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Relationships between area and perimeter: Beliefs of teachers and students

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Sunto. *In questa ricerca si esaminano le convinzioni di insegnanti e studenti a proposito delle relazioni esistenti tra perimetro ed area di una figura piana. La ricerca si inserisce in un classico filone, molto esplorato da oltre 60 anni, ma con molti fattori di novità. In particolare, si esamina la modifica delle convinzioni, il linguaggio usato per esprimerla, il grado di incidenza che hanno gli esempi forniti; si discute un'idea secondo la quale proprio le supposte relazioni tra perimetro ed area sono un esempio di comportamento in base al quale lo studente tende acriticamente a confermare maggiorazioni o minorazioni tra entità poste in relazione.*

Résumé. *Dans cette recherche on examine les convictions d'enseignants et étudiants au sujet des relations existantes entre périmètre et surface d'une figure plane. La recherche s'insère dans un filon classique, très exploré de plus que 60 ans, mais avec plusieurs facteurs de nouveauté. En particulier, on examine la modification des convictions, le langage utilisé pour la exprimer, le degré d'incidence que ont les exemples donnés; on discute une idée selon laquelle justement les présumées relations entre périmètre et surface sont un exemple de comportement sur la base duquel* *l'étudiant tend non critiquement à confirmer augmentations ou diminutions entre entités mises en relation.*

Resumen. *En esta investigación examinamos las convicciones de maestros y estudiantes en lo concerniente a las relaciones existentes entre perímetro y área de una figura plana. La investigación se inserta en una corriente clásica, explorada por más de 60 años, pero que hoy incluye nuevos factores. En particular, se estudia el cambio en las convicciones, el lenguaje utilizado para expresar dicho cambio, el grado de incidencia que tienen los ejemplos dados; y en particular discutimos la idea según la cual exactamente las supuestas relaciones entre perímetro y área constituyen un ejemplo de la actitud no crítica del estudiante que tiende a confirmar aumentos o disminuciones entre entidades puestas en relación.*

Summary. *In this paper we examine teachers' and students' beliefs connected to the relations that exist between the area and perimeter of a plane figure. The research joins, with many new features, a classic mainstream that has been explored considerably for over 60 years. In particular we examine the change of beliefs, the language used to express them and the degree of incidence of the examples we provide. We discuss an idea according to which the supposed relations between area and perimeter are an example of the student's behaviour that leads him to confirm, without criticism, increases and reductions between entities that are placed in relationship.*

1. INTRODUCTION AND THEORETICAL FRAMEWORK

The critical reflections on the problem of learning the concepts of perimeter and area of plane figures can boast of the fact of having been amongst the first to be studied. After concerning himself with the birth of thought and of language in infants, then with the acquisition – construction of the idea of number (with its various meanings), Piaget concerned himself, starting from the 1930's, with conceptual constructions having to do with Geometry. Amongst the various works which it would be possible to cite here, we limit ourselves to those in which perimeter and area specifically appear or where there are references to such concepts (Piaget, 1926; Piaget, 1937; Piaget, Inhelder, Szeminska, 1948; Piaget, Inhelder, 1962). In the 50's and 60's, these basic works were quickly followed by studies of pupils or followers of the school master from Geneva, based on the same certainties treated by genetic epistemology, for example, Vihn et al. (1964), Vihn, Lunzer (1965). We also mention Battro's study (1969) which repeats all of the celebrated experiments of the master.

These are the studies that have, for over 20 years, conditioned the successive analyses on the same theme. They were based, overall, on the failures of the young pupils at determined stages – ages. In particular, in this vein, the ideas of length and area, amongst others, were studied with great attention, highlighting the great difficulty on the part of the pupils to appropriate the idea of surface. Even more specifically, with the changing of the shape, the research highlighted how the young student tended not to be able to accept the invariance of the surface measurement. The difficulties tied to the false relationship between area and perimeter seem to continue up to the age of 12, according to this research, and they are even more connected to the linguistic development of the subject.

[It is well noted that Piaget's conclusions were subjected to severe criticism on the part of later scholars; so as not to make this work heavier, we refer back only to Resnick, Ford (1981, above all chap. 7)].

Following these preliminary and classic studies, abundant other research was done, so much so that it is impossible here and now to give the complete picture. We will limit ourselves (following a chronological path) only to those that, in some way, refer to the difficulty specific to the learning of the ideas of perimeter and area. These have, without doubt, conditioned the direction of our current research.

In Rogalski (1979) it is reported that one of the greatest problems in learning about surfaces is in the fact that there exist specific conceptual obstacles which reinforce one another.[1] Following are the most important difficulties: the changes in dimension, the specific statute of the units of measure, their relationships to the units of length and spatial measure.

In Gentner (1983), and with many cautions, the use of simple models is suggested for the first approaches to geometry in general and to the study of surfaces in particular.

The idea of the intuitive model is explained well in Fischbein (1985): «To create an intuitive support for intellectual research, for concepts and mental operations, we tend spontaneously to associate meaningful models from the intuitive point of view (…) *An intuitive model always has a pictorial - behavioural meaning and always induces effects of immediate acceptance.* (…)» (pp. 14-15). However, «Excessively insisting in supplying intuitive suggestions using artificial and too elaborate representations can cause more ill than good» (pg. 18).

A much more general discussion was proposed in Speranza (1987). Together with general considerations of extraordinary cultural interest, it is shown how the conceptual difficulties found in the primary school about questions connected to area and perimeter persist even amongst more evolved pupils, even up to the university. [We will see how true this is, thanks also to the present research.].

It is interesting the reflection proposed in Iacomella, Marchini (1990) which highlights how there is a contrast between direct (ex. with geoplanes, checkerwork, Pick's theorem) and indirect (ex. by the recourse to formulas, appealing to linear measurements) measurements of a surface and how this contrast can be a conceptual difficulty for its understanding.

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[1] In 1979 Brousseau's "theory of obstacles" was not completely diffuse (Brousseau, 1976-1983; 1986; 1989); therefore, the Authors used terms which today would be defined within this theoretical framework. It seems to us, however, that Rogalski's idea of "conceptual obstacle" can be compared to the "epistemological" one of Brousseau, but highlighting questions relative to difficulty in learning more than to facts connected with the history of mathematics.

In the interesting article by Tierney, Boyd, Davis (1990) one confronts the beliefs that primary school teachers have with respect to area. The evident fact is that it emphasises what follows: that such beliefs sometimes coincide with those of the pupils and how area is connected to the formulas for calculating it, more than to a general concept. In a certain sense, this work can be interpreted as the starting point of all those who investigate the beliefs of the teachers and therefore also of our present.

In Outhred, Mitchelmore (1992) there are some cases of children at the end of primary school who are able to carry out comparisons between the surfaces of rectangular figures, but are not able to pass from this experience to surface measurements. In general, the article is dedicated to specific difficulties in the conceptualisation of area and perimeter on the part of primary school pupils. Normally, in the activity of teaching it is taken for granted that, if a pupil learns to calculate the area of a rectangle, he is ready to learn how to measure the area of any geometric figure. Here is highlighted, on the contrary, how this is only an illusion.

An ample study, now considered by many researchers to be a classic, is that of Rouche (1992). In it is shown how the rectangle constitutes the most important departure point for the acquisition of the concept of surface, the crucial point, the sample figure, given that almost all the other figures which the pupil will know in the primary school are reduced to it and certainly the first ones (triangle, parallelogram, trapezoid,...). It also dwells on the fact that the determination of the area of a rectangle as the product of the measurements of two segments is an example of indirect measurement, difficult to accept and to construct conceptually.

There is an apparent contradiction between the last two pieces of research mentioned. However, it is not like that. In the first is shown that having learned how to handle a rectangle is not a sufficient condition for assuring mastery with other figures, as far as area is concerned. In the second is shown how, in any case, the preferred starting figure can be none other than the rectangle.

The research of Giovannoni (1996), where Piaget's famous experiments on the understanding of the concept of surface by 3-6 year old children are discussed and repeated, seems important to us. It vigorously shows that such a concept is not in and *of itself* beyond the reach of the children, as had been maintained in the past, but that this conquest depends on the surrounding conditions, overall referring to language and the proposal of specific adequate models (green sheets of paper as such and not green sheets of paper interpreted as pastures; surface area interpreted as such and not as pasture grass for cows). However, the possession of specific language has profound impact on the construction of such a concept. The use of an ambiguous adjective "large" is slowly and knowingly substituted with "extensive", bringing about notable success in learning even by 5 year old subjects.

In the works of Moreira Baltar, Comiti (1993-94) and Moreira, Baltar (1996-97) is highlighted the difficulty that the students of the last years of primary school encounter in recognising the measurements of a figure as one of the elements that establishes it and, in particular, in the first work, to separate the measurements of area and perimeter and, in the second, to acquire the idea of plane figure. In such works it is well highlighted how the aspects of the learning of different elements of the measure of geometric size are specific and diverse amongst themselves. The idea of the area of a plane figure is not always recognised as a characteristic of such a figure.

In Marchini (1999) he talks about the conflict between the two concepts and about the didactic way of confronting the subject to arrive at a solution. The article contains many considerations of great importance and strong impact; not only didactic, but also mathematical and epistemological.

In Medici (1999), the question of linguistic formulation of the texts of the problems in geometry, is discussed; if, that is, it is necessary to resort to a perhaps less precise language, but more accessible and one that does not use formulas excessively.

Another study of interest is that of Jaquet (2000); in which a problem proposed to 3rd and 4th year primary pupils, in the course of the Transalpine Mathematics Rally[2] in January and February 2000, was presented and very extensively discussed. That problem, original in its formulation, had to do with evaluating a comparison between the areas of non-standard figures, of which neither surface nor linear measurements were supplied. The approaches of many subjects were studied; showing the complexity of the processes used by the pupils, who mix direct and indirect methods evaluating areas and perimeters of the polygons that appeared in the figure. It is an interesting study which shows the complexity of the relationship between the two concepts.

A work which we have followed closely, also in its development is that of Chamorro (1997). The author there analyses 8 distinct aspects that determine the surroundings of learning for that which concerns measurement (in general), in agreement with the ideas of Guy Brousseau. They are: object support, size, particular value (or quantity of size), application measure, image measure, concrete measure, measurement, order of size. Chamorro's interesting research regards measurement in general and shows the complexity of such a theme, especially as concerns its learning. Amongst the specific examples that are made, there rightly appear perimeter and surface. «On the surface, because of the measurement produced, multiple conceptual obstacles converge. Amongst these, there is the relationship that the units of surface maintain with the units of length, being the first subsidiaries to the second as product measure. Such relationships can be understood only beginning from spatial relationships which, in their turn, must be coordinated with multiplicative relationships. The coordination between the linearity of each of the dimensions and the linearity of the surfaces must be able to be guaranteed through a geometric model that helps the visualising of such relationships».

Following the doctoral thesis of Chamorro, there was a long article that is a synthesis, but also a deepening of it, so much so, that we have translated and published it in its entirety in Italian, Chamorro (2001-02). Here analyses are done of the experiences

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[2] This has to do with a challenge between pupils from different countries on the basis of a test which a commission prepares. This competition has a fairly good diffusion, principally in Switzerland and Italy.

realised in the primary school with regards to the problem of the teaching-learning of measurement and in particular of perimeter and area. The aim of this study is to contribute to the realisation of masterly and well-aimed a-didactic situations and didactic engineering whose goal is that of eliminating or at least containing the well known learning difficulties.

A study by Montis, Mallocci, Polo (2003) confirms that which experience highlights, that is, that the young pupils between 6 and 8 years old identify the figure of greater expanse with that of greater length or with the higher one.

In the research done by Medici, Marchetti, Vighi, Zaccomer (2005) are highlighted the preconceptions and the spontaneous processes that the pupils between 9 and 11 years old (4th and 5th years of Italian primary school) put into play when they have to resolve problematic situations which concern area and perimeter. Turning to tests and interviews, the Authors insist on the fact that these two fundamental ideas constitute epistemological obstacles.

As can be seen, the scientific frame of reference, even within the limitations of content that we have proposed, is of extraordinary complexity and breadth.

2. RESEARCH PROBLEMS

It is evident, therefore, that the two geometric concepts: *perimeter / area of a plane figure*, have many common elements at the scientific level, but also many others that are simply supposed at the level of misconceptions; very diffuse amongst the students at every scholastic level.

The literature has amply shown (for example, see Stavy, Tirosh, 2001, and many of the above cited articles) how many students of every age are convinced that there is a close dependence relationship between the two concepts on the relational plane, of the type:

if A and B are two plane figures, then:

- if (perimeter of A > perimeter of B) then (area of A > area of B);
- likewise with <;
- likewise with = (for which: two isoperimetric figures are necessarily equi-extensive);

and vice versa, exchanging the order "perimeter – area" with "area – perimeter".

Rarely is this theme taken into didactic examination in an explicit way, also because of a supposed difficulty, according to the teachers.

We might wonder if on the teacher's part, at any scholastic level, there is full awareness of the theme or if, by chance, also for some teachers there are problems of conceptual construction. This obviously concerns the problems of the beliefs and conceptions of the teachers.[3]

The studies of the importance of the beliefs that the society, the common people, certain social groups, the teachers, the students have of mathematics, comprising also that which regards processes which go beyond teaching and learning have a quite a recent origin. Right from the start, these studies reveal the great importance that these considerations have for learning and teaching. Schoenfeld (1992) arrived at affirming that each individual conceptualises mathematics and places himself in the mathematical environment precisely on the basis of the beliefs that he has about mathematics. It is this belief that determines not only the modality of such a placement, but also the sensations that the individual feels after such a placement has been realised. From this one deduces the impossibility of separating knowledge (of mathematics) and beliefs (about mathematics) in the teachers (Fennema, Franke, 1992). Besides, this brings one to affirm, as an obvious consequence, that the decisions which the teachers take are determined by their beliefs, which explains the great importance that, for quite a while now, the research in the field of these beliefs has had (Thompson, 1992; Hoyles, 1992; Pehkonen, Törner, 1996; Krainer et al., 1998).

Interesting theoretical considerations on the structure of beliefs and on the current research on this theme can be found in Törner (2002).

On the other hand, it is universally recognised that beliefs form an important part of knowledge, given that they determine and condition it, as Schoenfeld (1983) had already noted more than twenty year ago.

It has been since the advent of interest in this sort of subject that a kind of analysis of the typologies of beliefs began. In Schoenfeld's work (1992), for example, distinction is made on the agent, and it is so that he distinguishes between beliefs

- of the student
- of the teacher
- of the society,

a distinction taken for granted, but not for this reason exempt from surprises and, in any case, the most followed precisely for its immediateness.

With regard to the third point, today we know that it is not possible to separate the analysis of an individual's beliefs from those of the social group to which he belongs, given that these are in every case the result of complex interactions between social

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[3] We deem it necessary to state explicitly that we use the following interpretations of such terms (also proposed at the opening of: D'Amore, Fandiño Pinilla, 2004), always, however, more diffuse and shared:

- *belief* (or credence): opinion, set of judgments/expectations, that which one thinks about something;
- the set of beliefs of someone (A) about something (T) gives the *conception* (K) of A relative to T; if A belongs to a social group (S) and shares with others belonging to S that set of convictions relative to T, then K is the conception of S relative to T.

Often, in place of "conception of A relative to T" one speaks of "the image that A has of T".

groups (Hoyles, 1992). Therefore, a study of this type must be placed in its social context.

A recent vast panoramic work on this theme, at least as regards the PME community, can be found in Llinares, Kraimer (2006).

In the years that go from the beginning of the study of beliefs (early 80's) to today, the methodologies for doing research on such beliefs have changed. At the origins, it had to do exclusively with individual typologies to which the subjects analyses belonged. Today it has as its aim that of finding a relationship between the beliefs of the subject analyses and his action in the classroom. We had precisely this last possibility that we followed as our research methodology, in particular in the case of teachers. The methodology followed was, therefore, an interview divided into two phases. The first, in which, with its management "as a mirror", was done in such a way that the subject manifested his own beliefs about the theme, the object under study. The second, of a "reflective" type, in which the subject is asked to compare his own beliefs with his life in the classroom.

This type of analytical methodology was introduced by Skott (1999) and successively re-elaborated in analogous situations (Beswick, 2004).

Once the beliefs are analysed, it is necessary to analyse the change of beliefs, after a specific event, for example, as we will see in our case, *after* the discussion with the researcher about the relationships between area and perimeter. This one is obtained by having the subject compare his same declarations, before and after.

To obtain the objective it is necessary that the subjects make their own beliefs clear before the research and these can be obtained either by turning to written declarations (D'Amore, Fandiño Pinilla, 2005) or by recorded individual interviews, or by collective interviews with 2 or 3 subjects, in such a way that each one can then testify to the affirmations of the others. We used all three methodologies according to the circumstances.

Besides, another factor, highlighted by Azhari (1998), is in play. We will try to say it quickly: if there are two relationships with some reciprocal ties, the student tries to apply the following "law of conservation":

- if the one thing grows, this other one related to it also grows (and vice versa).

Now, the example that connects perimeter and area to each other seems to fit well for Azhari's considerations (1998) (better still, this is exactly one of the examples offered in this work, cited by Stavy, Tirosh, 2001).

If we place the perimeters of two figures A and B into relationship, with their respective areas, it seems to us that a convincing way to highlight that the "laws" mentioned above are NOT valid is:

to show an example for each of the following 9 possible cases:

p S	**p S**	**p S**

> >	> =	> <
= >	= =	= <
< >	< =	< <

The first box **> >** says:

- find two figures such that, passing from the first to the second, the perimeter grows and the area grows

and so on.

To avoid difficulties, one can always give simple figures, such as a rectangle, when it is possible, carrying out the various transformations on it or on figures derived from it. It seems necessary to us to make sure that the figures treated are the most elementary possible, that is, the most usual figures that are found in textbooks and in classrooms, to avoid complications owing to the figure itself.

In the Appendix, the 9 examples as above are given in extremely elementary cases. These examples are never supplied firstly to the subjects who submit to the test that will be described later. Each of the subjects, on his own, must take care of finding appropriate examples, at least in the first instance.

3. QUESTIONS, RESEARCH METHODOLOGY AND HYPOTHESES OF ANSWER[4]

To a group of collaborators,[5] primary school teachers, middle school, upper school and university, we proposed, making themselves responsible for the above mentioned research, giving the following indications which are contemporaneously, the explicit demands of the research, the relative methodological indications and our hypotheses of response, subdivided into 3 points. We decided to do the research at all levels of instruction to verify if the results could have had to do, in a specific way, with a scholastic level or instead if the results could be encountered independent of the "scholastic level" variable.

POINT 1.

Research PROBLEM R1: We asked all the collaborators to put themselves to the test, in complete sincerity, and some of their colleagues in the primary, middle, upper

[4] Contrary to our usual practice, in this work we do not separate these three points because this time they are profoundly connected to each other.

[5] For "collaborators" we mean as follows: in Italy there are research groups in the didactics of mathematics recognised by the Ministry of the University and of Research at the different university seats. The Bologna group is called RSDDM (www.dm.unibo.it/rsddm). As part of these groups there are also teachers at every scholastic level who are officially called "research collaborators". In the group of collaborators, there are also university teachers or others at the same level.

schools, as well as university students in teacher training. The problem consists in checking, first in the collaborators themselves and then in the other subjects of the trial, if a change of beliefs happens relative to the relationship between area and perimeter.

Research QUESTION Q1: Is it true or false that one can find examples for all 9 cases? Is it true or false that it is spontaneous to think that at the increasing of the perimeter of a plane figure the area of it is increased, in general? Is it true or false that it is necessary to turn to the cognitive and to ones own experience, to *convince oneself* that things are not like this?

Answer HYPOTHESES H1: We believed that not only for many students, but also for some teachers and some collaborators there were deep-rooted misconceptions with regards to supposed necessary relationships between perimeters and areas of plane figures. That it was not so trivial to find the 9 mentioned examples (especially in the case in which the perimeter must decrease and the surface increase and vice versa). That even after having seen these examples, there was some resistance. As indicators of these deep-rooted misconceptions we thought we would take on the declarations of the same collaborators, of the teachers interviewed and of the university students.

POINT 2.

Research PROBLEM R2: We asked all of the collaborators to do some tests on the primary, middle, upper and university students (from every kind of faculty) and not only students in training as future teachers. Each of these was invited to introduce, in a conversational oral form[6], any discussion they wanted on the perimeter and area of a simple plane figure and try to carry out the transformations, verifying if the students

- accept spontaneously
- accept willingly after an example
- accept with difficulty after several examples
- …
- reject without discussion
- reject even after examples
- …

that all 9 relationships can be valid and that NOTHING can be said a priori about the ties between "increase (equality, decrease) of the perimeter" and "increase (equality, decrease) of the area of plane figures". Each collaborator had to speak to the students, as the first thing proposing the problem, listening to the first answer, making a note of it according to the said scale. As the second, proposing the 9 tests and assisting the students in their execution, listening to their comments. As the third, reaching an explicit formulation of his new belief, in case there is one.

[6] This introduction to the theme was the proposal of the so-called "Galilean problem of town squares" that we will see shortly.

We were interested in observing two things:

a) the *change of beliefs*; if, that is, after some examples, the students are willing to change belief, according to the hypotheses of the theoretical framework presented before, and if age has an influence on this; it thus became essential to have the subjects express their beliefs *before* and *after* the examples. To reach this aim, more than just doing the tests, it became essential to interview the subjects in small groups (2-3 per group) or individually;

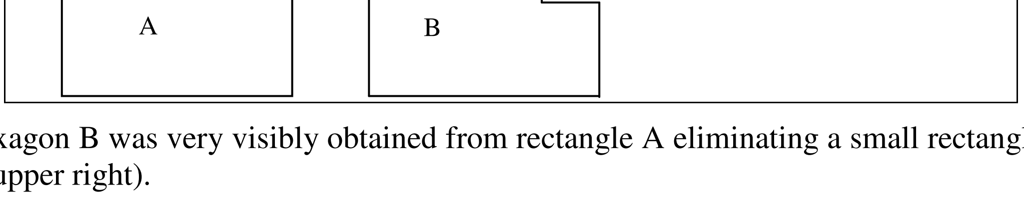
b) the language that the students use to explain their thought, before and after: examples, general discussions, sentences, …, use of drawings, of diagrams, …

Research QUESTION Q2: With how much naturalness and spontaneity do the students manage to accept that there do not exist obligatory relationships between the perimeter and area of plane figures? How does this acceptance vary with age? Does it turn out to be easy to accept the 9 examples? How do they express their beliefs? What kind of language do they use?

Answer HYPOTHESES H2: We believed that the students, at any age, would express great difficulty in accepting that which seemed anti-intuitive. That is, we thought that more than one student was convinced, before the test, that at the increase of the perimeter there *necessarily* corresponded an increase of the area, for example, and that the more he considered this was obvious and intuitive, the more he would have stated his own effort to accept the result of the test itself. On the basis of our research experience, we considered that with the increase in age this acceptance has a net rise. That the subjects would find some difficulty in accepting the examples. That they would have expressed their beliefs in a minimally academic way, given that they contrast with the scholastically built ones. That the language used would have tended to be the most colloquial possible, perhaps with the spontaneous use of graphics and schematic drawings.

POINT 3.

Research PROBLEM R3: We invited the collaborators to have different students, who were not part of the preceding test, undergo the following, during individual interviews. They had to give a card which contained the following two figures to the new subjects:



(Hexagon B was very visibly obtained from rectangle A eliminating a small rectangle in the upper right).

Now, to half of the students the following two questions were posed:

q1: Is the surface of A smaller than, equal to or greater than the surface of B?

Is the perimeter of A smaller than, equal to or greater than the perimeter of B?

To the other half, instead, they had to pose the following two questions:

q2: Is the perimeter of A smaller than, equal to or greater than the perimeter of B?

Is the area of A smaller than, equal to or greater than the area of B?

Research QUESTION Q3: Can the inverted order of the questions which characterised q1 and q2 radically modify the answers of the students? The pertinence of this question is described in the following lines.

Answer HYPOTHESIS H3: Our hypothesis was that:

- in q1 the students would have easily verified that the area of A is greater than that of B (because graphically it appears very evident) and they would have tended to say, without verification, that the perimeter of A is greater than the perimeter of B. The collaborators had only to verify if this tendency really existed;
- in q2 the students should have been embarrassed by the first question on the perimeter, which they should have verified with attention because it is NOT immediately evident. Once verified that the perimeter of A is equal to that of B, still, they should not have had problems saying that the area of A is greater than that of B. The collaborators had to make sure the students verified that the two perimeters were equal and said it, after which being careful of what they should have answered with regards to the areas.

If things had really worked in this way, we would have contradicted Azhari (1998) (cited and at least partially accepted by Stavy e Tirosh, 2001) on the basis of the evidence of the figures. Their supposed "law of conservation" would then have no value, but everything would be attributed to a fact tied to misconceptions and perceptive evidence.

In all, we had 14 *collaborators*:

7 primary school teachers
2 middle school teachers
3 upper school teachers
2 teachers from the university (or equivalent).

Each of these put himself and some colleagues to the test. In all, the number of *teachers* who underwent the test were:

26 from the primary school
16 from the middle school

13 from the upper school
2 teachers from the university,

for a total of 57 teachers.

The numbers of *students* who underwent test 2 were the following:

29 from the primary school (all from the 5^{th} year)
20 from the middle school (6 from the 1^{st} year and 14 from the 3^{rd})
21 from the upper school (8 from the first two years of the Scientific High School, 9 from the 4^{th} or 5^{th} years of Scientific High School, 4 from Professional Institutes)
13 from the university or analogous level (4 from the degree course in the Science of Education, 1 from the 3^{rd} year of the degree course in Mathematics, 8 from the Pedagogical High School)
for a total of 83 students.

The numbers of *students* who underwent test 3 were the following:

50 from the primary school (all from the 5^{th} year)
26 from the middle school (12 from the 1^{st} and 14 from the 3^{rd})
14 from the upper school (4 from the first two years of the Scientific High School, 5 from a Professional Institute, 5 from the 3^{rd}, 4^{th} or 5^{th} of a Scientific High School)
17 from the university or similar (4 from the degree course in the Science of Education, 12 from the Pedagogical High School, 1 from the 3rd year of the degree course in Philosophy)
for a total of 107 students.

It should be remembered that all the tests were carried out in the form of individual interviews, of a clinical kind, using the grid described in paragraph 3 of point 2. On the other hand, the interview, often in the cases in which the interviewee was one of the collaborators, was often a kind of *personal story*, given that we tried to have answers which revealed the changes of conscious beliefs. In this research we have made broad use of the written statements of the teachers involved. With various names, this technique has been profitably used a lot in an international context for a while. Proof of this is the precursory work of Gudmundsdottir (1996), in which is used the *metaphor of the iceberg* to illustrate how the emerging point corresponds to that which is declared as a teacher's (explicit) response to a question in the course of an interview, while the larger (implicit) part is that which is hidden under the water. It emerges thanks only to a personal narration.

The research described by Edwards, Hensien (1999) is particularly interesting to us given that, there, a group of teachers (from the primary and secondary school) were involved in research – common action aimed at discussing didactic action in the classroom, is analysed. Well, the teachers expressed themselves precisely by means of narration of that which happens and of the sensations experienced during that action.

In Gudmundsdottir, Flem (2000) is discussed, always using these 'narrative' techniques, how life in the classroom has changed in recent decades, what are the sensations and sentiments of the teachers with regard to this, while in Gudmundsdottir (2001)is presented the narration of a teaching experience in a school for children from 5 to 8 years old.

In Strehele et al. (2001) the technique is used to study the integration of technologies in didactic practice, while in Raths (2001) the beliefs about the teacher and teaching are analysed, also in view of the decision to modify personal teaching strategies. Also in Presmeg (2002) the focus is on the use of the autobiography to bring out the personal beliefs about mathematics and the relative changes with the passing of time.

The use of research, in Llinares, Sánchez García (2002), that is done on the written texts of the subjects analysed, secondary school teachers in training, is very interesting for determining their suppositions about mathematics, its teaching and learning and on the meaning of the scholastic tasks, in this direction.

Finally, we cite the work of D'Amore, Fandiño Pinilla (2005) in which the subjects analysed had to express themselves precisely by means of an autobiographical letter.

4. RESEARCH RESULTS, DISCUSION OF THE RESULTS AND ANSWERS TO THE RESEARCH QUESTIONS

4.1. Teachers to the test on perimeter and area

4.1.1.

As regards point 1., research problem R1, we made a distinction between the two handouts:

- we asked all the collaborators to put themselves to the test

and

- some colleagues from the primary, middle and upper schools, as well as students of university courses (specialisation courses in Italy and Master's courses in Switzerland) for the training of secondary school teachers (lower and upper).

In both cases the research questions Q1 were the following: Is it true or false that one can find examples for all 9 cases? Is it true or false that it is spontaneous to think that at the increase of the perimeter of a plane figure, the area of it increases, in general? Is it true or false that it is necessary to strain to *convince oneself* that things are not like this?,

while our answer hypotheses H1 were: we believed that on the part of some teachers (including some collaborators) there were deep-rooted misconceptions about supposed necessary relationships between perimeters and areas of plane figures. That it wasn't so banal to find the said 9 examples [especially in the case of (p<, S>) in which the perimeter must decrease and the surface increase]. That even after having seen the examples, there was some resistance.

In this paragraph **4.1.1.** we will examine the case in which the subjects who underwent the (self) test were our same collaborators in the research, while we put off to paragraph **4.1.2.** the case in which the subjects that underwent the test were colleagues of our research collaborators or university students from the faculties mentioned previously.

We have rather similar reactions from the 14 research collaborators as regards the modality of response:

- 1 subject (university teacher) limited himself to carrying out an exclusively mathematical analysis of the question, obviously correct, not responding to the personal question about his own difficulties;
- 13 wrote texts that go from 1 to 6 pages in response, sometimes rather rich with references to their own difficulties:
    - 9 collaborators (7 primary teachers, 1 upper, 1 university) confess their own difficulty at the moment of having to give form to their ideas, even if correct and conscious. They also admit that they had to force themselves to imagine all 9 situations;
    - 4 collaborators (2 middle teachers, 2 upper) stated that they had no problem immediately finding the answers and above all they stated their full awareness that the things had to work in that way.

[4 collaborators (2 primary, 2 middle) make full reference to their own pupils, not managing to answer in the first person only as subjects, but interpreting our question as an implicit invitation to think of a classroom situation].

The case that was declared almost unanimously as the most difficult was exactly that one (p<, S>) which we had supposed and its analogous one (p>, S<).

Our hypotheses H1 are therefore fully confirmed; even by people with a high level of education, like our collaborators, there are, at least at first sight, deep-rooted misconceptions about the supposed necessary relationships between perimeters and areas of plane figures. As indicators of such misconceptions, we had decided to take on either their own explicit admissions or the evident proof of their difficulties. For many, it wasn’t so banal to find the 9 examples mentioned [especially in the cases (p<, S>) and, a bit less, (p>, S<)], by their explicit admission. One of the collaborators stated explicitly in writing «(…) I had greater difficulties finding figures in the cases where the perimeter had to decrease and the area had to remain the same or increase», a sentence that we take as a prototype for many others of the same tenor.

One can well see how the self-declarations of difficulty are more numerous amongst the teachers of the first scholastic levels, perhaps because of the lower technical preparation (reported by more than one; many primary school teacher collaborators confess to having learned how to critically treat these questions within the framework of courses organised by the NRD of Bologna).

The choice of the figures for the 9 cases is more numerous, at least at the beginning, around convex polygons and specifically rectangles.[7]

4.1.2.

The 43 teachers interviewed (19 from the primary school, 8 from the middle, 10 from the upper, 6 in post graduate training as lower secondary school teachers) had very dissimilar behaviours, but also many reactions in common. The protocols of the interviews are available; here we will pick only the essential. We will report between «» the sentences that confirm our affirmations and that seem most representative.

A very diffuse reaction, at all scholastic levels, is the difference shown at the intuitive level upon the first contact with the problem as regards the change (sometimes strong) between the first intuitive response and the belief acquired at the end of the test.

As we said at the beginning, almost every interview began with the so-called "problem of Galileo": «A town has two squares A and B; the perimeter of square A is greater than the perimeter of square B; which of the two squares has the greater area?».[8]

Very many of those interviewed, decidedly the great majority, 40 out of 43, even university graduates and upper school teachers, affirmed that the square that has the greater area is that with the greater perimeter, except for then:

- spontaneously correcting oneself, affirming that "it isn't necessarily so", even before carrying out all the tests foreseen in the interview (and here one notes a greater gathering amongst the upper school teachers)

or

- accepting that their own answer might be criticisable or incorrect, but only after having carried out the tests (and here one notes a greater gathering amongst the teachers at the first scholastic levels).

Therefore, the *change of belief* is obvious, sometimes strong, and in many cases requires proofs and not insignificant reflection.

To the questions: «Is it true or false that one can find examples for all 9 cases? Is it true or false that it is spontaneous to think that at the increase of the perimeter of a plane figure, the area of it increases, in general? Is it true or false that it is necessary to strain to *convince oneself* that things are not like this?»,

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[7] A note, only as a curiosity, outside of the research. One of the collaborators stated having put some of his family members to the test:

- those involved in construction activities, who daily confront situations in which the cases (p>, S<) and (p<, S>) are recurrent, did not have problems, not only responding correctly, but also supplying examples;
- others, involved in more routine activities, showed that they tended to give the expected classical answers: there exist only the cases (p>, S>), (p<, S<), (p=, S=); the other cases are believed to be impossible: for example, it was not believed possible to find examples for the case (p<, S>).

[8] About the exact history of this problem as it is encountered really in Galileo, one can see D'Amore, Fandiño Pinilla (2006).

many of the teachers, and NOT necessarily only of the primary school, begin with a 'no' answer, which reveals that the deep-rooted misconceptions with regard to supposed necessary relationships between perimeters and areas of plane figures do not lie only with *some* teachers, as we believed, but with most of them.

For many of those interviewed finding the 9 mentioned examples was not the slightest bit trivial [especially in the case (p<, S>) or vice versa). We had many cases of teachers (even upper school and middle school) who found it necessary to take recourse to the (or some of the) examples supplied by the interviewer. [Many noted the symmetry of the requests and some showed intolerance in the case p=, S= for not simply wanting to apply an isometry or to leave the identical figures].

That which one infers, however, is that, after having seen the examples, either created by the person interviewed himself or proposed by the interviewer, (almost) all the misconceptions connected to intuition disappeared. One arrives at sentences full of awareness, such as the following: «Therefore, two equally extensive figures are not automatically also isoperimetric» [this perfect enunciation, was done with obvious surprise by a primary school teacher who stated having struggled a lot with himself to find the 9 examples, stopped by his own beliefs about it, a deep-rooted misconception which before he had never accounted for, that at the increase of the perimeter it might be necessary that the area increases also].

It appears very clear that the misconceptions revealed are due to the fact that almost all the figurative models that accompany these questions are realised with quite usual convex plane figures, which drive one to believe that it is possible to confront the problem ONLY with such figures. Better still, this consideration is confirmed by more than one of the same people interviewed: «It is possible beginning from a square; it is not possible beginning from a circle» (in other words, the square is considered an admissible figure for transformations such as those proposed by us, the circle no); to the proposal of a concave figure: «But this is not a geometric figure» [meaning to say: not of those usually used in didactic practice when one speaks of area and perimeter]; others consider possible only homothety, so: «…but with squares it is impossible» given that the homothetic of a square is still a square.

Very recurrent is the cross-reference that the teachers interviewed made to their own pupils; many of the questions and answers were in fact "filtered" through the experience with or of their own pupils: «They also do no see it» [that which I did not see]; «…they find it hard to imagine it»; «It is necessary to change many figures» [that is, pass from one standard figure to another, for example concave; in reality, it would not always be necessary, but the examples supplied by the interviewers (see Appendix) often are considered as unique].

The fact is interesting that some secondary school teachers (lower and upper) consider this kind of question to be closer to the world of the primary school, «because there one works with figures, more on the concrete, less on the abstract», almost to justify their own failure (and the potential failure of their pupils) in the task. Naturally, there is much truth in this; in the primary school, all to often images that should remain only partial become transformed into deep-rooted models. Often there is not even an awareness of the problem.

We will see in the next paragraphs **4.2.1.** and **4.2.2.** the progress of the research with the students and here we venture the hypothesis, that we will analyse in **5.**, that the obstacle which will seem evident with respect to the construction of a satisfying mathematical awareness on the relationships between "perimeter and area" is not only of an epistemological nature, but *rather much more* of a didactic nature.

The epistemological nature is obvious and has multiple aspects:

a) it is not by chance that stories and legends which connect area and perimeter are extremely old and are repeated in time, even at the distance of centuries (suffice it to think of the myth of the foundation of Carthage on the part of Dido and the celebrated riddle of Galileo). This is a sign, not more than a sign, obviously, of an epistemological obstacle; on the other hand, when a mathematical idea does not enter immediately as a part of universally accepted mathematics and is, on the contrary, the cause of arguments, contrasts, fights it can generally be considered as an epistemological obstacle in Brousseau's sense (1976-1983; 1986, 1989);

b) to complete these analyses, geometric transformations must be done on the figures; well, only at the end of the 19th century were these transformations, their power, their necessity, revealed to the eyes of the mathematicians. For millennia the staticity of the *Elements* of Euclid dominated. Even this delay in the introduction-acceptance is an obvious sign of an epistemological obstacle.

On the other hand, however, to these obvious epistemological obstacles are also grafted didactic obstacles. If rather profound, appropriate interviews were necessary to change the beliefs of the teachers themselves, how can one not think that their didactic choices used in the classroom with their own pupils don't influence the formation of misconceptions relative to this strategic theme?

4.2. Students to the test on area and perimeter

4.2.1.

We recall that at point 2., as research problem R2, we had asked all the collaborators to interview students from the primary, middle and upper school and university students of the different faculties. Each collaborator had to speak with the students, for the first thing to propose the "Galileo" problem, listen to the first answer, make note of it according to the scale mentioned. For the second thing, to propose the 9 tests and help the student during their executions, listening to his comment. For the third thing, to arrive at an explicit formulation of the new belief, in the case there is one.

Each of these was invited to introduce a discourse of any kind on perimeter and area of simple plane figures and to try to carry out the transformations, verifying if the students

- accept spontaneously;

- accept willingly after an example;
- accept with difficulty after several examples;
- …;
- reject without discussion;
- reject even after examples;
- …

that all 9 relationships can be valid and that NOTHING can be said a priori about the ties between "increase (equality, decrease) of the perimeter" and "increase (equality, decrease) of the area of plane figures".

Finding out two things interested us:

a) the *change of beliefs*; if that is, after some examples, the students are willing to change ideas and if age has an influence on this; it thus became essential to have the subjects express their beliefs *before* and *after* the examples. To reach this aim, more than just doing the tests, it became essential to interview the subjects in small groups (2-3 per group) or individually, according to the methodology explained previously;

b) the language that the students use to explain their thought, before and after: examples, general discussions, sentences, …, use of drawings, of diagrams, …

To this aim, the research questions Q2 were the following:

With how much naturalness and spontaneity do the students manage to accept that there do not exist obligatory relationships between the perimeter and area of plane figures? How does this acceptance vary with age? Does it turn out to be easy to accept the 9 examples? How do they express their beliefs? What kind of language do they use?

As a preliminary hypothesis, we believed that the students, at any age, would express great difficulty in accepting that which seems anti-intuitive. That with the increase in age, this acceptance would show a net increase. That the subjects would find some difficulty in accepting the examples. That they would have expressed their beliefs in a very non-academic way, given that they contrast with the scholastically constructed beliefs. That the language used would have been held to be the most colloquial possible, perhaps with the spontaneous use of graphics and diagrams.

The most sensational result of the research is tied to the fact that the most complex cases (p>, S<; vice versa; p>, S=; vice versa) were not widely accepted spontaneously with the increase in age and not even of the scholastic level.

More than 90% of the students interviewed, independently of their scholastic level, tended spontaneously to affirm that there is a close dependence between the increase/decrease of the perimeter and the increase/decrease of the area;

confronted with the task of supplying examples, the difficulties were concentrated overall in the cases mentioned now;

only a few succeed in this task and the positive result is not correlated to the age (therefore to the scholastic grade); amongst the university students there were some of the most sensational negative results.

Once shown, on the part of the researcher, that the 9 cases which exhaust all of the possibilities are truly possible, there are the following reactions:

- more than half of the students[9] showed surprise at the use of the concave figures. Someone added saying that «These are not geometric figures», that «They are not correct», that «At school they aren't used»,…; this behaviour is not relatable in a meaningful way to the age and therefore to the scholastic level or to the kind of school attended;
- more than half of the students understood the sense of the proposal and admitted to having undergone a change of beliefs. Also this behaviour, slightly superior at high levels of school attendance, is not however statistically connected to the age;
- in unsuccessful cases, the student often entrenches himself behind justifications due to the lack of development of this subject on the part of the teacher. This fact is much more present in the middle school. Numerous students of the upper school exhibit having understood well the *sense* of the research and reveal interest and motivation in giving the answers. Some recognise their own personal difficulty in completing appropriate transformations of the figures;
- it is interesting to see how some of the upper school students made themselves responsible for the problem without unloading on the teachers of the preceding levels the responsibility (on the contrary, we had teachers with university degrees who blamed their lackings and difficulties on their university studies, in which, they charge, this kind of subject was ignored; or on the textbooks for similar reasons).

Returning to the change of beliefs, in several cases whoever did this, did so in a surprised way, as if he unhinged an awareness given as already acquired.

Of the 13 university students interviewed (only one of them from the degree course in Mathematics) less than half of them stated spontaneously that the 9 cases were all possible; independently from knowing how to find them;
of the others, those who needed to do the tests, only half stated in a convincing way, at the end, to have changed belief; of these, some did so in a very explicit way;
many maintained that the misunderstanding in thinking that the increase of the perimeter brought about the increase of the area derived from bad didactics and they intended to keep account of it during their professional future, better still starting already from their training.
The acceptance is not always easy: «For me it was hard to accept it. I was convinced that they depended on each other. It's quite a surprise that I have to digest. It's hard».

[9] This is not meant to be a quantitative evaluation, given that we are in full qualitative field (as inferred by the fact that we cite sentences from those interviewed more than make statistical findings). We only want to note the dimension of the phenomenon.

As regards the language, enormous recourse was made to natural language, to terminological confusion (for example, even though one spoke explicitly about perimeter and area, many students, from the primary school to the upper, said "perimeter" in place of "area" and vice versa), to inadequate expressions from a lexical point of view.

Note that the recourse to a colloquial language of low formal or at least cultural, in a mathematical environment, profile is NOT a peculiar thing in the first levels of school attendance. There are, in fact, some university students, those who surprise us even more with adjectives and expressions that are not very consonant with official geometry: «If you make a little thin [figure]…», «If I make something very sharp-cornered, the perimeter…» etc.[10]

Many interviewee tried to resort to explicative drawings that illustrate, confirm, prove to be wrong their own thought. The result, however, is very disappointing. Very few students, with any distinction of scholastic level, truly know how to use a drawing to validate or negate their own assertions; they but they do not master this specific graphical language.

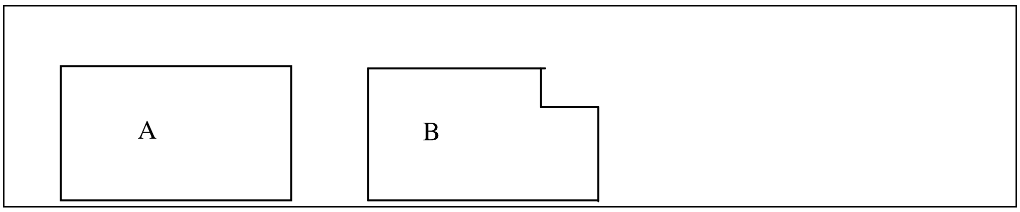
It is interesting to note how pupils of the 5th year of primary school in various Italian areas, who spontaneously respond well to question I (if that is, having a rectangle and a square of equal perimeter, there is necessarily equal area), state 'no' because the square is «more spacious», «it has more space inside», «it is bigger»,…

It is obvious that amongst the isoperimetric quadrilaterals, the square is the one with greater surface area and this fact is imagined, seen, known instinctively from the graphic point of view, more in the primary school than later. Naturally, there is no lack of cases of upper school students who show competence in these themes. In the 5th year, for example, we had cases of students who knew and mastered the relationships between surfaces of isoperimetric figures.

4.2.2.

Coming to point 3 of the research, we invited the collaborators to submit different students, who were not submitted to the preceding test, the following, during individual interviews.

They had to give these new subjects a card containing the following two figures:



[10] On this linguistic behaviour of the students we will have to take up the research again in a more specific form, given that it surprised us. Therefore, we do not deal with it here in great detail.

(Hexagon B was very obviously obtained from rectangle A eliminating a small rectangle in the upper right).

Now, the following two questions were posed to half of the students:

q1: Is the surface of A smaller than, equal to or greater than the surface of B?

Is the perimeter of A smaller than, equal to or greater than the perimeter of B?

To the other half, instead, the following two questions had to be posed:

q2: Is the perimeter of A smaller than, equal to or greater than the perimeter of B?

Is the area of A smaller than, equal to or greater than the area of B?

We had a single research question Q3: Can the inverted order of the questions which characterise q1 and q2 radically change the students' answers?

Our hypothesis H3 was that:

- in q1 the students would have easily verified that the area of A is greater than that of B (because it is graphically very evident) and they would then have tended to say, without verification, that the perimeter of A is greater than the perimeter of B. The collaborators only had to verify if this tendency really existed;
- in q2 the students would have been embarrassed by the first question on the perimeter, which they should have checked attentively because it is NOT immediately evident. Once verified that the perimeter of A is equal to that of B, they still should not have had problems saying that the area of A is greater than that of B. The collaborators had to make sure that the student verified that the two perimeters were equal and said it, after which being careful about what he would answer with regards to the areas.

If things had really worked like this, we would have contradicted Azhari's hypothesis (1998) (at least partially accepted by Stavy and Tirosh, 2001) on the basis of the evidence of the figures. Their supposed "law of conservation" would no longer have value, but everything would be taken back to a fact tied to misconceptions and perceived evidence.

The results of the tests done demonstrate our hypothesis, in an absolutely incontrovertible way. The order of the questions is fundamental in the answers but, in this case, the age (and therefore the scholastic level) effects it in a very statistically important way.

The correct answer to the first part of question q1 (the surface of A > the surface of B) was given spontaneously and immediately by all of those interviewed; of these 90-91% erroneously concluded, without reflecting, that, therefore, also: the perimeter of A > the perimeter of B;

the correct answer to the first part of question q2 (the perimeter of A = to the perimeter of B) was given spontaneously, without need of reflection, in very few cases, even at the high scholastic levels; once the collaborators pushed them to reflect on this answer, many of the interviewees, about 84-85% of the cases, acknowledged the equality of the perimeters.

The answers mistaken in the case of the second part of question q2, therefore, are not tied to the supposed "law of conservation", but to misconceptions tied to that which came out in the preceding paragraph and to perceived evidence which, in the case of area, is immediate and which in the case of perimeter it is not.

Azhari's hypothesis here is disproved.

The problems that are encountered are of different types and only in part expected:

- some students confuse area and perimeter in their terminology. This brings about the unacceptability of the statements of the one interviewed on the part of the researcher;
- difficulty in accepting comparisons between the two figures because one of the two is an unusual one, not considered amongst those to which the school has dedicated formulas;
- when one student, often at the first scholastic levels, tries to measure the perimeters, he does not always know what to do. It should be noted that, to answer the questions, it was not at all necessary to measure anything. Measurement was tried in the cases in which the subject interviewed held it necessary (which happened in more cases than expected and not only in the primary and middle schools; various students of the secondary school made use of quadrilaterals always specifying the size of their sides and calculating the area and perimeter of them).

5. CONCLUSIONS AND DIDACTIC NOTES

Seeing the progression of the research with the students, the obstacle which presents itself to the construction of a sufficient knowledge of the relationships between "perimeter and area" is not only epistemological, as is stated in many previous works on this field of research, but rather also of a didactic nature.

It therefore rests in the didactic choices:

- one always uses only convex figures causing the misconception that concave figures *cannot be used* or that using them is unacceptable;
- one always uses only standard figures, causing the misconception that is often expressed with the sentence: «But this is not a geometric figure»;
- almost never are the area and perimeter of the same figure explicitly placed in relationship. In fact, sometimes one insists on the fact that the perimeter is measured in metres (m) while the area is measured in square metres (m^2), insisting on the differences and never on the reciprocal relationships;
- almost never are transformations done on the figures in such a way to preserve or modify the area and perimeter; creating a misconception about the meaning of the

term "transformation". Many students, in fact, spontaneously interpret "transformation" to mean a change that only consists in a reduction or an enlargement of the figure (a homothety or a similitude). In the case (p=, S=) many students, as a consequence, refuse the identity or an isometry as "transformation".

The confirmation of the above also derives from the research carried out on the teachers. There happens, not only in the primary school, the case of the teachers who have reactions which are analogous to those of the students, that is one of surprise when confronted by a necessary change of beliefs. One teacher states: «But if no-one has ever taught us these things, how can we possibly know them?». This seems to us the confirmation of the fact that almost everything can be taken back to didactic obstacles.

The teachers' choices do NOT happen within a correct didactic transposition which lets them act transforming "Knowledge" (which for some of them actually there isn't) into a "knowledge to be taught", in a learned and aware way (often, unfortunately, there is not even an awareness of the difference between "Knowledge" and "knowledge to be taught").[11] Actually, at least in the field investigated by us, a scenario of a-critical questions, hashed and rehashed, is perpetrated following a pre-established script and consecrated by the textbooks. The confirmation is in the following facts: when the teacher changes belief, he does so

- insisting on the fact that this subject should explicitly enter into didactics
- sometimes spontaneously promising himself again to include it in his own didactic action of teaching/learning.

These last considerations allow us to insert the final result of our research on the teachers' change of belief in an important international context. It is true that beliefs can have deleterious effects on didactic action, but the opposite can also be valuable, as our case shows. We are encouraged in this also by the following affirmation: «Beliefs can be an obstacle, but also a powerful force that allows carrying out changes in teaching» (Tirosh, Graeber, 2003).

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[11] Here we mean the "didactic transposition" in the classic sense, as the transformation of Knowledge (savoir savant) into a knowledge to be taught (Chevallard, 1985). We will have to study in depth, in the future, this theme, in that there lacks, in may teachers, a competence in this type of theme. This lack does not permit them to truly transform Knowledge into a knowledge to be taught, but only to make reference to a personal knowledge which coincides with the institutional one expected by the curricular standards and by tradition.

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APPENDIX

p	**S**	**p**	**S**	**p**	**S**
>	>	>	=	>	<
=	>	=	=	=	<
<	>	<	=	<	<