideas.

The interviews provided the basic elements necessary to give an interpretation of misconceptions regarding the angle according to the framework we presented above: semiotic resources, meanings, discourse and reasoning. The analysis and interpretation of our data is presented in section 6.

First phase

Twenty primary school teachers, from different Italian cities, were interviewed individually and asked the following questions that triggered a discussion between the interviewer and the researcher. The first three questions were intentionally vague and broad to introduce the subject, allow the teachers' convictions and way of working in the classroom to emerge. We thus derived important information for this research.

1) What would you like your pupils to know with regard to the angle?
2) Where do you start from to reach this learning?
3) What do you propose to your pupils on this theme?
4) Do you have in mind a single definition of the angle to propose to your pupils or different ones?
5) What representation do you chose to speak about the angle in the classroom?
6) Why do you chose this representation?
7) Do you also provide other representations of the angle?

Second phase

In the second phase, eight 5th year primary school pupils, from each of the 20 teachers' classes for a total of 160, were interviewed individually. They were chosen by a drawing. Using an interview, these pupils were asked: We are in geometry... What is an angle, for you? This question was the starting point to understand more deeply the convictions of the pupils on the angle.

6. Research results

6.1. First phase. The teachers

Below, we report the answers of the 20 teachers to the seven main questions.

Question 1), 2), 3), and 4)

To the first question, 14 teachers answered by listing intrinsic utilitarian aims of mathematics, for the purpose of knowing how to manage the typical scholastic requests on this theme, as recognising the various types of angle: acute, right, straight, obtuse, full, …, knowing how to measure the size of an angle with a protractor, knowing how to solve problems with angles, knowing how to do comparisons of the magnitude of angles, …. Only 4 of these teachers referred explicitly to the reality outside of the school: «I would like my pupils to be able to solve problems that involve angles even when they are outside of the school». For the others, the learning of the concept of angle seemed to be exclusively within the school, tied to scholastic success, and without any relationship to the external reality. The other 6 teachers answered with more conceptual aims, reaffirming the importance of acquiring the meaning of angle in geometry.

To introduce the concept of angle, all of the teachers declare that they refer to the environment around the children, to allow the pupils living the experience in first person with their body, touching with their hands and looking for prevalently right angles that they compare with other types of angles. Only 2 teachers claim that they begin the learning of angle from the convictions of the pupils, but immediately after they state that they say what an angle in mathematics is, without further negotiating the meaning of such a mathematical object with the pupils. To sum up, rather than working from the pupils' convictions, they simply investigate what they think.

All the teachers claimed that they had in mind a single definition of angle they wanted their pupils to reach. No-one thought of providing different definitions of angle to the pupils or working from their definitions. Teachers declare that, in order to choose the definition, they show their pupils different standard situations taken from textbooks and, without accepting to negotiate different interpretations with the students, provide the definition to learn.

The teachers' assertions highlight that the definition of angle they proposed in the classroom is not the result of mediation or negotiation processes, carried out within the micro-society of the classroom, to arrive at a shared piece of knowledge but it is something imposed by the teacher.

In particular,

- 14 teachers out of 20 stated that they choose, as the definition of angle, to propose to their pupils: «The part of the plane included between two half-lines with a common origin». Such a definition is surely the most common in Italy amongst teachers at the primary school and as a consequence amongst the students. Its origin is uncertain and begins to appear from the XVII century in Europe.
- 3 of the 14 teachers forgot to speak about the common origin of the two half-lines, but from their gestures one knows instinctively that they were making this conceptual choice without knowing how to explain it in the correct way. Moreover, 4 of the 14 teachers stated that the part of the plane is unlimited. It is evident that in this situation, the adjective "unlimited" is pleonastic given that it refers to an "open" part of the plane, but the subsequent interviews reveal that 5 of the teachers who did not explicit it ignored the unlimitedness of the part of the plane implicit in such definition. As we will see later, they think of the angle as a limited part of a plane localized in correspondence to its origin.

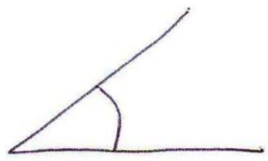
The other 6 teachers, instead, made the following choices:

- 1 spoke of inclination of two straight lines [a choice which recalls Euclid, $^{-}$III century or Proclus (412-486)];
- 1 considered the angle as a change of direction of two straight lines [a choice which recalls Eudemus of Pergamum (active in $^{-}$225)];
- 1 spoke of two half-lines with the origin in common [a formulation which recalls Hilbert (1899)];
- 1 spoke of the magnitude of two half-lines [therefore making an exclusively metric choice; a metric choice is present in Carpus of Antioch (II century) who defined the angle as the distance of the lines (…) which include it];
- 2 spoke of the rotation of two half-lines with a common origin one on the other (a choice that developed in Great Britain from the XVIII-XIX century).

[For a deeper investigation of the different definitions of angle in history, we used D'Amore (1985)].

Question 5)

12 teachers answered to this question saying that to represent the angle they use a “small arc” near the origin of the angle which limits a part of the plane. 10 of them asked to draw it to show what they meant.



Such a representation is not univocal in the Italian textbooks at any scholastic level because sometimes the angle is drawn with dotted lines and up to an imaginary arc or shading in extolling the unlimitedness of the part of the plane, or indicated with an asterisk, …, but for most of the teachers interviewed the representation by means of an arc was considered “the” representation par excellence, that which reflects the angle better than the others, without a conceptual motivation, but rather from convention and habit.

The other 8 teachers in 3 cases coloured in a limited part of the plane up to an arc, going back, in a way, to the same choice of the other teachers and the other 5 coloured in the part of the plane showing its unlimitedness.

The motivation that brought to the choice of this semiotic means of objectification is not related to the need to further highlight some properties they stated when answering the previous questions. Rather, in some cases, it is the conventional graphic representation itself that leads to lose the meaning of the definition that students should learn; which itself can be re-read as a semiotic means of objectification. Or rather, we highlight an inconsistency, in 17 teachers out of 20, between the explicit intention from the institutional point of view and the semiotic means of objectification chosen to introduce the angle.

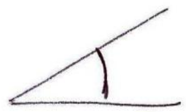
Inconsistency. Taking into account the teachers’ definitions of the angle, we analyse more deeply the inconsistency between their intention relative to the concept they want the classroom to reach and the semiotic means of objectification they have chosen. We also want to analyse the rational control described in section 2.2 behind their semiotic choices.

Part of a plane. Of the 14 teachers who stated that the angle is the part of a plane comprised between the two half-lines with the common origin, 9 choose as a semiotic means the arc, 3 choose the part of the plane coloured up to the arc, and 2 direct their attention to the unlimitedness of the part of the plane.

The 12 teachers who choose to indicate the arc or to colour the part of the plane up to the arc placed importance, with such graphic semiotic means of objectification, on the limitedness of the part of the plane and not on its unlimitedness; unlimitedness is instead contemplated in their definition because the part of the plane deriving from such definition turns out to be ‘open’. This inconsistency highlights also a lack of rational control both at the teleological and communicative level. If the teachers had a rational control they would have singled out the incoherence between the verbal definition and the one expressed with a figural representations. Through the teleological control the teachers would have seen that their definition implied the intrinsic unlimitedness of the

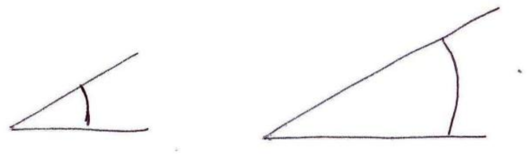
angle, thereby accessing the appropriate figural representation consistent with their intention. Furthermore, a control of the communicative component would have induced the use of more definitions and semiotics means of objectification involving the students in significative reflexive activities.
After the interview, the choices of these 12 teachers were divided into two categories: 5 relative to the *lack of awareness of the mathematical knowledge they bring into play* and 7 relative to *the lack of a critical sense with respect to their own choice.*

- We report a part of the interview regarding the *two types of inconsistency*. We begin with the *lack of awareness of the mathematical knowledge.*
R.: Why did you choose this representation?
C.: Because the angle is represented like this.
R.: In what sense is it represented like this?
C.: When you want to talk about an angle, you draw it like this:



and the children know that we are talking about an angle.
In terms of the cultural-semiotic approach the there is no synchronic use of semiotic means of objectification. The restriction to the “little arc” fixes at a strong embodied level an incorrect objectification of the mathematical object keeping the student away from a rich mathematical activity that traces back the historical and cultural evolution of the mathematical object. Again, the inconsistency between the mathematical object the student should objectify and the semiotic means suggested by the teacher are rooted in a lack of teleological and communicative components in the subject’s argumentative control. The answers to the researcher’s questions have no rational control, the subjects states “it is represented like this”, “you draw it like this”. Behind, there is probably a lack of awareness of the mathematical knowledge to carry out a rational control of the situation. Note how this choice appears unvocal in the eyes of that teacher. And yet, as Duval (2006, p. 598) maintains: «Opposite to this reduction of the semiotic representations to the simple role of surrogate of the mathematical objects or to expression of mental representations, we focus on that which constitutes the fundamental characteristic of every mathematical practice: the *transformation* of semiotic representations. *Because, in mathematics, a representation is interesting only if it can be transformed into another representation*. Only when they respond to this fundamental need, semiotic representations can represent something “real” and rationally explorable, that is, become the means of access to otherwise inaccessible objects».
The interview continues in the following way:
R.: Indicate, on this illustration, which angle you are speaking about.
(C. He indicates the part of the plane up to the arc).
R.: Up to where does the angle arrive?
C.: Up to here (he indicates the arc).
R.: Can you go beyond this arc?
C.: No, it goes up to here.

R.: Can't we go beyond the arc?
C.: In this case, no.
R.: And in which cases can we go beyond?
C.: If the angle is bigger.
(He draws another angle, apparently of the same amplitude, with longer half-lines and arc).



From this extract it emerges how misconceptions about the angle deriving from graphic representations, confirmed by classical research in the field and described in literature (Fischbein, Tirosh & Melamed, 1981; Foxman & Ruddock, 1984; Tsamir, Tirosh & Stavy, 1997) are present in some cases in the teachers themselves and therefore transferred to their pupils. The use of the "little arc" hinders the unlimited meaning of the angle that can be grasped at a higher level of generality that goes beyond the embodied meaning conveyed by this figural representation. The synchronic use of other semiotic means of objectification would allow to overcome this limit and access a disembodied meaning of this mathematical object. From this extract we can single out a self-referential effect of the lack rational control. On the one hand it is the cause for an inappropriate semiotic choice. On the other hand this inappropriate choice is the cause for a reasoning that leads to an inconsistent conclusion regarding the property of the angle, its limitedness and amplitude depending on the position of the arc.

The interview continued in the following way:
R.: Why did you choose this representation?
C.: Because this is the way to represent the angle.
R.: It is the way chosen by whom?
C.: By everyone, in all the books, it is like this.
R.: And do you like this representation?
C.: Yes, I have always done it this way, I don't see why I should change it.
R.: What, for you, is an angle?
C.: It is the part of the plane comprised between two half-lines that start from the same point.
R.: And how is this part of the plane?
C.: In what sense?
R.: What properties does this part of the plane have?
C.: I don't understand.
R.: Is this part of the plane of which you are speaking limited or unlimited?
C. He looks at his drawing, thinks a bit and then answers:
C.: It is limited by the half-lines.
R.: And here, how is it? (The researcher indicates the unlimited part of the plane).
C.: It arrives up to here (he indicates the arc).
R.: When I asked you what an angle is, why didn't you say that it arrives up to the arc?
C.: Because it isn't mentioned in the definition, but it becomes evident in the drawing.

Note how the graphic semiotic means of objectification is inconsistent with respect to the verbal one, even if the teacher declares that she wants students to learn the latter. From this excerpt the lack of rational control on the part of the teacher is completely evident. Toulmin's argumentative structure, data-claim-warrant, is completely absent at a teleological and communicative level. The subject declares that the angle is limited by two half lines that implies that the angle has an unlimited nature. Then she refers to the "little arc" to justify the its limitedness. The last answer to the researcher's question highlights a total lack of teleological control that yields the inconsistency between semiotic means of objectification and meaning of the mathematical object. The property that is not evident in the verbal definition becomes evident in the figural. The point is that the choice of the figural definition did not take into account the in a teleological sense the characteristics of the angle to justify the semiotic choice. Furthermore, from this same excerpt we can see how habits and beliefs hinder the rational and argumentative control of teachers choices. In fact, the subject declares that the choice of the semiotic representation was suggested by textbooks. We can also interpret this from Duval's point of view. There is no coordination of registers, and yet «(…) coordination of registers is the condition to master understanding since it is the condition for a real differentiation between mathematical objects and their representation. It is a threshold that changes the attitude towards an activity or a domain when it is overcome. (…) Now, in this coordination there is nothing spontaneous» (Duval, 1995, p. 259).

♦ Let's also consider the following extract of an interview of one teacher of the 7 who turned out to be inconsistent for *the lack of a critical sense with respect to their own choice.* The teacher chooses, as representation, to colour the part of the plane up to the arc, but she is aware of the unlimitedness of the part of the plane that characterises her definition of angle.

R.: In the definition that you choose, is the part of the plane limited or unlimited?
S.: Unlimited
R.: Why did you chose to represent the angle with a part of the plane up to the arc?
S.: I've always drawn it like this and it seems to me that the pupils see it (the angle).
R.: Don't you tell your pupils that they can continue to colour?
S.: Perhaps sometimes, but then we decided immediately to represent it like this.
R.: Does it seem to you a good choice?
S.: Now that you make me think about it, perhaps no, but it's a question of habit and we usually don't think about everything we propose.

This is a good example of lack of the teleological component in rational behaviour of the teacher. The teacher is aware of the unlimitedness of the angle but the figural representation she proposes is inconsistent with her objective. When the researcher draws her attention on the adequacy of the representation, the teacher is aware of the inconsistency of this semiotic means. The researcher's question activated the rational control on the part of the teacher. We stress that use of the in not itself a negative choice. It becomes so only when it is not synchronically used with other semiotic means of objectification that foster a reflexive activity consistent with the cultural meaning the teacher wants her students to objectify. In fact, as declared by the teacher, the "little arc" is effective from an embodied point of view. The problem arises if the there is no attention to higher levels of generality that take into account the unlimitedness of the angle. This excerpt confirms that the transition to higher levels of generality requires a rational control at the meta-level of argumentation to choose appropriate semiotic

means of objectification and involve students in reflexive activities that promote the communicative component of rational behaviour.
Also the teachers who choose the other definition, present several inconsistencies between the concept to which they wanted to direct their pupils' intention and the semiotic means they choose to objectify it. The 3 teachers who introduced the angle as the inclination of two lines, the change of direction of two lines or two half-lines with their point of origin in common, choose a graphic representation that highlights the unlimited part of the plane, even though it isn't an explicit characteristic of such definitions.
Instead, the teacher who defines the angle as the magnitude of two half-lines and who chooses the arc as semiotic means of objectification to highlight the measurement of the angle turns out to be consistent; an arc, which is displayed by the teacher also through a protractor, the tool for measuring the angle.
The same consistency emerges in the 2 teachers who define the angle as rotation of two half-lines with their origin in common and who brought attention to the arc as semiotic means which displays the dynamic process of rotation.
This consistency is not placed here in relationship to a judgment on the efficacy of the didactic choice, which goes beyond the scope of this article.

Question 6)
All the teachers we interviewed justify the choice of the graphic semiotic means of objectification because such representation is the one mainly used and conventionally accepted in Italy. For this reason, it is perceived as binding and often univocal; the "mathematically correct" representation. The means of objectification are so binding as to cause, on the part of the teacher, the loss the of critical sense of his proposals to the classroom. Besides, they do not turn out to be socially constructed in the class environment, but imposed. Amongst the motivations for their choices, two teachers also refer to the shape of the protractor which recalls the arc, a very superficial motivation, that confuses a concept with the tool of measurement used to evaluate its quantity.
Generally, we do not observe, on the part of the teachers, aware conceptual or personal choices tied to the concept brought into play. And yet, as D'Amore & Godino (2006, pp. 26-27) maintain: «It seems to us that we can confirm that the meaning of the mathematical objects begins as pragmatic, relative to the context, but amongst the types of use relative to that meaning, there exist some that allow orientating mathematics teaching-learning processes. These types of uses are objectified through language and end up with the construction of institutional lessical references». Such uses that orientate the teaching-learning processes are not fostered by the binding choices of the teachers we interviewed.

Question 7)
The representations chosen for modifying the initial request are found amongst the three already mentioned.
Specifically, it should be noticed that 5 teachers could not suggest a different way to represent the angle with respect to the arc that they considered in the second question. This behaviour highlights the rigidity of such a semiotic means that has become, by now, univocal in the minds of some teachers. The other 7 teachers who had chosen the arc, subsequently coloured a broader part of the plane, but there were still 4 cases bound by its limitedness. This is a sign that in this case the arc doesn't only refer to the part of

the plane that identifies the angle, but it also leads to visualize the limited part between the arc and the half lines.
The 3 teachers that initially colour the part of the plane up to the arc, in 2 cases change only the type of colouring; one hatches and one dots the limited part of the plane, and in 1 case, the arc is presented as the only representation of the angle.
The 5 teachers who colour the part of the plane attempting to make its unlimitedness stand out, 3 of them don't resort to any alternative representation a part from changing the type of hatching, while the other 2 remain stuck to the little arc.
One of the last two cases is a good example of a *change of meaning* due to treatment semiotic transformation, i.e. the passage from one representation to another representation in the same semiotic register (D'Amore, 2006; D'Amore & Fandiño Pinilla, 2008; Santi, 2010, 2011). This change of meaning is explicitly confirmed by one of the teachers: «If we represent the angle in this way, then we are describing the angle as a magnitude» (referring to the representation with the arc), while he had previously described the angle as part of the plane comprised between two half-lines with a common origin providing a representation that highlights the unlimited part of the plane. This example seems to confirm the results of D'Amore & Fandiño Pinilla researches in which we witness unexpected semiotic behaviours with respect to the conclusions which constitute the heart of the theory developed by Duval (1995, 2006). In fact, Duval considers *conversion* - the passage from one representation in a semiotic system to another representation in another semiotic system - the operation which, in mathematics, characterises a specific cognitive functioning and is the main source of learning difficulties. This example, instead, shows that also *treatment* is a cause of difficulties in the conceptualisation of a mathematical object. The example we are examining, shows a change of meaning of the mathematical object (the angle) that brings about a distortion of its general meaning. The teacher associates the magnitude of the angle to a representation R_1 and its definition to another representation R_2 without recognising the reference to the same conceptual object, thus confusing the mathematical object with its representation. The change of meaning can be interpreted as a misalignment between more intrapersonal meanings with respect to the interpersonal and general meaning; culturally and historically constructed.

6.2. Second phase. The pupils

The convictions about angle emerged thanks to the interviews carried out with 160 pupils interviewed from the V year of primary school and they fall within the following categories:
- *Angle as part of a plane limited by an arc.* 62 pupils maintain that the angle is the coloured part up to the arc used to indicate it. The majority of them asks to draw and visualize the colour, thus highlighting the limitedness of the part of the plane. In some cases, the arc is also indicated, in others it remains indirectly visualized by the coloured limited part that stays well defined within the two sides of the angle and it seems that it cannot extend beyond a certain limit. To the question as to whether it is possible to continue colouring beyond the arc, the pupils answer that the angle arrives up to there (in the sense that it is limited): G.: «It arrives up to here, otherwise it would go outside the angle».

This category already emerged with the teachers, but there is not a strict correlation between the proposal of the teacher and the answers of the pupils. In fact, several of these pupils were not students of the teachers who were in this category.
- *Angle as two consecutive segments.* 18 pupils claim that two consecutive segments represent the angle itself: «They are these two lines here».
- *Angle as arc.* 21 pupils declare and indicate with gestures on the table or on the drawing that the angle coincides with the arc itself:
S.: This is the angle (indicating an arc on the table near to one of its vertices).
R.: This thing? Show it better.
S.: This here (again indicating a curved line).
R.: What do you mean?
S.: From here to here (indicating a curved line that joins the two edges of the table).
- *Angle as length of an arc.* 9 pupils maintain that the angle is the length of the arc: «The angle is how long this is» (indicating the arc).
These last three categories are not present amongst those of the teachers and they highlight how much the graphic semiotic representation proposed by the teacher has taken the upper hand over the conceptual aspect, misleading its meaning. In this case the meaning given by the pupils to the mathematical object (the angle) turns out to be different with respect to that proposed by the teacher both in verbal and graphic terms. This shows how much caution is necessary to propose representations of a mathematical object, and above all how, important it is to investigate the interpretation given by the pupil of such representations.
- *Angle as unlimited part of a plane.* 34 pupils refer to the plane comprised between two half-lines with a common origin: D.: «It is the part of the plane comprised between two half-lines». To the question: R.: «Draw an example of an angle», 21 pupils draw two half-lines with a common origin and highlight the unlimitedness of one of the two parts of the plane: «It is all this part here» (indicating all the part of the plane). While the remaining 13, after drawing, in 8 cases, two segments with a common extreme and in the other 5 cases two half-lines with the origin in common, all indicate a limited part of the plane between the two segments or half-lines, in this way falling into the first category. We witness an evident inconsistency between the definition they use to describe the angle and its graphic representation; an inconsistency that also emerges amongst the answers of the teachers and commented on in paragraph 6.1. The verbal semiotic means used by the pupils is in contrast with the symbolic one, but such a contrast is not perceived by the students, nor previously by the teachers. In this case, there is a correlation between the convictions of the teachers and those of the pupils. In fact, these 13 turned out to be students of the teachers with this kind of inconsistency. This example highlights how the convictions of the teachers condition their practices in the classroom. That is, one perceives a causal relationship between convictions and misconceptions, because the misconceptions of the pupils seem to derive directly from the misconceptions of the teacher and from his convictions, according to the following sequence: conviction of the teacher $\rightarrow$ misconception of the teacher $\rightarrow$ misconception of the pupil $\rightarrow$ conviction of the pupil.
- *Angle as origin-point.* 12 pupils claim that the angle is the point where two segments or two half-lines meet, referring to the ones on the table or in a drawing: «It is this point here».
This conviction doesn't belong to any of the categories that we used to classify the answers of the teachers and it is spread in a uniform way amongst the different classes.

Such a category derives from the common language which conceives of the angle as a vertex.
- *Angle as magnitude.* 4 pupils spoke of angle exclusively as a quantity:
S.: It is a magnitude.
R.: What is a magnitude?.
S.: How big it is from here to here (indicating two edges of the table).
The 4 pupils are students of a teacher who conceives of the angle in the same way, demonstrating in this way a correlation between the answers of the pupils and the intentions of the teacher.

In general, the pupil's answers were not correlated to conceptual and cultural intentions made clear by the teachers, specifically the graphic semiotic means of objectification proposed by the teachers emerges with much greater strength than the conceptual goal they want to reach. In some cases, the graphic semiotic means takes the upper hand so much as to distort the teacher's intention itself, as in the case of the angle conceived as the length of the arc or the arc itself. In this case, the pupils confuse the graphic representation with the concept that was proposed. Furthermore, the pupils' answers did not belong to any of the categories foreseen by the teacher. They derive from the daily use of the common language (angle as a synonym for vertex) and from a limited interpretation of the few or sometimes univocal means of objectification proposed in the classroom. The univocity of the means of objectification proposed in the classroom is in contrast to the theoretical references relative both to semiotics and specifically to the angle.

7. Conclusions

Research results show that pupils' misconceptions about the concept of angle, pointed out by various investigations, also depend on teachers' didactic choices; choices that are often univocal, binding and do not take into account that mathematical objects usually have various definitions that history has elaborated, each one can gather one or more of the specific features of the object. Each definition tends to gather specific particularities of that object. In particular, in the case of angle, the different definitions which history has passed on are often essentially different, so much that one can conjecture that the object "angle" is the set of the characterizations that each definition highlights. If one of the definitions were epistemologically better, or easier, or closer to the identity of the object…, then we should do everything to propose it and to make it universal. In the case of the angle, however, each of the definitions that history has elaborated presents some problems even of intuitive acceptance.
In particular, D'Amore & Marazzani (2008) highlight that *all* the definitions that history has created are present, at an intuitive level, amongst the students we interviewed. A single mathematical object, has various interpretations and various models that tend to represent the characteristics of that object. Therefore, it would turn out to be didactically important to respect the interpretations of angle which come from the pupils. We are, in fact, in complete agreement with Mitchelmore & White (2000, p. 234) when they declare: «A third implication of our study is that verbal definitions of angle are unlikely to be helpful to young children. It is only when students have learned to recognise the

similarity between many angle contexts that they are likely to accept a definition which is expressed in terms of a single context as applicable to all angle contexts». And yet, from the interviews of the teachers, it comes out that the definition proposed to the pupils turns out to be univocal without any mediation or negotiation within the community of practices to reach a shared knowledge, but it is imposed by the teacher himself.

This article focussed, in a specific way, also on another cause of difficulty, the inconsistencies in the intentionality of the teachers, which derive from a limited use and unawareness of the semiotic means of objectification, with respect to the conceptual and cultural aspect of the knowledge they want their pupils to acquire. The complexity related to the learning of the concept of angle on the part of the pupils, highlighted by the reference literature, is therefore amplified by the choices of the teachers with regard to the didactic transposition and the educational design.

Intentionality attributes to the individual, in this case the teacher, a fundamental role in the possibility of giving meaning to mathematical objects, but such intentionality must be managed with awareness to make it didactically effective. In fact, the inconsistency between the explicit intentionality of the teacher, through verbal means of objectification and the graphic means of objectification, chosen to express this concept, can be the source of avoidable misconceptions in the mind of the pupil. The choice of the signs is not, in fact, neutral or independent. Radford (2005b, p. 204) claims that «semiotic means of objectification offer several possibilities for carrying out a task, designating objects and expressing intentions. (…) It is necessary, therefore, to know how to identify the semiotic means of objectification to obtain objects of consciousness», such an identification should be managed with a strong critical sense on the part of the teacher.

Referring to Husserl (1913), the results of this research highlight that the teacher, in classroom practices, too often creates inconsistency between the intentional act that determines the way in which the object is presented to consciousness (noesis) and the conceptual content of the individual experience (noema). Consistency and unity of the different intentional acts of the teacher do not seem to be always present in the classroom practices, when dealing with the angle. In particular, there is a lack of unity between the teacher's didactic intention, the semiotic means of objectification involved, and the direction of students' intentional acts and of the shifts of attention prompted by the teacher. This may result in lack of objectification on the part of the pupil that entails a fragile and meaningless learning.

The results of the research show that the decisions taken by the teacher to present the angle derive from proposals of school, curricula, established didactic habits etcetera, rather than from aware personal choices. In fact, he always provides the pupil only with univocal and conventional representations without analysing with the students their distinctive features. To create a consistency between semiotic means and the mathematical objects should objectify requires on the one hand a synchronic use of artefacts, gestures, bodily movements, language, figural and symbolic representations, organized in a semiotic node. On the other hand it is important to involve students, through the creation of a semiotic node in a meaningful mediated reflexive activity that encompass their personal experience and the socio-cultural features that are constitutive of learning: in particular negotiation and communication regarding the characteristics of the activity and the mathematical object. From a structural and functional approach, we can take into account also Duval's claims, that. «understanding begins when the subject articulates two registers of representation. In other words, we cannot consider one kind of representation better than another, if the individual is not able to master, alone and in the two directions, the conversion from one kind of representation, proposed by the teacher, in another kind of representation» (Duval, 2006, p. 613). The teacher has the delicate task of prompting adequate shifts of attention (Mason, 2003, 2010), guiding and supporting the student in the coordination of heterogeneous semiotic means of objectification – each of them articulated and difficult to be managed. This avoids the pupil or the teacher himself to confuse the mathematical object with one of its representations.

The semiotic means of objectification must not become *a priori* choices without any relationship with the classroom environment, without any critical analysis on the part of the teacher. We highlighted the importance of a rational control on the part of the teacher addressing Boero's et al. that coordinates Toulmin's model of argumentations and Habermas components of rational behaviour. In particular, the control, on the part of the teacher, of the teleological and communicative components allow both a consistency between his intentions and the semiotic choices, and negotiation with the students in suitable reflexive activities. D'Amore & Fandiño Pinilla (2009) claim that a mathematics teacher needs a strong mathematical competence acquired through a deep personal study that goes well beyond the discipline, in order to take into account an historical and epistemological point of view for each object, so as to reflect, compare, analyse, control from a rational point of view, and avoid the situations described in this paper.

This research brings a contribution to the topic of misconceptions in the learning of mathematics. It ascribes students' difficulties not only to the mathematical content and concept but also to a complex and rich activity that is the substance of knowledge and cognition: *semiotic reflexive activity* that we analysed as *objectification* processes. Therefore a possible way to overcome *unavoidable misconceptions* and prevent *avoidable misconceptions* could be to provide a great variety of semiotic means of objectification appropriately organised and integrated into a social system of signification fostered by mathematical practices shared by the pupils. The teacher has the delicate responsibility of an aware and consistent management of this cultural and semiotic complexity.

In this paper we presented the learning of the angle as objectification, a meaning making process driven by the teacher who directs the student's intentional acts and prompts shifts of attention to the mathematical object. This research focuses mainly on the use of natural and symbolic language. Further investigation is required to broaden the range of semiotic means of objectification involved to include, for example, also gestures, bodily movements, glances etc. A research based on videotaping allows to take into account such semiotic means and identify student's intentional acts and shifts of attention when objectifying the mathematical object. Therefore it would be possible to more thoroughly investigate the coherence between the teacher's intention, the semiotic means of objectification involved and the intentional acts and prompting of shifts of attention.

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