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Articulation and Change of Senses Assigned to Representations of Mathematical Objects[1]

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ABSTRACT: *This article documents the phenomenon related to the difficulties encountered by some students to articulate the senses assigned to different representations of a mathematical object obtained by semiotic transformations of treatment. It is presented a description and an analysis of the process of assigning senses achieved by students regarding specific tasks, where is required making such treatments between representations*

This paper is situated in a semiotic context, and studies in general the relationship semiosis-noesis in the construction of mathematical knowledge by students from grades 9th and 11th of the secondary education (Colombia); this study, without being exhaustive, includes aspects of mathematical activity, communication of mathematical objects emerging and cognitive construction of mathematical objects.

Keywords*:* *Mathematical object, Semiotic representation, Treatment transformation, Meaning, Articulation of senses.*

INTRODUCTION

The language has been constituted for human being as a way of describing the world and understanding their productions, generating at the same time the need to culturally build meanings. From the statement from Bruner (2006), language is acquired as used, in interaction processes, where the functions and communicative intentions are established. Such acquisitions are quite sensitive to context.

The form of human life, as suggested by the author, depends on shared forms of discourse, on meanings and shared concepts[3], which are published in each culture,

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[3] Perhaps it would be more appropriate to say that such meanings "are taken as shared."

which enables human beings to negotiate differences of interpretation and meaning, recognizing that the symbolic systems used by individuals in the construction of meaning preexist, and are rooted in the language and the culture. It is through symbolic systems that humans construct and give meaning to the world.

People live publicly, that is, not in isolation, through meanings and interpretations that must be publicly accessible. If meanings are not shared with others, they are useless. As Bruner (2006) puts it, the culture and the search for meaning within culture are the true causes of human action. In other words, it is culture that shapes human life and mind, giving meaning to the action.

The relationship between human beings and the world, as suggested by Bagni (2009), has no observational cognitive character, but practical-comprehensive. In relation to the school context, as recognized by this author, students must give sense to sentences or propositions, to parts of speech that sometimes they barely know or ignore at all. They must give sense to a particular signs, must "make speak" these signs. So, learners must move in the plane of the active interpretation of signs, in terms of hermeneutics. What has been discussed so far, emphasizes the fact that is assumed in this study: the subjects of a group, in processes of interaction around a specific task, necessarily stem from interpretations different to words, signs and representations. It is in such interaction, that they make explicit the assigned senses and consensually build the required meanings for tackling the task.

The present investigation documents the phenomenon related to the difficulties faced by some students to articulate the *senses* assigned to semiotic representations of the same mathematical object[4], obtained by transformations of *treatment*, in other words, by transformations within the same semiotic system of representation (Duval, 1999). A description and analysis of the processes of senses assigning of nine students, six of grade 9th and three of grade 11th, was made. It was based on work done by them in three small groups in relation to specific tasks[5], in which the meaning assigned to certain semiotic representations is explored and it is required to make treatment transformations. A qualitative research approach is assumed, making a descriptive-interpretive analysis, using two theoretical perspectives: Ontosemiotic (Godino, 2003, Godino, Batanero, & Font, 2007) and Sociocultural (Radford, 2006).

The situations presented below allow highlighting the complexities associated to semiotic transformations of treatment, related to the difficulty in articulating the senses assigned to a mathematical object, which were reported by D'Amore (2006):

> *Situation 1.* Proposal to 5th grade students in basic education (in Italy, mean age 10 years). Calculate the probability of the next event: throwing a die to obtain an even number.

[4] In principle, the meaning is taken here as a partial meaning (Font & Ramos, 2005), associated more with the contextual and even temporal. Each context helps generate sense, though not all possible senses. The meaning of an object (institutional/personal), following Godino and Batanero's ideas (1994), is the system of practices (institutional/personal) associated with the field of problems from where such object emerge at any given time.

[5] In Colombia the school previous to the university is organized into eleven grades, grouped in three levels: Basic Primary Education (5 degrees, ages 6 and 10-11 years), Basic Secondary Education (4 degrees, between 11-12 and 14 -15 years) and Vocational Secondary Education (2 degrees, between 15-16 and 16-18 years).

After working in small groups, with the teacher's orientation, students shared that while the possible outcomes when rolling a dice are 6 and those who make true the event are 3, the answer is 3/6. They also recognize that this probability can be expressed as 50%, while accepting the equivalence between 3/6 and 50/100, given by the teacher. Even some of the students recognize that speak of 50% means that *we have half of the probability to verified the event relative to the set of possible events* and, therefore, must be valid as response the expression ½, which is accepted and validated by the other students and the teacher, that is, the senses assigned to the mathematical objects are shared.

Once the class session is concluded, the researcher poses to the students that the fraction 4/8 would also be an appropriate answer, taking into account that it is equivalent to 3/6. Students and teacher say they do not agree. The teacher of the course says that the fraction 4/8 *cannot represent the event because the faces of a dice are 6 and not 8.*

In this case, what explains this «change» in the assigned senses to representations of mathematical objects shared before? Or rather, why not "articulate" the different senses assigned to the representations? If 4/8 is a result of treatment of 3/6 well dominated by students and teacher, why the sense of mathematical object "probability of obtaining an even by number throwing a dice" is not "preserved" with 4/8?

Situation 2. The sense assigned by a group of university students (in Italy) to the equation $x^2+y^2+2xy-1=0$ is "a circle" and the equation $x+y=\frac{1}{x+y}$ is "an amount that has the same value as its converse." They recognize that by transformations of treatment, they can pass from the first equation to the second, but the question, does the second equation represent a circle or not? Find answers as follows.

Student A: Absolutely not, a circle should have $x^2 + y^2$.

Student B: If simplified, yes!

While the first equation does not correspond to a circle, it is not this paper interest to tackle this aspect. The interesting thing here is that in the first case, "being circle" is associated with a certain expression, which is seen as an icon, and in the second, the semiotic transformation (of treatment) of the one who gives or not sense or to the expression; to perform the transformation generates a "change of sense." The mathematical object "circle" is accepted and related to the first equation, but it is not accepted by the second equation, although this one is obtained by treating the first by the students themselves.

From what is described in the above situations, the senses assigned to each of the specific representations of a mathematical object, apparently, have no connection with each other to enable their articulation. There is evidence in a variety of situations, at different levels of schooling (D'Amore, 2006), about a "change of sense" when a semiotic representation is transformed into another, within the same record of representation. Events similar to those described above have been made evident in the Colombian context by the author of this research.

THEORETICAL FRAMEWORK

Science, seen as human activity, allow interpretations for understanding the phenomena of the world, without necessarily assuming that there are privileged ways to do it. The problem has many facets and, there are different ways to interpret and address them; ways that has to be analyzed, contrasted, criticized, endorsed, or restated according to its relevance and effectiveness. This current research assumes a philosophical pragmatist approach, in which experience, typically human,[6] is necessarily referred.

From a pragmatist approach as Rorty's (1991), the *ideas* are not only taken as guide for *action*, but its validity and importance derived from the utility and effectiveness in a given situation or problem that satisfies the needs or requirements from a subject or society[7]. In particular, following approaches by this author, the *reality* is described by using languages, but not preexisting to them, it develops with them, born with them, makes sense with them. Human language is contingent -it can happen or not, can be one way or the other-, and reality is setting in, and through languages. Therefore, the descriptions of the world and the truth cannot be independent from human beings. The reality is a set of agreements between humans. The world, meanwhile, is a set of events, of facts rather than things. That is, there is nothing that can be considered as "objective reality", but human groups with different discourses and "objectivity" should be seen as a desire to persuade and agree unforced. There is no hierarchy between disciplines or discursive genres of science or of the humanities; *scientific* language is only one of the possible languages. Using Wittgenstein's terms, scientific language is only one possibility in the *language games*.

Semiotic Representations and Types of Transformations

In recent years have returned with some force studies about semiotic representations and their relationship with cognitive operation, among which stands out the one developed in the last two decades by Duval (1999, 2004). In certain everyday contexts, and in some fields of scientific knowledge, it is possible to access the objects directly through perception, the use of instruments or, indirectly, using representations of such objects. In other fields, access via representations is not only useful but mandatory; such representations are produced using different systems of representation of different nature[8].

In mathematics, in particular, learning of objects is primarily conceptual, which requires the appropriation of semiotic representations, in other words, representations by signs.

[6] From the ideas of Wittgenstein (1958/1999), it is not about examining the use of words that subjects do in specific situations, but also about recognizing the existence of social rules of use of signs in language games in certain contexts.

[7] What is contingent, it is in a certain place and time and may not be useful in another time or in another situation. Temporality and usefulness validate each other, mutually in the field of contingencies (Dáros, 2001).

[8] As stated by Duval (2004), such systems may be *non-semiotic* -neural networks (such as the different forms of memory), physical instruments (such as microscopes and telescopes) -, or may be *semiotic* (by signs). Producing a semiotic representation is necessarily intentional. This is, for some authors, the difference between semiotics and semiology.

The subject does not come into direct "contact" with the objects, as these are not accessible perceptual or instrumental. While the ostensive references of such objects are not possible, it is necessary to use representations[9].

In the process of teaching and learning mathematics, it is essential the use of representations of objects in a variety of semiotic systems of representation, more specifically, in a variety of semiotic *registers* (Duval, 1999). In particular, it is necessary to seize opportunities to transform a semiotic representation of a mathematical object into another representation of the same object. Such transformations between semiotic representations occur within the same record of semiotic representation, called *treatments*, as well as between different registers, called *conversions* (Duval, 1999).

It is usually stated that cognitive problems are related to the conversion while any related to treatment is not usually seen as a relevant issue for the construction of the mathematical object. Duval (2004), for example, recognizes the conversion as one of the fundamental cognitive operations for the subject's access to a true understanding, explicitly highlighting the complexity involved in the recognition of a same object through completely different representations, as produced in heterogeneous semiotic systems, and focuses his gaze on the difficulties of learning mathematics in this process. Nevertheless, it does not highlight the complexity associated to transformations made within the same semiotic system of representation. However, in mathematics, treatment transformations between semiotic representations -within the variety of records used-, not only are essential but can be a source of difficulty in understanding processes of mathematics by students.

In the international context there are several research papers that specifically address issues related to the semiotic transformations, among which highlights those by Duval (1999, 2004, 2006); D'Amore (2006); Godino, Batanero, and Font (2007), Font, Godino, and D'Amore (2007), D'Amore and Fandiño Pinilla (2008), and Santi (2011).

Sociocultural Approach

Even if it is recognized the epistemic importance of language, as mediator of human activities, it is argued that only in terms of discursive practices cannot adequately describe the ways of thinking, understanding and conceptualizing. In the pursuit of knowledge, as suggested by Radford (2006), human beings speak, gesticulate, write, use artifacts, and grab objects appealing to a variety of culturally arranged semiotic systems. While the signs and artifacts used mediate knowledge acts, alter the cognitive ability to be affected by things, and make this capacity, and therefore knowledge, be culturally dependent.

Inspired in the anthropological and historical and cultural schools of knowledge, Radford (2006) suggests elements of a culture theory of objectification, supported by an

[9] These representations may be *discursive* -using natural language or formal languages- or may be *non-discursive* -through Cartesian graphs or geometric figures-.

epistemology and ontology unrealistic. This author rejects the mentalist conceptions of thought, going for a characterization because of its nature semiotically mediated, and its mode of being as reflexive praxis. Radford (2006) stresses the role of cultural artifacts (systems of signs, objects, instruments, etc.) in the social practices since they are constituent parts of thought and it not only aids, that is, humans think with and through these artifacts. It recognizes, in particular, that individuals require becoming aware of cultural objects, and the impact cultural meanings have in the way an individual thinks and knows the objects of knowledge, while they not only guide the activity in which this happens, but they give certain "form." To operationalize his theory, Radford (2006) introduced a fundamental concept of semiotic-cognitive nature, which he called as *objectification*, precisely this subjective awareness of the cultural object.

From a socio-cultural approach, knowledge is related to the activities of individuals in a particular context, in other words, culture is consubstantial from knowledge.

Ontosemiotic Approach

Since the proposal targeted by Godino (2003), the concept of system of practices is conceived as a set of significant practices to solve a problem area. Mathematical activity, organized in systems of operational and discursive practices, has an essential role in the generation of mathematical entities (cultural/mental). For this author, mathematical objects are conceived as emerging from a practices system, as complex entities progressively constructed that enriches and completes itself from reflective activity in resolving certain problem areas. Emphasizing that these are the result of human construction, they evolve and depending on the persons or institutions may be provided with diverse meanings, shifting the focus to the action of individuals in contexts[10], mediated by instruments.

For Godino (2003), the theoretical notions, *system of practices* and functional categories of *primary entities* or types of objects (language, situations, procedures, definitions, properties and arguments), the five *dual facets* (personal/institutional, ostensive/non-ostensive, copy/type, elementary/systemic, expression/content) from which these entities can be considered, as well as the notion of *semiotic function (*expression/content, every expression refers to a content) constitute an adequate possibility to analyze the human cognition[11]. For this work, the analysis focused on the dimension or facet *expression-content.* In this approach the semiotic transformations are an emerging aspect of a semiotic function that relates a representation R (antecedent), in the couple configuration-practice system of objects with a representation S, in another couple practices -configuration system of objects.

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[10] *Context* is seen as a set of extra/inters linguistic factors, that support or determine the mathematical activity and, therefore, the form, adaptation and significance of objects placed in its game.

[11] This notion, as Godino remarks (2003), comes from Hjelmslev (1943), who called *sign function* to the relationship of dependence established between the parts of a text and its components, as well as between the components thereof; notion that was later described by Eco (1979) as *semiotic function*. See also: D'Amore & Godino, 2006, 2007; Font, Godino & D'Amore, 2007.

In the performing of any *mathematical practice*[12], subjects make use of basic knowledge, and it triggered a set of relationships between different types of objects (primary entities): problem-situations, language, definitions, procedures, properties, and arguments. In other words, personal mathematical practices activate a network of emerging and intervening objects (Figure 1), in other words, the *cognitive configuration* on action (Godino, Batanero, & Font, 2007).

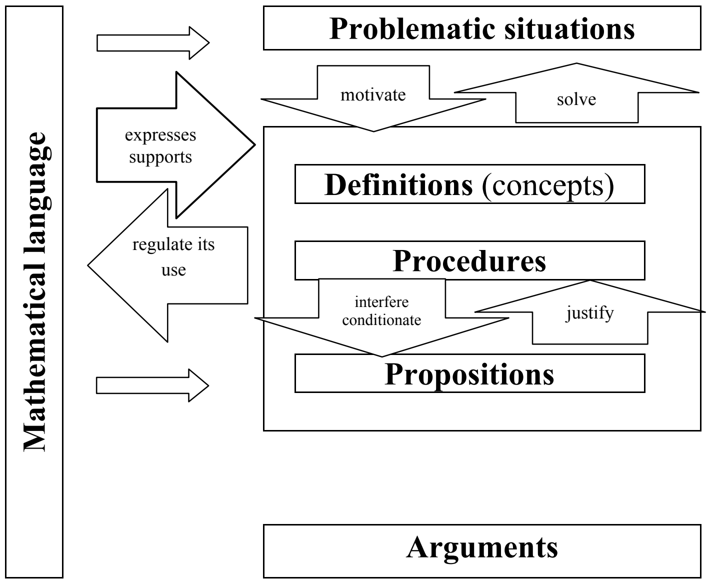

Figure 1. Configuration of Primary Object.

Objects, Meaning and Sense

Learning mathematics and mathematical activity itself, while require appealing to the transformation of signs within semiotic registers, culturally given, are inherently a semiotic activity. However, from a social and cultural approach, to understand the use of signs, we should take into account the reflective activity mediated that underlying the coordination of semiotic registers. Thus, everything associated with the meaning moves from *object* that the signs *represent*, to the practice that they enable and mediate (Santi, 2011). Therefore, we pass from the coordination of semiotic systems to the integration of systems of the practice from which meaning emerges, in other words, to cognitive configurations that are activated by such practical systems.

Thus, the meaning cannot be identified only by the relationship representation semiotic-object of reference. To account for the complexity of mathematics as a cultural and individual effort, it is not enough to reduce learning and mathematical thinking to a coordination of different representations with a common denotation. The way we get to

[12] From the EOS, mathematical practice is regarded as any action or manifestation, not only linguistic, formed both in solving mathematical problems and communication of other found solutions, in order to validate or generalize to other contexts and situations or problems (Godino, Batanero & Font, 2007; D'Amore, Font & Godino, 2007).

think and know the objects of knowledge is framed by cultural meanings that go beyond the content of the activity, within which occurs the act of thinking. That is, the meaning attributed to a mathematical object depends on both; the subject and the context in which it is addressed, and therefore, it is somewhat flexible, dynamic, and moving[13]. The meaning of an object, as it is assumed by Radford (2006), is attributed by culture and has an existence that transcends the subject, it is more stable. One could say that the meaning is more decontextualized and general[14]. Nay, the sense is relative to several sensory and semiotic modalities, and it is associated more to the pragmatic, while the meaning is associated more with cultural semantics. The sense of an object can be considered as a contextual meaning of that object.

Sense of a Primary Mathematical Object

Given a primary mathematical object, the *sense* of such object is the content of the semiotic function that has such primary object as an expression of the semiotic function (Figure 2).

Expression	*Content*
Primary object	Sense of the primary object

Figure 2. Sense assigned to a primary mathematical object.

A single parent object can have different senses (Figure 3). For example:

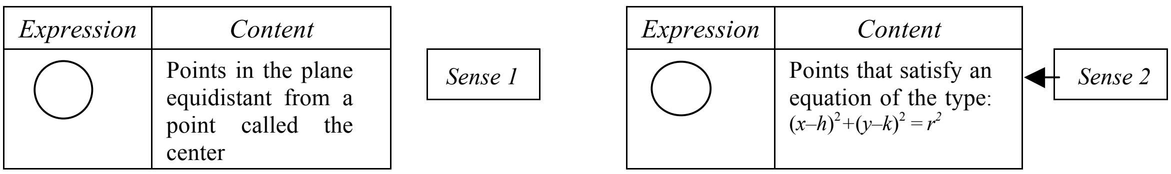

Figure 3. Different senses of an object, which is institutionally expected to construct apprentices.

Articulation of Senses

There is a joint of senses when it is established a semiotic function between two different senses of the same primary mathematical object. This is, when one of the senses (content) of the primary object becomes the expression of a new semiotic function whose content is another sense of such object. Thus, for example, in the case of Figure 3, the joint produced may be synthesized in a diagram as follows.

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13 The subject can be an individual, a group of individuals or an institution. The sense can be assigned individually, be the result of a negotiation within a community of practice, or acceptance of an institutional nature.

14 For him, mathematical objects are "fixed patterns of reflective activity (...) embedded in the world of constant changing of the social practice mediated by devices" (p. 111), where such devices can be objects, instruments, systems of signs, etc.

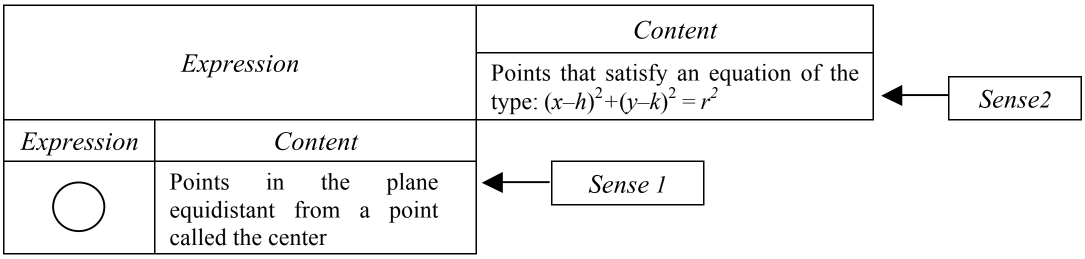

Figure 4. Joint of senses.

This joint is the result of the concatenation of the two-presented semiotic functions. We have a primary mathematical object with two different senses assigned. In other words, there is a joint of senses when a new sign-function is established in which one of the two preceding semiotic functions plays the role of expression (Figure 4). Such combination of semiotic functions can be simplified into a single semiotic function that relates (articulates) both senses:

Expression	**Content**
Points in the plane equidistant from a point called the center	Points that satisfy an equation of the type: $(x–h)^2+(y–k)^2 = r^2$

Figure 5. Joint of senses (simplification).

Thus, the joint of senses means that contents of semiotic functions previously considered as different (no explicit relationship) are now considered, in some ways, equivalent (explicit relationship). From what stated above, two primary objects (especially two representations), which are considered syntactically equivalent, while one of them is obtained from another as a result of a treatment process, they can be associated to the *same sense* (the same content), that is, also retaining the semantic equivalence. When the sense assigned to a semiotic representation does not articulate with the sense assigned later to another semiotic representation obtained from this one by treatment while the originally given sense is "abandoned" and a new sense is assumed, it will be said that there is a *change of sense.*

METHODOLOGICAL DESIGN OF RESEARCH

This research is part of an approach of qualitative research, from the type descriptive-interpretative, analyzing in real context of the described phenomenon, related to the change of sense. It makes use of the structured interview based on task (Goldin, 2000), conducted in small groups of students in grade 9th (basic education) and grade 11th (Middle education), from five schools, two official schools, located in the outskirts of the city of Bogota, in areas of low socioeconomic status, considered vulnerable, and three private schools, located in not peripheral areas of the city, where attend students of middle and high socioeconomic status.

Instruments

We worked with three semi-opened instruments, each one with a task associated with a specific topic: probability, equivalence of expressions and conical. The first two were proposed to students in grades 9th and 11th and the third only to grade 11th. Each of them is with three items, and a similar design to what is shown below.

Questionnaire 2 (Equivalence of expressions):
Hereinafter, assume that n represents an integer either. Please answer in the order in which the points appear and go to the next only when you have fully answered the previous point.

(1) Say what does it mean or you assigned interpretation of the expression $3n$. Can be interpreted as *a number tripled.*

(2) State whether the following equality is valid or not: $(n-1)+n+(n+1)=3n$

   a. Mark with an **X** the answer you think is correct ***Yes*** **( )** ***No*** **( )**

   b. If yes, check equality, if not, give reasons why not met.

(3) Can the expression $(n-1)+n+(n+1)$ be interpreted as *a number tripled?*

   a. Mark with an **X** the answer you think is correct ***Yes*** **( )** ***No*** **( )**

   b. Explain or justify below with as much detail as possible, your response:

Gather of Information

In addition to the inquiry by developing tasks (proposed in the questionnaire) and the content of the notes taken by the researcher, there are transcripts of audio-taped interviews, which were made with each of the small groups, selected based in the responses to the proposed task in the different questionnaires.

Once recognized the importance for students to get involved in the development of the activity, the tasks were initially worked individually by each of the students from different courses (each with about 40 students) and then in small groups (2 to 4 students). Later, under the guidance of the teacher of the institution in charge of the course, there was a discussion of some of the responses given by small groups[15], which were selected by the teacher based on the observation made by him during the students working time.

From research in social psychology of learning, it is recognized that in such situations it is important to take into account that students must not only fulfill the proposed task, but also respond to the complexity of the social situation, in this case, to understand the expectations of the researcher and the nature of the problem, in addition to understand their role, and their peers' role in the interaction.

The selection of small groups to interview was conducted from the responses to the task, which contained three items, based on the following criteria:

[15] It recognizes the importance of who socialize a response, does it on behalf of the group; not only to realize the group process but also that his speech can be supported by the other group members and reduce some of the tension that usually can be generated when an outcome is defended individually.

(1) That in the first item, the investigation made by students was compatible with the "institutional meaning" (Godino & Tanner, 1994) assigned to that object.
(2) That in the second item at least one of the students can recognize explicitly the syntactic equivalence between the given expressions (arithmetic or algebraic), that is, that at least one student can perform a treatment process that allows him to get one of the expression from the other (making use of the properties into the arithmetic or algebraic register).
(3) That in the third item at least one of the students who recognized the syntactic equivalence in the previous item of expressions, answers negatively.

From such small groups of students four were selected, one for each of the four institutions, i.e., 16 small groups in total, taking into account the availability and the interest shown by the students to participate in the interviews. Of these 16 small groups were finally selected 3, one for each proposed task, whose oral and written production constitutes the main source of data reported and analyzed in this research.

Interviews were conducted in a different space from the classroom where were present only respondents (one group at a time) and the researcher. For interviews was prepared the following general script:

- Explicitly confirm the names of those interviewed at the beginning of the interview and request that, when possible, students mention their name in the interventions made (to record the voices, in addition to the researcher's notes).
- Check if group members share at least one interpretation from the worked expression(s) in the proposed task(s).
- Check if group members show a mastery of the transformations of treatment required for getting one of the expressions from other(s), i.e., if they recognize a syntactic equivalence between them.
- Investigate the possibility for respondents to recognize that the interpretation given to one of the expressions can be assigned to another expression syntactically equivalent to it, that is, if they can or cannot articulate the assigned senses to the referring expressions.
- Check if changes occur with respect to interpretations initially given to worked expressions, that is, if a change is generated in the sense assigned to the primary mathematical objects.

Although the researcher had a script for the development of the interview, incorporated some retrospective questions, and even some hints to supplement the inquiry and was attentive to recognize situations of interest for the purpose of research, asking specific questions designed to get more information.

Data Analysis

It was made an analysis of the individual production of each student, from the three small selected groups, in relation to the task, although the focus was on the production that was generated during the interview with the groups, in which was possible the intervention of each student, the interaction between them and, in particular, the choice to agree or disagree with the statements of the members of each small group in relation to the item(s) of the proposed task(s). Group work was privileged, while it is explicitly recognized the interest in enabling students to both, share views and consider the statements of others about the work done on the task, as well as the possibility to recognize a "shared sense" in the development of the task.

It is assumed that the verbalization of processes of thought and action provide important information, not only from written materials, such as those obtained by instruments of inquiry (tasks or questionnaires), but also from interaction processes, like those generated in the work with small groups or by interviews, in the context of a given task. The interview to small groups, unlike individual, offers more opportunities of interaction that allow recognizing the different interpretations made regarding mathematical objects involved in such task and identify the senses assigned to the expressions, in addition to recognize some reasons that make possible or not the assignment of senses and the articulation of these. It also offers a less formal and tense environment for each of the respondents, due to the interviewer's attention is not permanently focused on the work of a single individual. Respondents can interact with each other, welcome or call into question the claims and arguments presented by their peers, as well as having the opportunity to meet, analyze and have additional elements relating to the proposed tasks and the arguments initially considered.

The analysis of the student productions reported in this research is made from various perspectives, oriented from three theoretical proposals, with different levels of use thereof. First, the structural-functional approach from Duval (2004), used primarily to identify the field of research, owing to that allows making denominations that allow to see and to describe the phenomenon under study. Moreover, sociocultural approaches from Radford (2006) and onto-semiotic from the group of Godino (2003), used to explain and understand the student productions, because their facts not only recognize cognitive facts but also cultural and historical. For these two authors, the use of formal systems of signs is an emerging phenomenon of the systems of practices social and cultural framed. To perform the analysis of the transcripts of the interviews, it is done a thematic segmentation, which is itself segmented in each of the interventions of the participants in the interview, making an enumeration.

DEVELOPMENT OF THE RESEARCH

It is presented, with some detail, the work done by two of the interviewed small groups, in two of the proposed tasks; the task about probability and the task related to conical, both worked with students from grades 9th and 11th.

Task About Probability

This section describes some aspects of the work of a group of 9th grade students (Questionnaire 1), from the CHA institution, integrated by Pablo (E_4), Daniel (E_5) and Jonathan (E_6). In his individual work, Pablo establishes three semiotic functions. One between the expression 50% and content "favorable cases on possible cases" another between the expression 50% and the content 3/6, and another between the expression 3/6 and "favorable cases on possible cases," that is, it can articulate the senses assigned. However, for him the probability cannot be 4/8, because if so, raises, *the given information would be wrong and the dice would have given 8 faces*. In the Figure 6 it is presented the cognitive configuration of primary mathematical objects attained by Daniel, obtained from work both individually and in group.

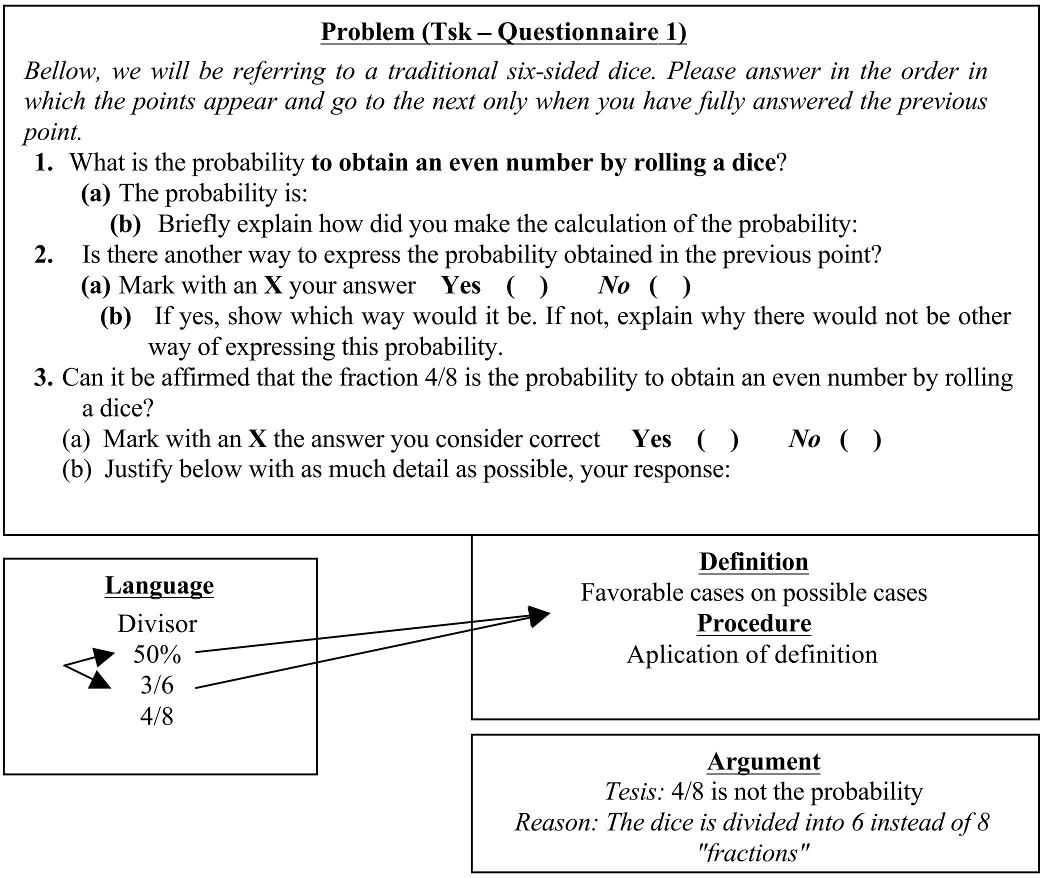

Figure 6. Initial configuration of Daniel (E5).

As a group (G-2), students found the probability to obtain an even number by throwing a dice, which they represent in two different ways, using the numerical expressions 50% and 3/6. However, none of them recognize that the fraction 4/8 can be interpreted as such probability. That is, they fail to articulate the senses assigned to the previous numerical expressions. In the small group work these students maintained the interpretations individually made regarding the proposed task. The reason why they do not accept that the fraction 4/8 is the requested probability, is *that the number of sides of the dice is 6 and not 8*, due to the sense assigned to the fraction 4/8 is "anchored" to the dice, to the concrete object referred to in the proposed task. The data obtained is summarized in the Table 1.

Table 1

Grillage Synthesis (Initial) –Pablo, Daniel y Jonathan– Grade 9th, School CHA

	Pablo (E_4)	Daniel (E_5)	Jonathan (E_6)	Group
Recognizes several ways to represent the probability	Yes	Yes	Yes	Yes
Articulates senses assigned	No	No	No	No

Interview G-2

The segments of the transcript of the interview are reported below in three columns. In the first one is numbered each intervention, in the second one the person involved (interviewer: P and students: E_k) and in the last one the accompanying text. These students, in their individual questionnaires and in the small groups' questionnaire had recognized more than one way to represent the probability of the event in question, proposing expressions like 50%, 3/6 and ½. Nevertheless, all of them started saying that the fraction 4/8 does not express the probability of such event.

[1] P Jonathan … What do you say Jonathan?
[2] E_6 Well, … as the main theme is a dice, it is recognized that the dice has six sides
[3] E_6 …
Yes, if we take the pairs out it would be three... 4/8 then is not as representative
[4] E_6 of the dice, since the dice has neither eight faces, other four even numbers.
[5] P Looking at it from the way I see it, the four eighths is not accurate, then ...
[6] E_5 That´s it
I understand. What do you say Daniel?
[7] P Well ... if it's a 6-sided dice, the division is three, three of the probability [refers
[8] E_4 to the fraction 3/6], it can't be the reference number four [refers to the fraction 4/8].
[9] E_4 All right… and Pablo?
Well, I think there is no..., not, because I think it wouldn't work, I think the fraction would be misconceived to how the problem should be solved, based on the sides of the dice.…
A dice is never going to have eight sides ... I think that.

The students focus on the proposed situation, specifically in the dice as object. While Jonathan (E_6) recognizes that the fractional 3/6, ½ and 4/8 are equivalent: *they can give the same*, insists that 4/8 *is not accurate* ([4]). He explains that fractions have a "generator", which in this case would be ½, but focuses on the role of the denominator. Later, as they interact, they start to partially modify their opinion, and begin to recognize that it is possible to accept the answer 4/8, but there are still several questions. For example, Pablo states:

But then, to be the fraction of the dice it wouldn't work because it is not specific what is sought ... 3 to the even numbers and 6 to the sides of the dice, instead in 4/8 it would not be clear.

[10] E_4 Well ... what I say is that the fraction ... yes, the same thing, that the fraction itself work but it´s misconceived in detail, why? Because it gives bad information about the probability and about the sides of the dice.

[11] E_4 If they specify and give for example a 3/6, it would be a detail already, what happens is that the fraction...

[12] P When you say, "it would be a detail already" what do you referred to?

[13] E_4 Rather, the fraction used is not well specified but why? Because ... 4/8 is basically the same as 3/6, right?

[14] P Mmju *[I understand, continue]*.

[15] E_4 … But then, to be the fraction of the dice it wouldn't work because it is not specific what is sought ... 3 to the even numbers and 6 to the sides of the dice, instead in 4/8 it would not be clear.

After listening to the ideas presented by their peers, particularly by Jonathan (E_6), Daniel (E_5) returns the statement [4] and tries to overcome the difficulty on not accuracy, proposing a dice with more sides that includes numbers on their faces with decimal figures. For him, it is physically easy to build a dice that work for this purpose. However, sometimes hi mentions an 8-sided dice and sometimes that it can be 6-sided as long as the sum can be 8, if you change the given usual numbers of the dice with decimal numbers. He fails to clearly express how this dice would be, which he recognizes different but possible:

[16] E_5 Decimal numbers, for example 2.5 plus 2.5, 4 so it may result, … to get 8, two point five, point six, bla, bla, bla... *[etc.]*.

[17] P Oh, ok! … that the numbers of the dice are not from 1 to 6, but others.

[18] E_5 Exactly.

[19] P … but, assuming that you don't have different dices and you know that you have been ask probability of getting an even number with a traditional dice, without changing the dices...

[20] E_5 No, because there is no accuracy.

[21] E_5 Rather, it would be illogical that one would say… ehhh; there is a 6-sided dice, which is divided into four eighths.

[22] P Mmju… *[I understand, continue]*.

[23] E_5 Unless you can split it, because… it is a dice.

[24] E_5 Well, for example, yes, physically it is very easy, we make a 8-sided dice, but… rolling decimal numbers that the sum of all is 8… and that at the same

[25] P time can be 6, yes? … Imagine this dice, the sum is 8, but the sides are still 6.
[27] E_5 Yes, and, how would it be?
Well, for example… I don't know … a different dice, how can I say it? For example, that the numbers change or … that work with decimals.

Subsequently, to investigate a little more about the potential "anchor" to the initial situation, the dice as physical object ([21] and [24]), the interviewer (P) decides to ask indirectly, by going to an argument of another student, who claimed that fractions like 3/6, 4/8 or 15/30 are equivalent to half, and so any of them could represent such probability as well as, for example, 10/20:

[28] P What would you say? Would you agree? Or would you see a problem in
[29] E_6 this statement?
[30] P I would agree.
[31] E_5 Daniel, I see yo thoughtful ... *[Smiles].*
Not exactly when… I don't contradict that it is the half and that it'd be the
[32] E_5 same… *[in his gestures and his voice tone there is evidence of doubt].*
… But if you search accuracy, if it is a dice of 6 sides, I would work with
[33] P the right numbers; 3 of 6.
[34] E_5 Three of six … *[continue]*
… Because it wouldn't be clear that, for example, I call 4/8 in a 6-sided
[35] P dice, I don't think, no! … Then it is an 8-sided dice, yes?
[36] E_4 What do you say Pablo?
[37] E_4 Well, I also agree that ten twenty *[fraction 10/20]* is the same as 3/6.
... But if I'd formulate the …, the question… , if for example you are
[38] E_4 asked Can be stated that the fraction 10/20 is the probability that rolling a dice it is obtained an even number?
[39] E_4 You would be blocked, because … how come 10/20? Since when have a dice 20 sides, you know? then [that is] what I don´t get *[laughs a little]*… What happens is that in the common sense of the people … of the … of everyone, that would not be understood.

Finally, the interviewer asks again to Daniel (E_5), who has been quietly listening intently to his classmates, if now he would accept the argument of the student:

[40] E_5 Right now, yes … After discussing all of this.
I mean, for someone common, no, but…
[41] P And what made you change your mind?
[42] E_5 Because…, here for example, I answer in the first question 50% *[points to the first item of the questionnaire]* … from 100 it would be the same, a half
[43] P *[he means that 50 is a half of 100]*
[44] E_5 So you say, 4/8 would be a half, so it's the same, it doesn't matter.
Equivalent yes … but not looking at the sides, or at the dice, but at the half.

During the interview, after several interactions with Pablo (E_4) and Jonathan (E_6), his group mates, and the interviewer (P), Daniel (E_5) recalls that in its Questionnaire, he had responded that the probability is 50%, which is *half*, and recognizes that 4/8 is also half; then he accepts that the fraction 4/8 is *equivalent* to 50% ([42]). Namely, when he manages to decenter from the object, of the sides of the dice, and focuses its attention on the formal expressions representing half, he gets to recognize that the fraction 4/8 expresses the desired probability and so articulate senses assigned ([44]). Meanwhile, Pablo and Jonathan do not accept the last argument given by Daniel and although the interviewer says that the issues raised by him seems to be a good argument, they do not change their minds and insist that even if the fraction 4/8 is *equal* 3/6, this fraction is not the desired probability. In fact, for Pablo (E_4) the dice should have as many sides as the digit in the denominator ([38] and [39]).

Cognitive Configuration of Primary M*athematical Objects*

Below is presented the cognitive configuration achieved by Daniel (E5) after the interaction process during the group interview, in relation to the work from the task of probability (Figure 7). In this diagram, by a solid line, are pointed the semiotic functions initially established by the student, between an expression and a content. By a dashed line are pointed new semiotic functions, shown during the interview in the interaction with peers in the small group.

In his individual work, Daniel had established three semiotic functions (Figure 6). During the interview he explicitly establishes a new semiotic function between 4/8 (expression) and 3/6 (content), although he initially suggested that in the specific situation of the dice, *if it is sought precision* he rather work with the right numbers that are: 3 of 6 ([32]). Then, after about three minutes listening intently to Pablo and Jonathan's interventions, and the interviewer's questions (P), he does "separate" from the given concrete situation of the dice and establishes a new semiotic function; this time between the expression 4/8 and content "number of favorable cases divided number of possible cases", accomplishing a joint between the different senses assigned ([40]).

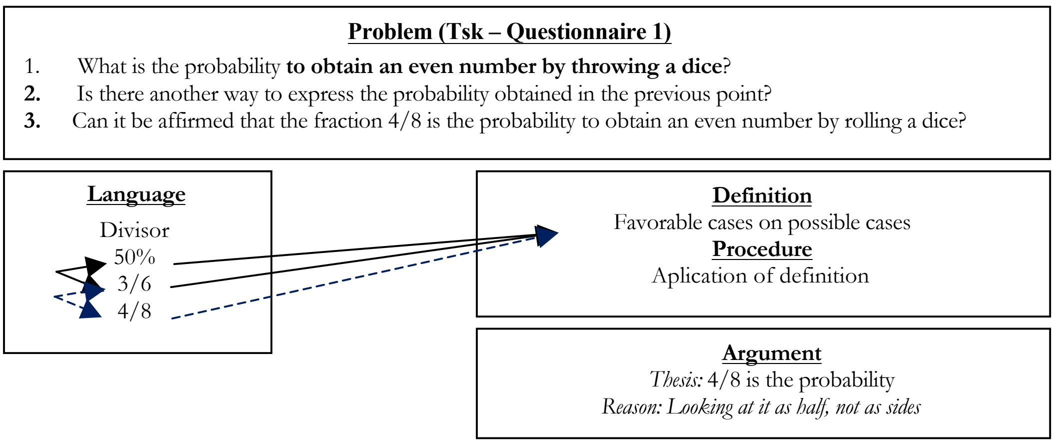

Figure 7. Final configuration of Daniel (E5).

To summarize. The process of interaction during the group interview led to changes in the initial interpretations made by students. On the one hand, the three of them explicitly recognized the equivalence between fractions 3/6 and 4/8, and even that any fraction equivalent to 3/6 could represent the desired probability in the task, but their "anchor" to the object, to the dice and the number of sides, do not allowed them to articulate the senses assigned to such terms and, therefore, Pablo and Jonathan do not accept that 4/8 is this probability. Daniel (E5), meanwhile, managed to "separate" from the concrete situation and recognizes that the fraction 4/8 represents the probability and achieves to articulate the senses assigned to different numerical expressions. Table 2 summarizes the information obtained.

Table 2

Grillage synthesis–Pablo, Daniel y Jonathan– Grade 9th (School CHA)

	Pablo (E_4)	Daniel (E_5)	Jonathan. (E_6)	Group
Recognizes several ways to represent the probability	Yes	Yes	Yes	Yes
Articulates senses assigned	No	Yes	No	No
Change (recognition of equivalence between 3/6 y 4/8)	✓	✓	✓	✓
Change (articulation of senses)		✓		

Task About Conics

This section describes some aspects of the work of a group of students of the institution CHA, grade 11th (G-1), composed of Maria Elvira (E_1), Daniel A. (E_2) and Daniel D. (E_3), in relation to a task about conics. In the individual work was evident that the sense assigned by each of the students to the equation $x^2+y^2+2xy-1=0$ was the circle[16].

In their individual work, Maria Elvira (E_1) established three semiotic functions, one between the expression $x^2+y^2+2xy-1=0$ and the content "a circle" another between the expression $x^2+y^2+2xy-1=0$, and the contents (equation) $x+y=\frac{1}{x+y}$, and the third semiotic function between the expression $x+y=\frac{1}{x+y}$ and the content "circle." So, Maria Elvira achieved to recognize the *syntactic equivalence* between these equations, by making the transformation of treatment required for getting an equation from the other, and to articulate the senses assigned to them.

Daniel A. (E_2), meanwhile, established a semiotic function between the expression $x^2+y^2+2xy–1=0$, and a content "circle", but does not recognize equivalence between the

[16] Although the quadratic equation $x^2+y^2+2xy-1=0$, does not represent a circle, but a "degenerated conic" (two parallel lines), for purposes of the analysis that presented this section, is it not relevant if the equation is erroneously interpreted by the students as circle.

two equations. He states that in one of them appears a product (the term $2xy$), which then becomes a sum ($x+y$), "which is meaningless". Also he states, that in one of them the variables are squared and in the other do not, "Square root is taken to the entire equation, which cannot be".

In the work done individually by Daniel D. (E_3), he established two semiotic functions, one between the expression $x^2+y^2+2xy-1=0$ and the content "circle," and another, between the expression (equation) $x^2+y^2+2xy-1=0$ and the content (equation) $x^2+y^2+2xy-1=0$. He argues that the equation of the circle must have squared variables, and because in the equation $x+y=\frac{1}{x+y}$ they are not squared, then it cannot be circle, i.e., although he recognizes the syntactic equivalence obtained by treatment the two given equations, fails to establish a semiotic function between the expression $x+y=\frac{1}{x+y}$ and the content *it is a circle,* then, he asserts that this expression has not the variables squared. The cognitive configuration of primary mathematical objects initially achieved by Daniel D. (E_3) is obtained from the work both individually and in groups, it is as follows.

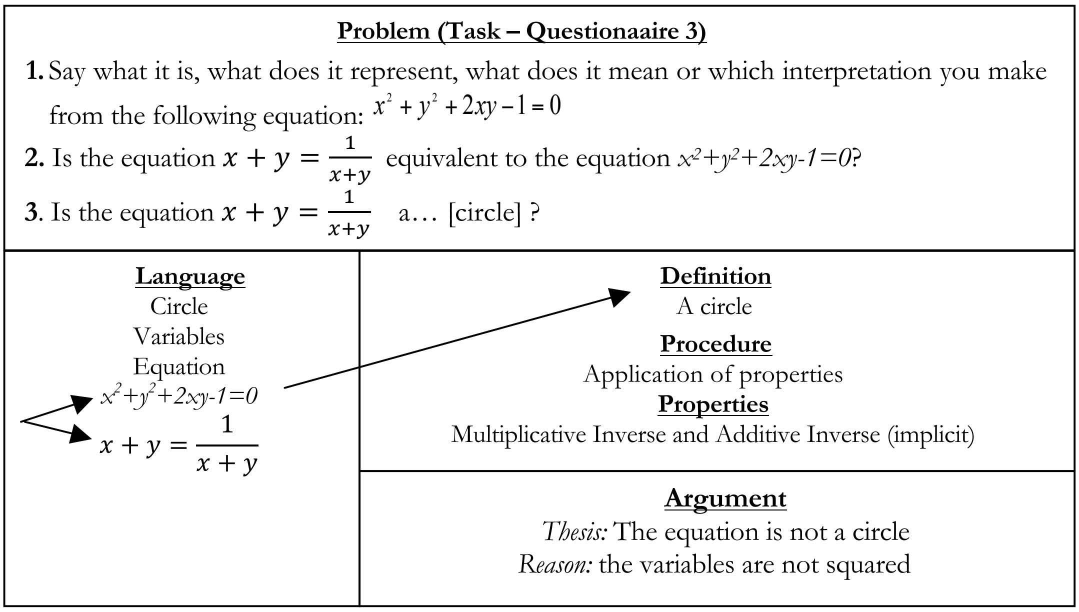

Figure 8. Initial configuration from Daniel D. (E_3).

After, Daniel A. (E_2), who had not recognized the possibility of making transformations of treatment to establish the equivalence between the given equations, in his work with Maria Elvira (E_1) and Daniel D. (E_3), he managed to do it. Thus, as a group, these students shared the sense assigned to the equation $x^2+y^2+2xy-1=0$, that of being "a circle", and recognized a syntactic equivalence obtained by treating between such equation and the equation $x + y = \frac{1}{x+y}$, but they did not share the possibility to articulate the senses assigned to such equations.

In connection with the recognition of the transformations of treatment required to obtain the equation $x^2+y^2+2xy-1=0$ from the equation $x + y = \frac{1}{x+y}$, that is, from the syntactic equivalence between these two equations using properties of real numbers, as well as the recognition of semantic equivalence, as possibility of articulating senses assigned to them. We have the information that is summarized in the Table 3.

Table 3

Grillage Synthesis (Initial) –Maria Elvira, Daniel A. and Daniel D. – Grade 11th, School CHA

	Ma. Elvira (E_1)	Daniel A. (E_2)	Daniel D. (E_3)	Group
Recognizes syntactic equivalence	Yes	No	Yes	Yes
Articulates senses assigned to equations	Yes	No	No	No

Interview G-1

From this group, only two of the three students recognized in their individual work the syntactic equivalence obtained by treatment between the given equations. However, in their work as a group they were able to perform the required treatment to one of the equations (expression) for getting the other (content) and so, to recognize this equivalence, but they failed to articulate the assigned senses to the two expression. That is, they did not managed to establish a semiotic function between the expression $\mathrm{x} + \mathrm{y} = \frac{1}{\mathrm{x}+\mathrm{y}}$ and the content "a circle"; while one of them (E_1) had achieved to establish such function, and therefore articulate the senses assigned to the two equations, as a group they focused on the "form" of the expressions in each equation, explaining that "in one of them the variables are squared and in the other are not."

[45] P Is the equation $x + y = \frac{1}{x+y}$ *a circle?* You had to say yes or no, you said no. I like to hear the arguments, What are the reasons why you say it's not a circle? Who would like to start? … Maria Elvira?

[46] E_1 No, well no, I wouldn't say it is a circle, because when I relate it, is when there is a…, when ehh… *x+y* is dividing or multiplying, I mean when there are two, because when adding twice then it would be 2*x*, then no, it wouldn't fit and when the two variables are squared, well, that's the principal thing to make it a circle…

[47] P

[48] E_2 And in the case of Daniel A.?

I say it is not a circle because if I pose this equation like that,
[49] E_2 $x + y = \frac{1}{x+y}$ then I don´t see it as a circle
… because one know that the circle as basic equation is ehhh …
[50] P when both variables are squared, right?
[51] E_2 Mmju … *[yes, I understand, continue]*.
… If you find that, you can make the procedure to… it is possible
that you can determine from this … I mean, I do, if I move the
[52] P values, change the values, each to another, it is possible that I find
the … the equation for the circle, but …
[53] E_2 But, initially what … what you look at to decide whether or not it is
[54] P [a circle] What is it?
[55] E_3 That, let's say, that the variables are squared.
… That they are squared … Daniel D., what do you say?
That Maria Elvira is right, if you see it like this *[points at the*
equation $x + y = \frac{1}{x+y}$ *],* you see it as a normal equation, but if you
[56] E_3 think about this *[indicates the denominator on the right side of the*
equality] on the other side are the squared variables, so like that you
don't see it as a circle.
But the main point of a circle is that the two variables are squared.

It is evident that in this group predominates the perception of the equation as an icon associated with the "circle," characterized by having squared variables ([46], [48], and [55]), about the proof of the syntactic equivalence initially made by E1 and later worked as a group. Basically it reflects a change in the interpretation of Daniel D. (E_3), while Daniel A. (E_2) maintains its initial interpretation.

[57] P Well, but the answer would be no, because they are not *[squared*
[58] E_3 *variables]*, or would it be yes?
It is *[a circle],* but you don't see it like that, I mean one does not
assimilate it just like that, because one don't, like in the mental process
[59] E_1 you don't go directly to multiply, *indicates the expression x+y in the*
denominator, to the right of the equation].
The thing is that when you see it just like that *[indicates the equation*
[60] PE_1 $x + y = \frac{1}{x+y}$ *],* I mean when you don't, don't see variables explicitly
[61] squared, then you don't start immediately to think what that is…
Aha *[yes, I understand, continue]*
Instead… I assimilate it, when I see it dividing the same variable, then
well … it is equalizer, I started to multiply right away, so that's how I …
I assimilate it to squared or the same when I have them expressed the
two of them … I mean the two variables multiply, then is when I do it
right away.

Daniel A. (E_2) insists that in the first instance he would say that the equation ($x + y = \frac{1}{x+y}$) is not a circle, because "from what is seen," it is not explicit what it is, since one have to perform transformations "to move [...] the parts of the equation," the variables, which he does not normally do, while if you give him an expression like $x^2+y^2+2xy-1=0$ " you know right away it's a circle." However, Daniel D. (E_3) in relation to the above said, confirms that for him it is a "circle", that the problem is that he treats it different because "don’t see this form that we were taught, as a circle has two square variables [...] on both sides of equality.” It means, as a group reaffirm the iconic look that they make of the equations, particularly of the expression associated with a circle, as a result of a misconception, possibly derived from an interpretation associated with a *classroom history*, different to the one institutionally intended by the teacher.

Cognitive Configuration of Primary Mathematical Objects

Below is presented the cognitive configuration finally achieved by Daniel D. (E_3), after the process of interaction during the group interview, conducted in relation to the work from the task on conic (Figure 9). In this diagram, by a solid line, are pointed the semiotic functions provided by the student, between expression and content. By a dashed line, is indicated the new semiotic function, evidenced during the interview, in the interaction with peers in the small group. In his individual work, Daniel D. had established two semiotic functions (Figure 8). During the interview he explicitly establishes a new semiotic function between the expression $x+y=\frac{1}{x+y}$ and the content "it is a circle", thus, achieving a link between the two senses assigned ([58]).

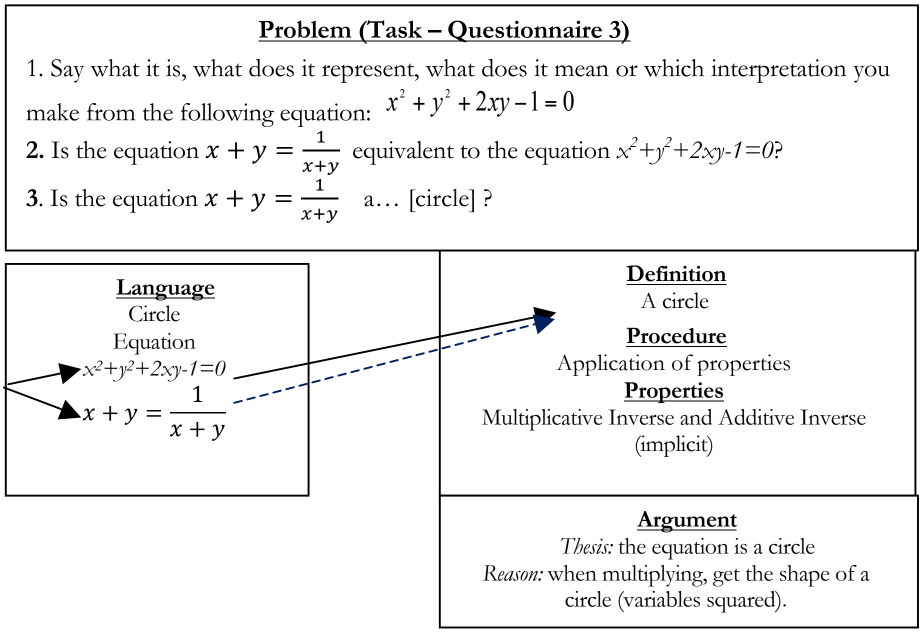

Figure 9. Final configuration of Daniel D. (E_3).

To summarize. As evidenced above, the individual work of each of the three grade 11th students, only one of them (Maria Elvira) could assign the same sense to the two equations. In the small group work, two changes were generated; on the one hand, Daniel A. accepted the syntactic equivalence between the two equations, as he acknowledged that by the changes in treatment he could obtain one equation from the other, and secondly, Maria Elvira welcomed arguments from her two mates regarding that, despite the performed transformations of treatment, the absence of one of them from squared variables, make it discarded the option of "view" as a "circle." Later, during the interview, the arguments given by each of them allowed them to share the sense initially assigned by Maria Elvira to the equation $x + y = \frac{1}{x+y}$, in addition to achieving an articulation of the different senses assigned to the two given equations. It is important to stand out, that for two of the students apparently continued prioritizing an iconic image of the circle, whereby both variables should be squared.

After the interview, and in relation to the recognition of the three students referred in this section of the syntactic equivalence between the equation $x + y = \frac{1}{x+y}$ and the equation $x^2+y^2+2xy-1=0$, as well as the recognition of the semantic equivalence, changes occur in the recognition of the syntactic equivalence and the articulation of the senses assigned to the equations. The data obtained is summarized in Table 4.

Table 4

Grillage Synthesis (Final) –Maria Elvira, Daniel A. and Daniel D.– Grade 11th, School CHA.

	Ma. Elvira (E_1)	Daniel A. (E_2)	Daniel D. (E_3)	Group
Recognizes syntactic equivalence	Yes	Yes	Yes	Yes
Articulates senses assigned to equations	Yes	Yes	Yes	Yes
Change (recognition of equivalence)		✓		
Change (articulation of senses)		✓	✓	✓

It is important to stand out that the process of interaction during the group interview led to changes in the interpretations made initially by the students; on the one hand, it made possible that one of the students (E_2), who by failing to focus his gaze on the form of the two given equations, recognizes the syntactic equivalence of these equations, and, second, that two of them (E_2 and E_3) achieved to articulate the senses assigned to them. Also, it made possible that E1 retake the interpretation she had done in her individual work, which had temporarily changed in the first work in small groups, to reaffirm the joint of the senses assigned to the two given equations.

RESULTS AND CONCLUSIONS

In the work done by students from 9th grade (last grade of basic education) and students from 11th grade (last year of pre-university education), regarding three specific tasks, it showed the difficulty that have many of them articulating several senses assigned to expressions associated with a mathematical object. Even though some of them recognize the syntactic equivalence obtained by treatment between two or more expressions, they are not always able to articulate the senses assigned to such expressions and may even change the initial sense assigned to one of them. The difficulties that students find to articulate the assigned senses to expressions can be grouped mainly into four groups, which are described below.

Iconic recognition of expressions. The sense assigned to the expressions in some cases is based on an iconic recognition thereof. For example, when considering that the "basic equation" of a circle is one in which the two variables are explicitly squared and are on one side of equality. In relation to the expression $(n-1) + n + (n+1)$, although several of the students were able to perform the treatment for getting the expression $3n$, they state that it is an addition and cannot be interpreted as triple a number. They suggest that each of these expressions incorporates procedures that differentiate them, although the second one is the result of the processing performed with the first. This reflects a cultural fact, the allocation of senses associated with each form of algebraic expressions[17]. Evidence is provided that these interpretations are entrenched, with some frequency, in school work.

It is important to stand out that similar investigations to those reported here have been conducted, informally, with university students taking courses related to training of mathematics teachers. For example, several students who were in fourth semester college degree in the area of math, recognized the equation $x^2+y^2+2xy-1=0$ as a "circle", but despite accepting the syntactic equivalence between the equation $x + y = \frac{1}{x+y}$ and the equation $x^2+y^2+2xy-1=0$, they did not recognize a "circle", because they did not "see" in the equation $x + y = \frac{1}{x+y}$ that the variables were squared.

Another case is that of a professor of secondary education, with university education in the area of mathematics and teaching experience of several years in grades 8th through 11th, who face the question: Can the expression $(n-1) + n + (n+1)$ may be, represent, or be interpreted as, three times a number?, she raised initially, and categorically, that there was not a number three times, because "the triple is $3n$, ... while the given expression is the sum of three consecutive numbers"; later, once she made transformations of treatment to the given expression: $(n-1) + n + (n+1) = n + n + n + 1 - 1 = 3n$, she thought for a few seconds and with a surprised expression said: *"This strikes me as strange, I never thought about the possibility that the sum of three consecutive numbers could be three times a number, ... I never thought so".*

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[17] Many teachers, in their courses, insist on such facts, as they consider that stress in the form of expressions or equations constitute a "help" for their students.

Anchor to given situations. It is evidenced a tendency to make interpretations linked almost exclusively with the proposed situation, i.e. it is evidenced some "anchor" to the given situation in the task, like in the proposed case to find the probability of rolling a dice to obtain an even number. We found a strong forwarding to the specific object "dice." Thereby is recognized a cognitive problem associated with the use of *concrete models* in building of objects of school mathematics. Even if the concrete material can provide an effective support for the mathematical intuition, in some cases it may be an obstacle. Even though it is just a model for the teacher, the student may be a learning object (Maier, 1998).

Interaction and changes in the interpretation. The interviews with small groups evidenced the importance of interaction spaces, as opportunity to meet other people's arguments, questions, ways to organize their ideas, that allows strengthening or modifying the initial interpretations. It is important to pin down that the interaction options, particularly the semiotic functions explicit by some, are not necessarily recognized or assumed by their peers, so it does not always produce changes in their interpretation, in assigning new senses or joint thereof. Even if the interventions and arguments from other peers of the small group can influence changes in the interpretation of some of the members, when the arguments are not clearly accepted, such changes can occur for short periods.

Mathematical language difficulties. There were some difficulties that students find regarding interpretations of the given expressions and performing of treatments of such expressions, particularly in the algebraic context. One of the difficulties is related to the generalization from particular cases[18] and difficulties processing transformations of treatment of the algebraic expressions.

To summarize. We present evidence that confirms the phenomenon reported by D'Amore (2006) on difficulties encountered by students to articulate senses associated with expressions recognized by them as syntactically equivalent and elements that allow making explicit for causes of this difficulty articulating the senses, associated to three fundamental facts. One, that although students "manage" the basic properties of number systems that enables them to make the transformations of treatment required establishing the syntactic equivalence of the expressions, they find it difficult to associate senses different from the given expressions. Two, the tendency to anchor in specific situations arising in the context of the proposed task and, three, the "look" basically iconic of algebraic expressions. Also, it highlights the importance of the interaction processes as a key element to enable the articulation of senses assigned to syntactically equivalent expressions. There is not only some time to socialize and recognize the arguments made by others but also, and especially, to analyze the arguments presented by each other, which are not assumed uncritically.

[18] In this regard, Radford (2008) discusses the problem of naive induction. He reports that, from particular cases, students tested with formulas until you find the right formula that allows them to calculate any term of a given sequence, abduction processes are explained but as "guessing". Therefore, he states a need to distinguish between algebraic generalizations and arithmetic generalizations.

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