1005. D'Amore, B., & Santi, G. (2021). Teaching and Learning of Geometry in Classroom as a process of Objectification. Language and attitude of students; definitions expressed in natural language; spontaneous recourse to figures. *Rematec*, 16(39), 119-141.
Scaricabile da: https://rematec.net.br/index.php/rematec

Teaching and Learning of Geometry as a process of Objectification: conditions and obstacles to argumentation and proof. The role of natural language, specific language, and figures.

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Abstract. This paper examines some examples (taken from research conducted over the years) that show students' linguistic attitudes in geometry tasks. The examples are framed within the Theory of Objectification with reference to the notion of sensuous cognition, semiotic means of objectification and levels of generality. We show the struggle students live, at higher levels of generality, in intertwining natural language, specific language and the spontaneous use of geometrical figures, bound to perception and kinaesthetic activity. Within the networking paradigm, we coordinate the Theory of Objectification and Duval's semio-cognitive approach to frame the interplay between the ideal and the material that occurs in geometrical argumentations and proofs.

Key words: Geometry task, natural language, use of figures in geometry, objectification, sensuous cognition.

1. Introduction

In D'Amore and Santi (2018) we have analysed the interference between natural language and specific language. Within a semiotic perspective, we have shown how and why such an interference entails the emergence of intuitive models and stereotypes in students learning mathematics.
In this article we tackle the issue of language within a broader perspective that includes students' proving attitude and the use of figures in geometry.
The type of language used in the classroom for teaching geometry has been studied by various authors, amongst whom we recall here Laborde (1982, 1995) and Maier (1993). We will make explicit and continuous reference to these papers in what follows.
To fully understand the specific teaching and learning processes in geometry it is essential to examine the *attitude* that the students assume (that we shall call "proving") which is clearly inspired by imitating the teacher, as studied for example in Balacheff (1982) and D'Amore (1993).
The need to use *figures* is a unique aspect in the teaching of geometry, but we must realise that with regard to both students and teachers, it has been taken little into account, as indicated in the research of Eisemberg (1992) and Kaldrimidou (1995). There is only one circumstance in teaching when the student is required to use drawings to represent a situation described in words - that is in geometry problems. However, in this case the student does not accept to use a figure willingly, reckoning that

it does not help, since it is perceived as a background annoyance. This is probably due to the lack of attention to this issue we mentioned above.

It is well known that, from the beginning, many students must face difficulties in understanding the language used by teachers and textbooks in mathematical lessons. Often the students are invited by teachers to explain, in natural language, proofs, ideas or definitions. We are aware, however, that these invitations are doomed to produce failure, because a clause of the didactic contract (Brousseau, 1980; D'Amore, 2003; D'Amore, Fandiño Pinilla, Marazzani, & Sarrazy, 2010; Narváez Ortiz, 2017) seems to impose on the students the use of a "specific language" to do mathematics, which is hardly, or not at all, close to natural language. The students resort to a naive use of a spontaneous language to overcome that restrictive clause.

The argumentative and proving activity typical of geometry takes place at the intersection of mathematical activity, the use of a rich arsenal of semiotic resources and the alignment of the student's personal meaning and the cultural-institutional meaning of geometrical entities (Godino & Batanero 1994; Radford, 2006).

In regard to proving, we should take into account:

- the peculiarity of proofs as objects emerging from specific activities that belong to the realm of argumentation.
- the use of the specific language of geometry and its relationship with natural language;
- the use of geometrical figures as a semiotic system characterizing geometrical knowledge and their connection with semiotic resources with an embodied nature, such as material objects, gestures, icons etc.
- the ontological statute of mathematical objects that do not allow ostensive references, thus the intrinsic need of semiotic representations and the unavoidable identification, on the part of the students, of the mathematical object with such representations.

The aim of this article is to single out the conditions and the obstacles that define the learning in geometry with particular attention to argumentation and proof. To accomplish such an aim, we shall focus on Italian middle school students (grades 6-9) facing geometry tasks. The experimental setting is inspired by research conducted by D'Amore and Sandri (1995) who expose students to mathematical practices that require students to remove themselves from the standard context of the classroom and to the usual role of student. They pretend that they are someone else dealing with questions in some way connected to those posed in mathematics. We want to refer to two episodes of the five related to the situation mentioned above.

The analysis is conducted within a semiotic framework that networks Duval's semio-cognitive approach and Radford's theory of objectification.

In section 2, we describe our theoretical framework and define our research questions. In section 3, we present the methodological setting and we analyse 6 protocols of students dealing with 2 tasks out of the 5 we proposed during the experimentation. Finally, in section 4 we answer the research questions and draw some conclusions of our study.

2. Theoretical Framework

As we mentioned in section 1, having recourse to semiotic representations is the only way to access mathematical objects. Mathematics education research has developed two understandings of the use of signs in mathematical thinking and learning: signs as mediators of mathematical practices and signs as something that stands for something else, to someone's interpretation.

We present the basic tenets of both stances in relation to geometry, focusing on Radford's Theory of Objectification and Duval's semio-cognitive approach respectively.

2.1. The theory of objectification

The theory of objectification pivots around the notion of mediated reflexive activity that conceives thinking as *praxis cogitans*, that is, "a mediated reflection in accordance with the form or mode of the activity of individuals" (Radford, 2008, p. 218):

• *Activity* refers to the individual and social agency towards shared goals, significant problems, operations, labor etc., within a cultural dimension that provides a system of beliefs, conceptions about truth, methods of inquiry, acceptable forms of knowledge etc.

• *Reflection* refers to the dialectical co-production between the individual and his cultural-historical context; a movement of the individual consciousness between his personal thinking, interpretations, emotions and feelings, perceptions and a historically and culturally constituted reality; on the one hand individuals are affected by their context, on the other hand they react agentically to such a context. Reflexivity refers to the fact that cognition is envisaged as a social, cultural and historical dialectical movement between a sense-making individual who creatively responds, acts, feels and transforms the world' (Radford, 2008, 2020).

• *Mediation* refers to the artifacts that, in a Vygotskian sense, are constitutive and consubstantial to thinking since they allow us to carry out activity, i.e., they mediate activity; within the theory of objectification, the system of artifacts that carry out activity are termed as the *territory of artifactual thought* and it includes objects, artifacts, gestures, natural language, symbolic language, icons, drawings etc.

The theory of objectification characterizes the cultural-historical context in terms of symbolic superstructures called *Semiotic Systems of Cultural Signification*: they include the nature of mathematical objects, epistemological stances, forms of rationality, conceptions about truth, method of activity and inquiry, legitimate ways of knowledge representation etc. The Semiotic Systems of Cultural Signification give rise both to *modes of activity* and *modes of knowing*, as they are consubstantial to activity and the territory of artifactual thought.

Advocating a pragmatic ontology, in the theory of objectification mathematical objects are "*fixed patterns of reflexive human activity incrusted in the ever-changing world of social practice mediated by artifacts*" (Radford, 2008, p. 222, emphasis in original). In this view, mathematical objects lose any intrinsic, a priori, realistic nature. Nevertheless, as fixed patterns of mediated reflexive activity they acquire, within the cultural-historical dimension, a form of ideal existence:

Radford (2016, p. 3) conceptualizes activity in terms of joint labor:

> The idea of joint labor seeks to restore to activity its most precious ontological force, namely, the dynamic locus where human existence creates and recreates itself against the backdrop of culture and history. Yet, with its utilitarian and consumerist orientation, contemporary mathematics classroom activity tends to produce and reproduce alienated students. It is argued that the search for non- alienating classroom activity requires a reconceptualization of the classroom's forms of human collaboration and its modes of knowledge production. (See also: D'Amore, 2015, 2018).

Learning is a specific *praxis cogitans* that Radford (2008) terms a *process of objectification*. In its etymological meaning it refers to the process that allows the student to notice, find and encounter the cultural object. The artifacts that accomplish the objectification processes are called *semiotic means of objectification* (Radford, 2003) and cover the whole range of ideal and material artifacts mentioned above:

> An opening movement towards others and the objects of culture. (...) To learn is not merely to acquire something in the corrupted sense of possessing it or mastering it, but to go to culture to find something in it. This is why the outcome of the act of learning is not the construction, re-construction, re-production, re- invention or mastering of concepts: its true outcome is to be found in the fact that, in this encounter with the other and cultural objects, the seeking individual finds herself. This creative process of finding or noticing something (a dynamic target) is what I have termed elsewhere a process of objectification (Radford, 2002). (Radford, 2008, p. 222)

The theory of objectification can arguably be set into the strand of embodied cognition in mathematics - for an overview we refer the reader to Radford, Arzarello, Edwards and Sabena (2017). Radford (2014), resorting to a dialectic materialistic stance, conceives embodiment as a *sensuous cognition*, that is, a multimodal sentient form of responding to the world sprouting from cultural and historical activity. Cultural and historical activity intertwines, in *sensuous cognition*, senses, feelings, materiality, and the conceptual realm. The materiality of cognition is not something subsumed in the mind to acquire the nature of a concept, but the material is consubstantial to the conceptual. Senses, feelings, materiality and the conceptual realm culturally and socially develop into what Radford (2014) terms "highly sensitive cultural objects - *theoreticians*" (p. 353; emphasis in original), in which the material and the ideal are tuned into the objectification of mathematical generality.

The theory of objectification allows us to outline *levels of generality* (Radford, 2003) at which the student objectifies the mathematical concept. The level of generality specifies the degree of *embodied activity*, according to the artifact that realizes the process of objectification. Radford (2003) outlines three levels of generality.

Factual generalization - characterized by perception, feelings, movement, spatial and temporal elements of the students' physical environment - is accounted for mainly by gestures, bodily movements, material objects and deictic and generative use of natural language.

Contextual generalization intertwines material perception, movement and feelings with a new perceptual field in which emergent objects are detached from mediated sensory perception. Students start introducing more ideal semiotic means of objectification, such as new linguist terms, natural language and the first elements of symbolic language.

In *symbolic generalizations*, perception is no longer embedded or related to a concrete space-time context but in a new abstract and relational "space" where mathematical activity is carried out mainly by symbolic language.

Semiotic means of objectification determine the mode of existence of mathematical objects in the pupils' experience, i.e., they determine how the intentional "arrow" attends to such objects. Referring to Husserl (1913/1931) they intertwine the noetic-noematic phenomenological layers that altogether result in the full meaning of the mathematical object. For example, we can deal with the circle through the kinesthetic movement of the compass, the definition in natural language, and using a second-degree equation in the algebraic symbolism. The theory of objectification allows us to outline levels of generality (Radford, 2004) at which the student objectifies the mathematical concept. The level of generality specifies the degree of embodied experience involved in the reflection mediated by a particular semiotic means of objectification. Recalling the example mentioned above, the compass mediates the circle with a lower level of generality with respect to the second-degree equation. The demand of higher levels of generality, as the individual and cultural meanings converge, obliges the pupil to live a rupture with his/her embodied experience that can bewilder him and lower his personal implication and involvement in the learning process (Radford, 2003). We stress the fact that mathematical objects are stratified not only in their levels of generality but also in the different modes of activity that allows Fandiño Pinilla (2020) to identify 5 fundamental learnings in mathematics: conceptual, strategic, communicative, algorithmic and semiotic. Argumentation and proving, which are the objective of this article can be cast mainly in the communicative and strategic learning.

The role of natural language as a semiotic means of objectification is the turning point in bridging the gap between the embodied experience of the pupil and the interpersonal meaning of the cultural object. Research in this topic (Radford, 2000, 2002, 2004) has shown how, in the generalization of algebraic patterns, the indexical use of natural language, in space and in time, triggers and enhances the shift from the sensorimotor experience to the algebraic symbolism. Exposing the students directly to the algebraic language would result in a shallow learning. Furthermore, as we reach higher levels of generality natural language allows us to keep the relation with the core of meaning that lies in the individual embodied experience. Natural language plays a key role in driving movement, organizing activity in time and space, in triggering rhythm, in singling out and individualizing objects, in activating and supporting schemata. This broad set of possibilities is the inerasable basis for the recognition of operational invariants, thus accessing higher layers of generality. A thorny issue is the relation between natural language and the use of the specific language of mathematics. It requires attention and awareness on the part of the teacher. The use of the specific language of mathematics is a learning objective that allows the student to reach a further layer of generality. It is achieved when embodied meaning has a solid basis to sustain the leap to specific mathematical language that objectifies definitions, generalizations, algorithms, inferential thinking etc. Without an underlying significant reflexive activity in the student's personal experience, the use of specific language can hinder the learning of mathematics. Furthermore, the specific language of mathematics has a semantic density (D'Amore, 1999) that can disembody meaning just as it happens with symbolic language. In this situation the combined use of symbolic language and specific language can foster an unbridgeable gap between the individual meaning of the student and the cultural one, thus entailing a lack of personal implication and the emergence of intuitive models (Fischbein, 1987 ,1992) and the didactical contract (Brousseau, 1980, 1997). For an in-depth discussion of this topic, we refer the reader to D'Amore (1999, pp. 251–261).

In regard to the learning of geometry and the argumentative activity, the situation is even more tricky. Given the interference between specific language and natural language mentioned above – in the case of geometry the specific language refers to the semantics and syntax used in proofs – the pupil has to handle also the interplay and distinction between drawings and geometrical figures. This learning obstacle is connected to the well-known notion of figural concept introduced by Fischbein (1993), which distinguishes geometrical concepts amongst mathematical concepts:

> The properties of geometrical figures are imposed by or derived from definitions in the realm of a certain axiomatic system. From this point of view, also, a geometrical figure has a conceptual nature. A square is not an image drawn on a sheet of paper. It is a shape controlled by its definition (though it may be inspired by a real object). A square is a rectangle having equal sides. Starting from these properties one may go on for discovering other properties of the square (the equality of angles which are all right angles, the equality of diagonals, etc.). A geometrical figure may then be described as having intrinsically conceptual properties. Nevertheless, a geometrical figure is not a mere concept. It is an image, a visual image. It possesses a property which usual concepts do not possess, namely, it includes the mental representation of space property. (…) The triangle, the circle, the square, the point, the line, the plane, (…) in general, all the geometrical figures represent mental constructs which possess, simultaneously, conceptual and figural properties. Certainly, when we imagine a circle, we imagine a drawn circle (including, for instance, the color of the ink) and not the ideal, perfect circle. But the mathematical circle, which is the object of our mathematical rreasoning, has no color, no material substance, no mass, etc. and it is supposedly ideally perfect. It has all the properties of a concept, it may participate as it is in a mathematical reasoning and this despite the fact that it still includes the representation of spatial properties. The objects of investigation and manipulation in geometrical reasoning are then mental entities, called by us figural concepts, which reflect spatial properties (shape, position, magnitude), and at the same time, possess conceptual qualities -like ideality, abstractness, generality, perfection. I do not intend to affirm that the representation we have in mind, when imagining a geometrical figure, is devoid of any sensorial quality (like color) except space properties. But I affirm that, while operating with a geometrical figure, we act as if no other quality counts. (Fischbein, 1993, pp. 141-143)

This approach to geometrical concepts is strictly connected to the intrinsic inaccessibility of mathematical concepts, geometrical ones included, and to the tension between the general and the particular. When dealing with geometrical entities, students tend to confuse the drawing with the geometrical concept since they identify it with the figural aspect of figural concepts. To grasp the true meaning of geometrical concepts, students have to handle the dual and complementary nature of geometrical entities as holding a conceptual aspect (its definition) and a figural aspect (its spatial properties).

In a purely rationalist conception of cognition, generalization is subsumed in the activity of the mind, what Descartes would call the *res cogitans*. The particular belongs to the realm of the *res extensa*, whose reality and action is driven not by perception but by reason. Kant in the *Critics of Pure Reason* questioned the basis of the subject's possibilities to know, synthesizing rationalism and empiricism. The philosopher conceives the issue of generalization as the relation between a priori concepts and our sensible experience; knowledge is within this relationship which is the only possible access to the per se object (the noumenon) as a phenomenon.

As for mathematics, Kant considers the a priori concepts, the general, and the sensible experience, the particular. He establishes a relationship between the general and the particular. According to Kant the general and the particular join together in the schema. Through the schema, the mathematical a priori concept descends into the world of sensible experience without losing its essential characteristics, i.e. generality. When we draw a rectangle, we have a sensible experience of this geometrical entity. The drawing is a particular case that betrays the generality and the a priori nature of the concept. Nevertheless, the essence of the concept is untouched in the operational invariant of the schema that allows us to draw it and is always beyond the drawing. Fischbein's (1993) notion of figural concept can be interpreted as a paradigmatic example of this tension between general and particular. The importance of schemas in mathematical practice is strictly related to the role of and confidence in signs as bridges between sensible experience and a priori concepts. We owe to Kant the merit of ascribing an epistemological role to signs (Radford, 2004).

Kant's epistemology, basically essentialist, however, erases the dialectical co-production between the ideal and the material realized in cultural-historical activity, thus disregarding the accomplishment of *sensuous cognition*, advocated by the theory of objectification. Within a socio-cultural perspective, *sensuous cognition* bridges the gap between the ideal and the material nature of mathematical objects conceived as emerging from social and cultural historical activity. Radford (2008) considers mathematical objects conceptual forms of historically, socially, and culturally embodied, reflective, mediated activity. The generality of mathematical objects is consubstantial with and is derived from human activities. Ilyenkov (1977), quoted in Radford (2006). clarifies the cultural and historical origin of generality.

> "Ideality" is rather like a stamp impressed on the substance of nature by social human life activity, a form of the functioning of the physical thing in the process of this activity. So, all the things involved in the social process acquire a new 'form of existence' that is not included in their physical nature and differs from it completely – [this is] their ideal form. (Radford, 2006, p. 86)

The tension between the general and sensible experience is still vibrant in the TO. Such a tension is not between a priori concepts and sensible experience but between the individual's sensible experience and the reflexive mediated activity condensed in the ideality of the historical and cultural object. Within a socio-cultural perspective, the study of generalization requires taking into account not only the ontological and epistemological dimensions but also the anthropological and socio-cultural ones.

Concerning geometry, the synchronic use of natural language and drawings must evolve towards the synchronic use of specialized language and more structured forms of visual representation to accompany the pupil from embodied factual generalizations to contextual and symbolic ones. The outcome is the development of a *theoretical eye* (Radford, 2010), specifically a "geometrical eye" able to grasp the conceptual-figural duality that allows the student to master argumentations and proofs in geometry. We highlight that proving is a cultural object that the student should objectify in the triangulation of reflexive mediated activity, SSCS and his sensuous experience. A form of knowledge that we cannot take for granted. The student needs to objectify this aspect of geometrical knowledge as a form of sensuous cognition, by going through the different levels of generalization, from factual to symbolic, and synchronically coordinating natural language and drawings that evolve in the specific language of geometry and structured figures respectively. If the student is exposed directly to specific language and structured figures, he will miss the material side of sensuous cognition and the ideal one will be meaningless to him, with the ensuing distance between the personal and cultural meaning.

We analyzed the TO within a pragmatist ontology of mathematics. D'Amore (2003) provides a detailed account of realist and pragmatist theories and concludes that there is not a definite boundary between the two perspectives. Ullmann (1962) highlights two complementary features that characterize the development of mathematical objects: the operational phase and the referential phase. On the one hand mathematical objects and their meaning emerge from and are objectified by reflexive activity, on the other hand it is necessary to linguistically refer to the entities that emerge from such activity. The dual nature of mathematical objects – as patterns of activity and as existing ideal entities in the culture – implies that also meaning and semiotics have a dual nature. We therefore need a semiotic perspective that accounts for the need, in the referential phase, to nominalize and transform signs in order to create relations, generalize, carry out calculations and proofs. Raymond Duval (Duval, 1995, 2006, 2008, 2017; Iori, 2017, 2018) introduced semiotics in mathematical thinking and learning and devised a structural and functional approach to the use of signs.

2.2 The semio-cognitive approach

The TO accounts for the emergence of mathematical objects as fixed patterns of cultural-historical activity and learning as a process of objectification. When mathematical objects assume an interpersonal reality as archetypes of the cultural-historical dimension, we need to linguistically and semiotically refer to such emerging forms of activity. Wittgenstein (1953), in the *Philosophical Investigations*, observes that the denotative character of language is one of its possible "uses", that is, one of the *possible linguistic games* – in the terminology of the TO, one of the *possible forms of activity*, which is suitably outlined by the semio-cognitive approach. The need to acknowledge the denotative use of semiotic resources prompted the insertion of Duval's semio-cognitive approach in our discourse according to the combining/coordination (Prediger et al., 2008) networking strategy, Thus, we can get a multi-faceted insight into argumentation and proof as the empirical phenomenon at stake in our study.

The denotative character of semiotics in mathematics can be identified with a complex coordination of several representation registers.

A semiotic system is devised by (Duval, 2006; Ernest, 2006):

- a *set of basic signs* that have a meaning only when *opposed to or in relation with* other *elementary basic* signs (for example the decimal numeration system);
- a set of o*rganizing rules* for the production of signs from the *basic* ones and for the transformation of signs;
- an *underlying meaning* resulting from the relation between the *basic* signs that form structured semiotic representations.

A *representation register* is a semiotic system that also accomplishes the functions of *communication*, *objectification* and *treatment* (Duval, 1996).

D'Amore (2001) identifies conceptualization with the following *semiotic functions*, which are specific for mathematics:

- c*hoice of the distinctive traits* of a mathematical object;
- *treatment,* i.e. the transformation of a representation into another representation of the same semiotic register;
- *conversion,* i.e. the transformation of a representation into another representation of another semiotic register.

The very combination of these three "actions" on a mathematical object can be considered as the "construction of knowledge in mathematics". But it is not spontaneous nor easily managed and represents the cause for many difficulties in the learning of mathematics when students struggle with the *cognitive paradox.* (See also: D'Amore, Fandiño Pinilla, Iori, & Matteuzzi, 2015; D'Amore, Fandiño Pinilla, & Iori, 2013).

While in the operational phase natural language plays a prominent role in sustaining the leap to higher levels of generality, in the referential phase natural language sustains the coordination of representation registers via treatment and conversion.

The discursive functions of natural language are responsible for an appropriate control of the semiotic functions at a cognitive and metacognitive level. The learning of geometry, in particular argumentation and proof, requires a coordination and control of the following semiotic systems:

- spontaneous and narrative use of natural language;
- specific natural language that implies the control of the 5 discursive functions mentioned above, with particular attention to the discursive expansion and the discursive reflexivity functions;
- figural register where drawings are not mere isolated icons but form a semiotic system that allows treatment and conversion operations.

On the one hand, in TO, when synchronically used with other semiotic means of objectification, in particular figures, natural language allows students to move along the different layers of generality, from argumentations at a factual level to the more refined and formal proofs at a contextual and symbolic level. On the other hand, within a structural and functional approach, at higher levels of generality, argumentation and proving require a diachronic use the discursive functions of natural language and figural semiotic systems via treatment and conversions.

The following quotation, taken from a work of Peirce, has been envisaged for algebra but we believe that it can be extended to geometry.

> It has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. Various have been the attempts to solve the paradox by breaking down one or other of these assertions, but without success. The truth, however, appears to be that all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. (…) As for algebra, the very idea of the art is that it presents formulae which can be manipulated, and that by observing the effects of such manipulation we find properties not to be otherwise discerned. In such manipulation, we are guided by previous discoveries which are embodied in general formulae. These are patterns which we have the right to imitate in our procedure, and are *the icons par excellence* of algebra. (Peirce, 1931-1958, 3.363)

The teaching design of geometry cannot underestimate the need of personal meaning that drives the student's activity both when learning at school and in his every-day experience. There are two basic constitutive elements that contribute to personal meaning:

- *operational invariants* of activity;
- a system of *convictions and interpretations.*

Often, teaching strategies can hinder the encounter of the student's personal meaning with the interpersonal cultural historical one. The student accomplishes his need of meaning by having recourse to appropriate operational invariants, enhanced by beliefs and interpretations, that make him feel self-confident and self-effective in a situation of cognitive and emotional dismay. Mathematics education refers to the operational invariants as *intuitive models* because of the sense of globality, immediacy and self-evidence they convey. The system of beliefs and interpretations that intertwine mathematical knowledge, teacher and pupil is referred as *didactical contract.*

An appropriate use of natural language and geometrical figures provides students with the cognitive and metacognitive strength to handle argumentation and proof as a cultural object. If we disregard the transition from natural language to specific language and from figures as drawings and icons to figures as belonging to a semiotic system, the student can turn to inappropriate intuitive models (Fischbein, 1987, 1992), stereotypes and the didactical contract (Brousseau, 1980, 1997).

3. Methodology and research problems

A group of 61 Italian Middle School students (grade 7) from the province of Bologna (Italy) have been singled out by their teachers according to their willingness to take part in the study. They have been exposed to situations in which they remove themselves from the context of the classroom and pretend they are someone else dealing with questions in some way connected with those that are posed in mathematics.
The students had to produce written texts relative to the following situations. The experimental setting aimed at initiating students' objectification process and deploying semiotic means of objectification with attention to natural language, specific language, and the spontaneous use of geometric figures.

T1. Imagine you are a trader ...
A lady has bought things and spent 3700 lire; He gave you 5000 lire and you turned him around just. She, however, protests and says you had to give her 1700 lire. You calmly explain that you're right.

T2. Pretend to be a primary school teacher ...
You wish to explain to your third-year primary school students (grade 3) how the area of a rectangle is found using the base and the height.

T3. Imagine you are a rigger (or a rigger) ...
This is the blueprint for an apartment you've drawn right now; but the buyer doesn't understand how he'll be able to live in an apartment so small that it fits on a folio. You explain well that this is a scale drawing. [The plan of an apartment is provided at scale, without any indication about it; the actual measurements are nevertheless given, in meters, but without any indication about the unit of measurement].

T4. Imagine you're a railway worker ...
A gentleman asks you what average speed a certain Intercity train travels to from Bologna to Milan (220 km), since the journey takes an hour and a half. You answer him correctly, but he says it's impossible and he wants you to explain how you've done the math.

T5. Imagine you're a dad (or a mom) ...
Your child, who is 7 years old, overhearing someone saying that every triangle has 3 heights, asks you: "Father (or mother), what does this mean?". There is nothing worse than avoiding questions from little children so, therefore, decide how to reply.

For our study, we will focus only on T2 and T5. The research problems we report below aim at scrutinizing the basic elements that contribute to the processes of objectification in geometry: the students' positioning towards cultural knowledge as they free themselves from the stereotypical role of the student and the interplay between natural language and geometrical figures in moulding sensuous cognition. Possible changes in the attitude of the students towards mathematical practice and the role language and geometrical figures in objectification processes might shed some light on conditions and obstacles in argumentation and proof in geometry.

Research problems
P1. In lower secondary school (grade 7) there are many teachers today who enhance a proving attitude to introduce geometrical properties. What is the relation between the personal meaning of proof objectified by the student and the cultural one? What is the role of the teacher's attitude and personal meaning of proving in the students' objectification of proof? How does objectification occur when the students are asked to assume the role of a teacher or a parent?
P2. If the student decides to make use of his natural language to describe geometric entities, it is highly probable that he feels free to avoid repeating the classical definition, heard in the classroom

(or perhaps learnt) and he will propose something that is probably very close to their embodied experience that allowed the objectification of that entity. Their use of natural language allows us to identify the student's personal meaning? Is it scholastic or from their fantasy? Is this acceptable?
P3. In this situation, which is so de-contextualised, will the recourse to figures be spontaneous or not? In their role as teacher or parent, will the pupil believe that the figure should have a certain significant role? What is the role of figures in underpinning sensuous cognition in the learning of geometry and how do they interact with other semiotic means of objectification?

4. Results

The proposal had an unexpected success, given that several children were willing to play a different role from their usual one. This allowed them to feel free, at least in part, from the restrictive clause of the didactic contract and to explain themselves using a slightly formal scholastic language (given that they are addressing younger pupils) or actually in natural language (in the case of T5).[1]
This research has highlighted a series of interesting considerations on language, on the student's attitude and also on the use of figures in mathematics education in general; but here we will only deal with specific applications to geometry.
The proving attitude
Of the 61 pupils who have confronted T2, a little more than a half attempted something that could be called a proof; often this was nothing more than an explication of the formula, other times the proofs involved questions of a formal type, or figures that, evidently, had always created difficulty for the students in the calculation of the area. For example, many focused on the fact that between the letters A and B of the segment AB there was a horizontal straight line (but hardly anyone made the distinction between a segment and its length); it seems that they were using an empty convention, connected solely with an explicit clause of the didactic contract, that is, "what the teacher expects is that I know how to draw to scale". There were then different cases, amongst which stands out the protocol of a pupil who explains that we measure the height of things at the centre, whereas the height of a rectangle is measured at the extremity; and then he "proves", with two convenient figures, that there is no difference between the measure of the height at the centre or at the "extreme" edges, exploiting a translation of the height (central) towards the right and towards the left. It is an interesting fact that in no case the area was expressed in m^2 or in cm^2; almost all units are cm and many protocols don't give any indication of the unit of measurement at all.
In the protocol of Nigel, a particular rectangle having 2 and 3 as the lengths of its sides is examined and the area proposed is 6^2; the 2 as exponent is explained because there are "two measures, the base and the height". Many attempts at proving are hindered in their path and end resorting to a principle of authority: "One does so and enough", "This was discovered by a great mathematician" and similar things.
The protocols that provided something that could be considered a true and proper proof, at least as far as attitude was concerned, were the two following (both without figures).

1. The rectangle is formed from two rectangular triangles. They are called thus because they have a right angle. We divide the rectangle by a diagonal into two equal parts. Since the sum of the internal angles of a triangle is 180°, to find the area of the rectangle we multiply base × height.

1 The detailed analysis of the results of the research are reported in D'Amore and Sandri (1995); here we limit ourself to the least necessary for the scope of this article. Since we described above how the research was carried out, we will avoid discussing methodology issues.

2. First of all to start, this figure is called like this because all its angles are 90° that is right angles. Its sides are in equal pairs: $\acute{A}B$ and $\acute{C}D$ and $\acute{A}D$ and $\acute{B}C$. Therefore, to find the area we multiply base × height.

In both cases one speaks of the pertinent use of language terms, with an evident attitude that could be called a proving one; but one clearly sees that there is a separation between the attitude and the coherence of the two reactions. The two pupils make claims that call in question elements of the rectangle that to them are well known and they use them much as a hypothesis for the final deduction, without any critical comment. This shows that a premature proving attitude on the part of the teacher has only created imitators without any self-reflective criticism.
Also in Balacheff (1982) cases of proofs in the middle school (*collège*) are studied. While everything may suggest that more maturity should guarantee more deductive coherence, this is evidently not the case. D'Amore (1993) shows a significant protocol (we call its author E) that seems to be the fatal outcome of an "empty" and incorrect proving attitude. The protocol of E comes from 3rd year of high secondary school (grade 11, in Italy: 16-17 years).
The pupil has to show that in the figure below the angles at the circumference, at A and E, are equal, given that they are subtended by the same chord BC.

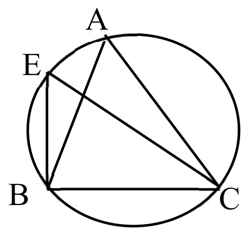

Fig 1.

After various considerations which are more or less locally correct (nothing to do with what was to be shown), E constructs the following figure:

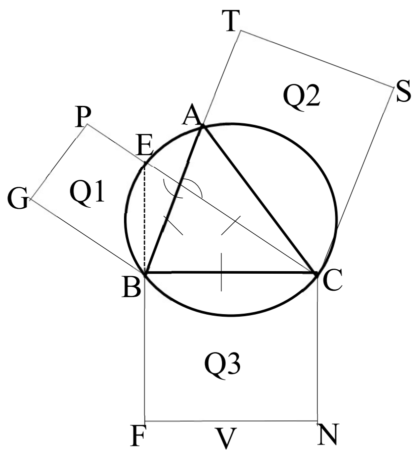

Fig 2.

and writes:

... From Pythagoras' Theorem, I can say that Q1 ≅ Q2 and Q2 ≅ Q3 so Q3 ≅ Q1. Having shown that the squares are congruent from this I can say that the base DB of triangle DEB is similar to base BC of triangle EBC and so (...) always using Pythagoras' Theorem, BEPG and ATSC are 2 congruent quadrilaterals (...) therefore EB and AC are similar.

These protocols have a common denominator in that they show the lack of what we termed above a "geometrical eye" as an outcome of sensuous cognition. There is a total absence of an argumentative and proving attitude that emerges as a complete inconsistency between the students' personal meaning and the cultural one, as far as argumentation is concerned. It seems that there is a lack of objectification of the basic geometrical concepts as fixed patterns of reflexive activity. Natural language and figures are used in a very specific sense without any connection to perception and the sensorimotor activity. For example, in 1. and 2. the students use specific language completely unrelated to any sensuous understanding of the concept of area, thus they are unable to develop any form of argumentation regarding the area of a rectangle. In the high school protocol, the student has recourse to a drawing where what is referred to as a square does not have right angles. In all the protocol, in regard to the semio-cognitive activity, there is no use of the discursive functions of natural language and no connection between natural language and the figural semiotic systems through treatment and conversion. We suppose that sometimes the teacher proposes a proving attitude without carefully explaining, with an abundance of examples and also of counter-examples, how a proof works. Thus, the student misses the link between his personal reflexive mediated activity and the cultural meaning of proving. Exposing pupils, as sometimes happens in Italian high schools (in Italy: 14-16 yrs., grades 9-10-11), only to a few formal and empty routines is not enough to explain what proof really is. We should rather reflect explicitly on the functioning of proofs using concrete examples (and perhaps counter-examples). The protocols examined here could well serve as counter examples.

Natural language

In reality, the great majority of children choosing to be a teacher (61) or a parent (62), decided to provide a definition of a geometrical entity (rectangle, or height of a triangle). They did this resorting to the language used in class, imitating as closely as possible the language of the teacher. We have therefore a series of attempts to claim that the height of the triangle is a segment, or line ... that it goes from one vertex, from vertices, from a point, from an extremity ... to the other side, the opposite side, to a point of the opposite side, to the centre of the opposite side ... The syntax is often pedantic compared to what it should be, perhaps an indicator of a clause of the didactic contract (as it is naturally interpreted by the student), according to which it should be a little pedantic ... They often resort to figures.

There are interesting cases in which the student occasionally responds in natural language, without any formalism, falling therefore into the "trap" set by us, in the hope of reading written responses in natural language without imitating the language used by the teacher. Below three protocols that are particularly fascinating, one relative to T2 and two to T5.

T2 (Anna):

I do not think that I can pretend to be a teacher of primary school, I can always try, however, there is always a first time.

First of all, if I really had to be a teacher, I would be very natural and pleasant in order to make the dialogue with my pupils simple and direct.

I would like to have a friendly entertaining relationship, in fact, if I had to explain how to find the area of a rectangle, given my craving for sweets, I would imagine the rectangle as a bar of chocolate. I have tried, but it has not worked. I am not capable of explaining that you find the area of a triangle by multiplying base times height. I leave this exercise to true and real teachers.

T5 (anonymous):

You must not believe everything that is said to you.

T5 (Simona):
My son, you don't know geometry but I wish to explain to you what height means. Like you, dad and I have a height that you measure from head to feet, so triangles also have one, but theirs is measured from the vertex, that is the tiny point, to the base that is like our feet. But given that they have 3 points (vertices) they have 3 heights because they have 3 pairs of feet. And given that we have only one head and only one pair of feet, we have only one height.
In all three protocols we have the breaking of the usual didactic contract, certainly caused by offering a de-contextualised situation.[2]
In the first case, apart from the pretty lesson in general pedagogy, there is the explicit reference to a concrete model, but also the hidden timidity in not having understood or respected the task. The protocol testifies the need to enhance and start from the material side of sensuous cognition as a robust basis for the development of the ideal one. The pupil is aware that she cannot accomplish it and such an awareness could be exploited for her further learning. There is no use of specific language and a strong connection with the material side of cognition, but the lack of the necessary cultural and ideal features of thinking hinders even a first encounter with the cultural mathematical object, both the notion of rectangle and proving.
In the second case, there is a very beautiful explicit declaration of the fact that to put 3 heights in a triangle is absurd.
In the third case there is a conscious complete rupture with the didactic contract: in the use of natural language and in the use of a non-stereotypical model; as with simplified models of entities using one height, this anthropomorphic view of triangles is an amusing didactic finding and perhaps reveals the mental model that Simona has spontaneously made of the situation at the time. This is a beautiful example of sensuous cognition that allows the student, at a factual level of generalization, to objectify the height of a triangle with a personal meaning that encounters the cultural one.
The other protocols are characterized by a blind imitation of the specific language used by the teacher that is devoid of any relation with the personal experience of the student in terms of perception, movement, the use of gestures etc. This approach to learning often hinders the development of sensuous cognition and a meaningful objectification of geometrical concepts. A suitable agentic space that intertwines ideal and material allows the introduction of specific natural language for a meaningful and robust learning of geometry.

The use of figures
In general, the spontaneous use of figures on the part of students is somewhat limited. D'Amore (1995) explicitly confronted the spontaneous use of figures on the part of students in the activity of problem solving and we refer the reader to this work for more details. In his study, D'Amore shows that it is much easier and much more common to find spontaneous recourse to figures in primary school (in Italy 6-11 yrs, grade I-V) than in *middle school* (in Italy: 11-14 yrs, grade VI-VIII).[3]
Moreover, even though primary school students draw the geometrical object (pictures evoked from the situation narrated in the text) in secondary school the drawing is seen more in a formalised external model. For example, in the case of a problem that concerns 2 trains in a long journey from A to B, the majority of the drawings that were done spontaneously in primary school were of trains, whereas the largest percentage of spontaneous drawings of children in secondary school (when there were any) drew only the track AB represented as a segment.[4]
Let us return to T2 and T5. In T2 almost, all students drew rectangles (excluding the case of the height drawn from the centre to the extremity). In the rectangular triangles obtained by dividing the rectangle

2 This is thoroughly analysed in D'Amore and Sandri (1995), here we pass over it.
3 In Baldisserri et al. (1993) we have proved how in kindergarten (in Italy: 3-6 yrs.) the apparent spontaneous recourse to drawings should (at least in part) be considered as a clause in the didactic contract, even more when they are explicit.
4 This did not necessarily guarantee that the problem posed was correctly solved.

by a diagonal not one figure appeared. In T5, however, we have some more figures. Some were used to show the child what are the three heights of a triangle. We have then, for example, the following figures.

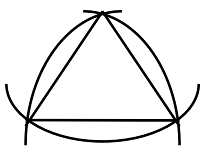

Fig. 3

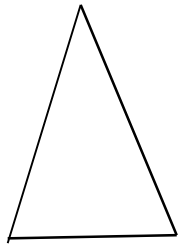

Fig. 4

There are therefore cases of misunderstanding; for example, a pupil drew the 3 ways in which it is possible to present the height of a triangle: all inside, coinciding with a side, all outside. But the most interesting thing is that, when a figure appears, then there is NO text (or the text is reduced to one or two words). The figure seems to take the place of the text. The clearness of the images perhaps makes the written words superfluous.

5 Answer to the research problems

With regard to problem P1, it is perhaps useful to remember that the proof has different purposes. Even the ancient Chinese mathematicians distinguished the modality *bian* (that is to convince) from the mode *xiao* (to understand) (Barbin, 1988).

Now, if it is true that there is a profound connection between these two functions ("to understand" is necessary in order to accept a concept and therefore it seems to be preliminary to "conviction") then it should be said that this distinction is not always so clear in practical didactics. The experiences given above seem to indicate the need to transform the proof from simple argumentative attitudes to true and proper objects of knowledge; it is not enough to simply give the method and to wait for imitation to do the rest. It seems much more efficient to produce examples of proofs and then to analyse them, as well as discussing counter examples. In essence: proving not as a method but as an object of study, as a part of knowledge to be transmitted. A knowledge that has to be objectified as a result of sensuous cognition that intertwines the ideal and the material. At higher levels of generality, formal proofs require the coordination through treatment and conversion of natural language, with its discursive functions, and the figural geometric register.

In relation to problem P2, we have seen that it is very rare for a student to do mathematics using the common language even with the help of the teacher, notwithstanding that the right climate may exist. They are invited by the teacher, however, to make use of an artificial language. In rare cases in which the game of decontextualization reappears, very personal models emerge, of great efficacy, but also interpretations that can greatly help the teacher to understand what is going on in these more or less positive actions. If to the implicit request to explain that in a triangle there are 3 heights, many students reply in a non-relevant way, for example listing entities inside the triangle even if these entities have nothing to do with the heights, showing that the concept is difficult to assimilate and accept. In our view, the best way of overcoming this difficulty is not questioning, practising, testing, but a descriptive activity which makes it possible to make use of languages different from the usual one. An inadequate reply to a question tells us nothing about the real nature of the non-competence of the student. This type of analysis, however, gives effective diagnoses with much clarity. There is,

however, the issue of whether accepting or not these invented personal "descriptions", with respect to standard ones. The question is very delicate. If on the one hand these descriptions can effectively replace the standard definitions in the student's personal comprehension (creating stable and appropriate mental models) it should also be said that according to mathematics education it is necessary to support the pupil's capacity to read, comprehend and make use of such an appropriate language. There are, however, various points to discuss. For those students (future citizens) is it really indispensable? If the possible result is that seen in some previous protocols, is it not perhaps the case to try to encourage the use of natural language, even at the loss of a supposed "more correct language"? Fostering the use of natural language in argumentations and proofs within sensuous cognition could be a condition for objectifications at higher levels of generality that involve the use of specific language and its discursive functions in coordination via treatment and conversion with other semiotic systems, with particular attention to the figural one.
With regard to problem P3, it is interesting that many tried proofs either without drawing figures or drawing a rectangle which had no relation to the proof; it seems to indicate that the function of the geometric figure is as "an allusion" and not as something on which to base the proof. Those students, however, who confronted text T5 showed more tendency to use figures (always triangles) and always with an attempt to "show" what are the heights of a triangle or what is the height of a triangle. In this case the figure bears a "descriptive" role substituting a description in words that the students reckoned difficult. This suggests that teachers should dedicate more time making the student familiar with the figural semiotic system, with its own syntax and semantics and coordinated with natural and specific language. The student should become familiar with the code of figures, making more use of figures with a descriptive role, by including them in our didactic heritage. This could help many students in difficulty. But this also carries a change of attitude in the approach to figures. It is necessary here to recall that in Arab and Indian medieval mathematical treatises of great scientific value there were often no proofs of a proposition, but rather a good diagram, with underneath: "Look" or "So". For students with linguistic difficulty this might be a good didactic strategy. In fact, students do not spontaneously refer to figures, except for particular cases - for example when the problem proposed does not have a standard scholastic solution and then reasoning on the figure that illustrates the situation seems to be the only method for creating a solving strategy. We refer the reader to the collection of examples in D'Amore (1995).
We remark that proving is a cultural object that the student should objectify in the triangulation of reflexive mediated activity, SSCS and his sensuous experience. A form of knowledge that we cannot take for granted. The student needs to objectify this aspect of geometrical knowledge as a form of sensuous cognition, by going through the different levels of generalization, from factual to symbolic, and synchronically coordinating natural language and drawings that evolve respectively in the specific language of geometry and structured figures, the figural semiotic system. If the student is exposed directly to specific language and structured figures, he will miss the material side of sensuous cognition and the ideal one will be meaningless to him, with the ensuing distance between the personal and cultural meaning. On the other hand, fostering sensuous cognition as an interplay of ideal and material develops the students' "geometrical eye" that will access the higher levels of generality necessary for geometrical proving by handling the complicated network of semiotic transformations that involve natural language, specific language and the figural register.

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