

Regularization in Inverse Problems

With a focus on image reconstruction



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Image Reconstruction

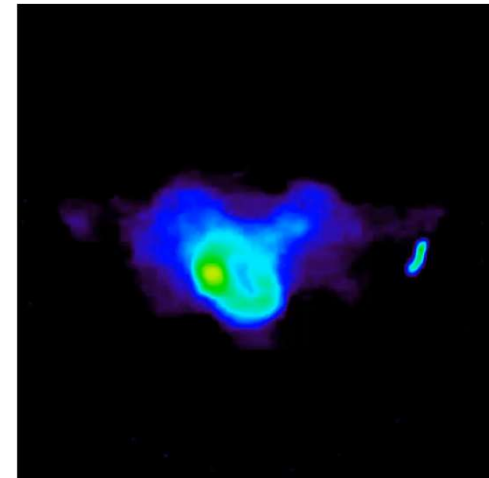
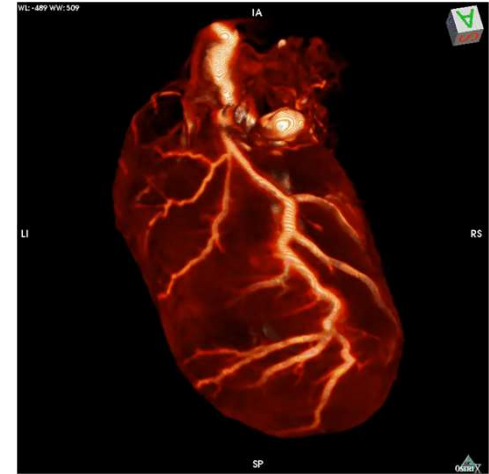
The underestimated part of imaging

Images (Videos) and their manipulation are part of our daily life

First step of image formation often underestimated, although often the enabling part, cf. **CT = Computed** Tomography

Information / quality **loss in** image formation / **reconstruction can hardly be recovered later**

Strong demand on methods for reconstruction and uncertainty quantification in many application fields, from nano to macro



Emission Tomography

Active / Passive

Idea: detect photons emitted e.g. from radioactive decay, with some kind of directional information

- Coincidence based (e.g. PET)
- Collimator based (e.g. SPECT)
- Energy based (Compton effect)
- ...

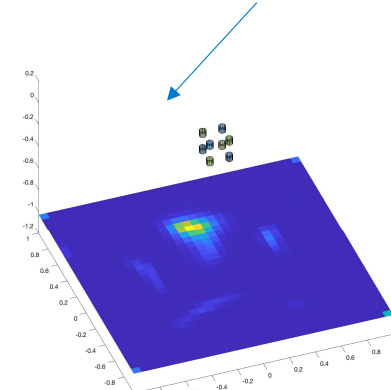
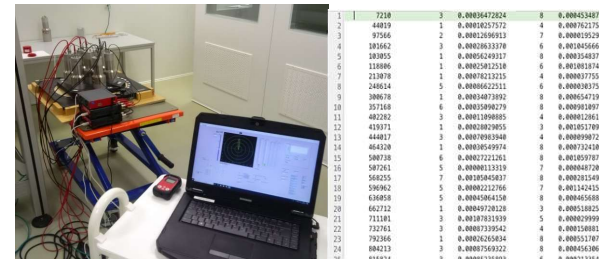
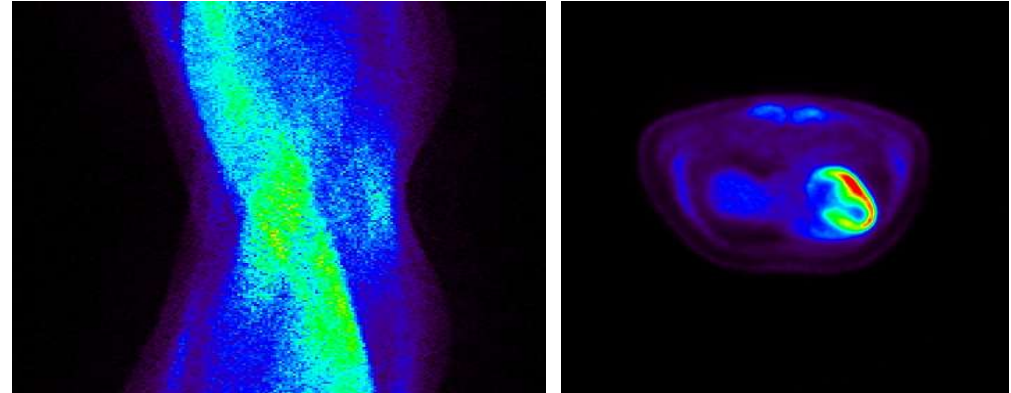
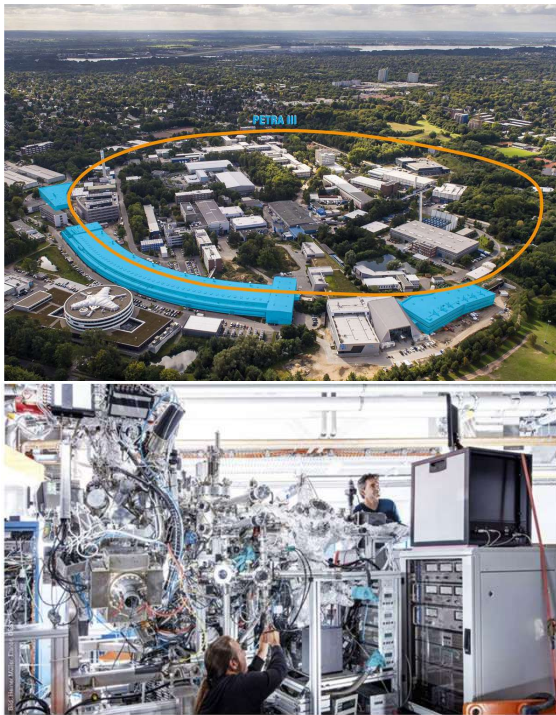


Image reconstruction from synchrotron x-ray sources

Ptychographic / Holographic Tomography

Wittwer et al 2023



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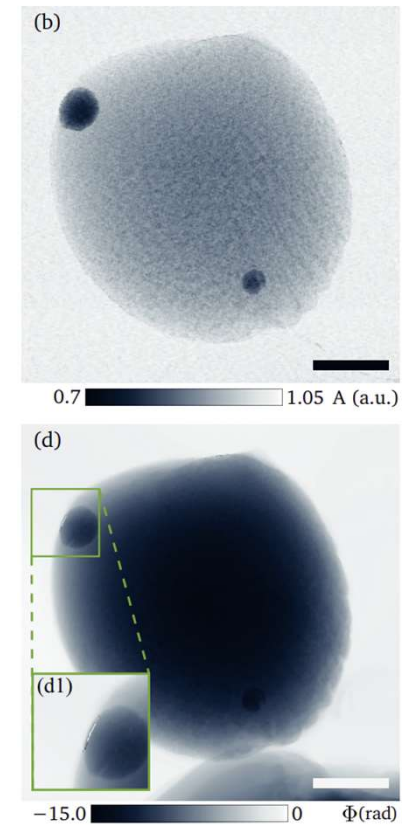
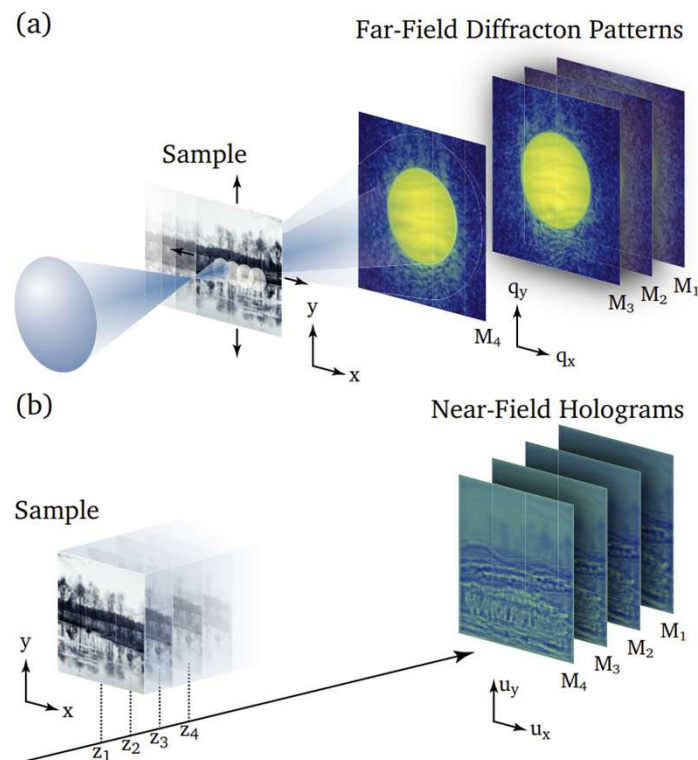
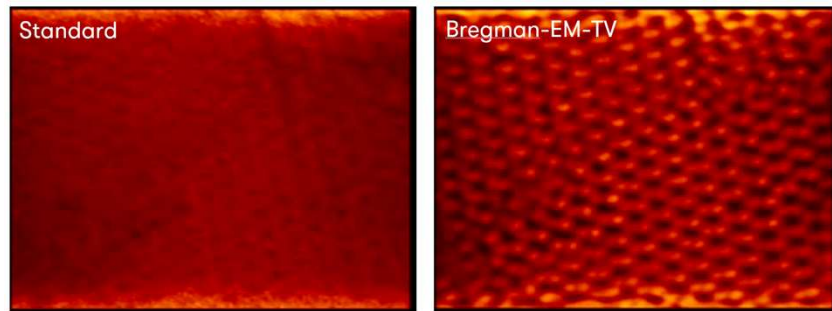
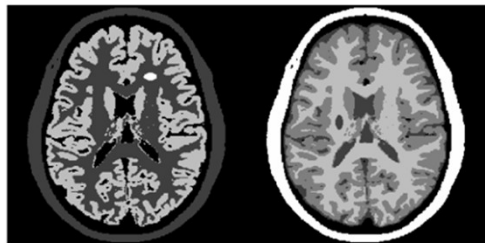


Image reconstruction across scales and planets

From nano to macro, from intracellular to outer space

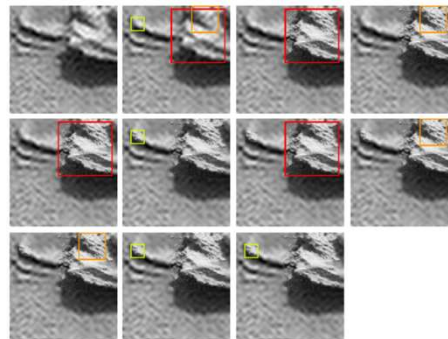


STED Deconvolution of Bead Crystal Structure (with Hell Lab, Göttingen)

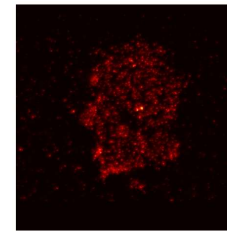


PET-MR, Rasch-Brinkmann-Burger 2017

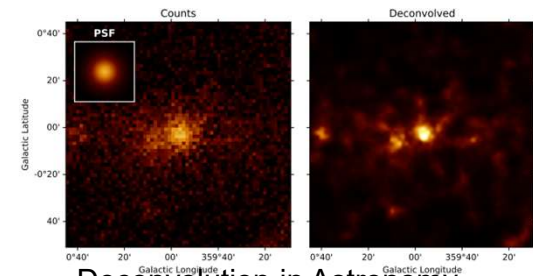
DESY.



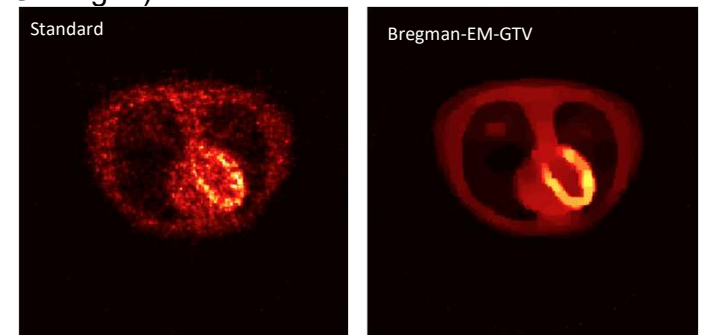
Energy Efficient THZ Imaging on Mars, with DLR Berlin



4Pi Deconvolution of Syntaxin PC12 (with Hell Lab, Göttingen)



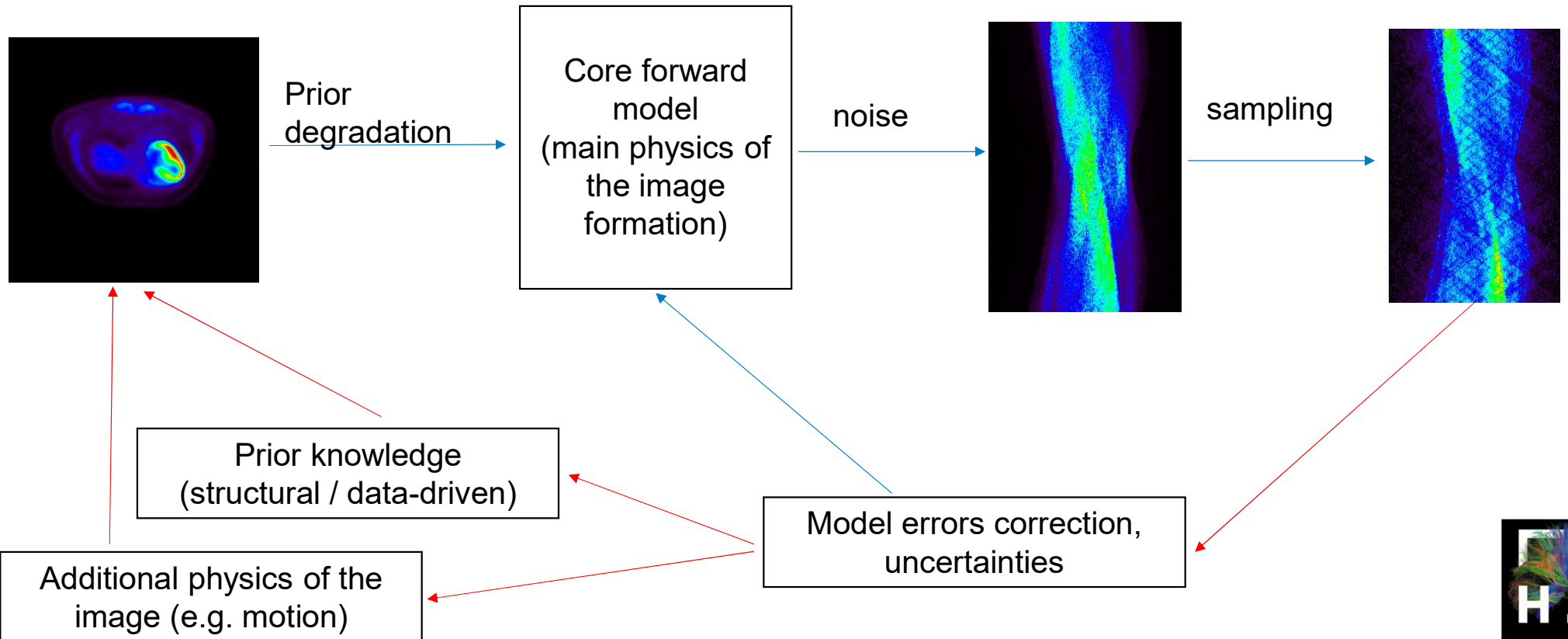
Deconvolution in Astronomy
Donath et al 2022



^{18}F -DG-PET Reconstruction from short time data
(with Nuclear Medicine, Münster)

Modern image reconstruction

Model based view



Model based approaches

The classical way of image reconstruction

Formulation as an inverse problem

- Derive physical model of (idealized) forward operator mapping from image to data
- Derive statistical model of noise (e.g. Poisson distribution for photon counts)
- Derive mathematical model of favourable images and structures (e.g. sparsity)
- Possibly add uncertainties

Condensed in Bayesian posterior model

$$\pi(u|f) = \frac{1}{\pi_*(f)} \pi(f|u) \pi_0(u)$$

Likelihood (from u to f) includes forward and noise model, **prior** includes model of favourable images

Model based variational methods

Point estimates

Bayesian **MAP** estimate

$$\hat{u} \in \arg \min_u (-\log \pi(f|u) - \log \pi_0(u))$$

Related to **variational regularization method**

$$\hat{u} \in \arg \min_u \left(F(Ku, f) + \alpha J(u) \right)$$

Simplest case: Gaussian likelihood / prior = quadratic functional = linear equation

Forward operator K , data fidelity F , regularization functional J

Forward operator: physics (examples: convolution, Radon transform, wave propagation, ...)

Data fidelity: stochastics (examples: additive Gaussian noise, Poisson distribution, ...)

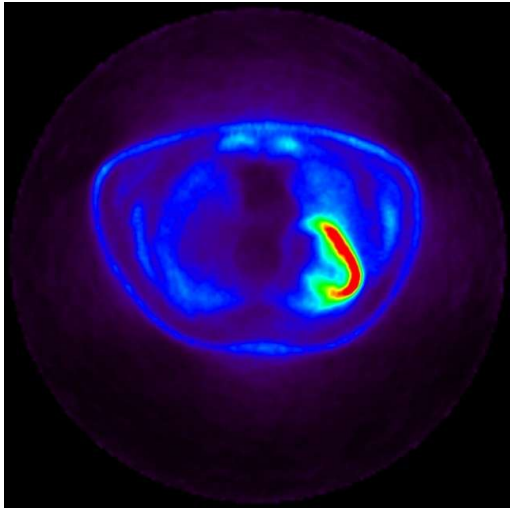
Regularization: art ? How to translate structural properties into a functional ?

Model based variational methods

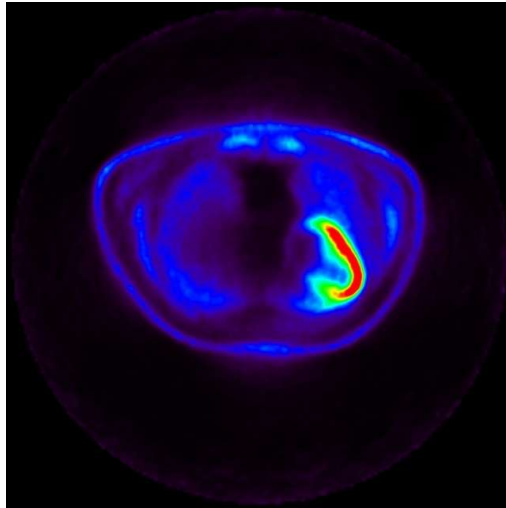
Improving forward models

Example: PET

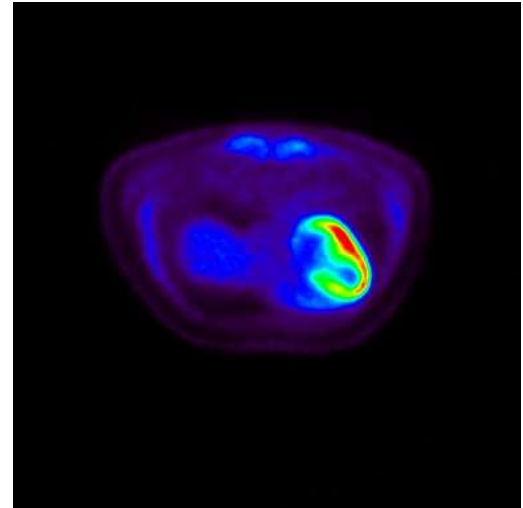
Radon + photon count
noise



Radon + photon +
scattering



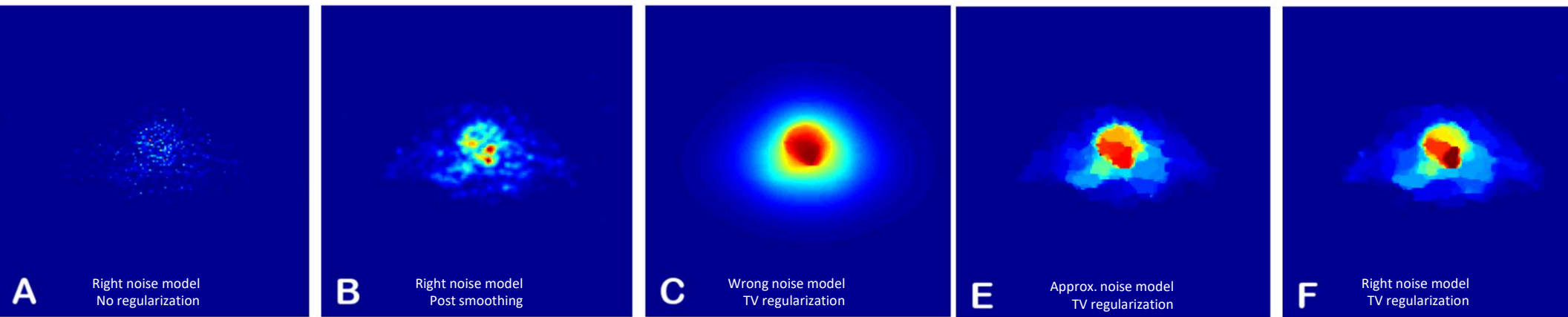
Radon + photon +
scattering + attenuation



Model based variational methods

Improving noise models

Example: PET



Cardiac $^{15}\text{H}_2\text{O}$ PET: Sawatzky, Brune, Müller, Burger 2009

Variational Models

The role of regularization

Recall variational model

$$\hat{u} \in \arg \min_u \left(F(Ku, f) + \alpha J(u) \right)$$

Optimality condition

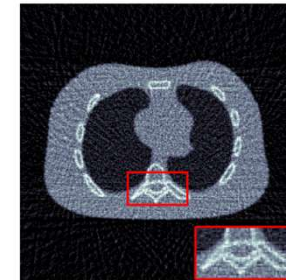
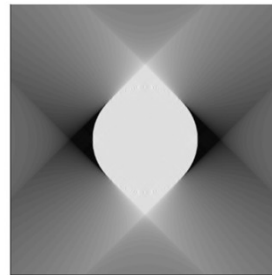
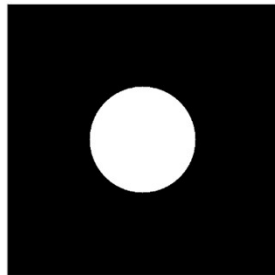
$$K^* \partial_x F(Ku, f) + \alpha p = 0, \quad p \in \partial J(u).$$

Every solution satisfies the source condition (range condition)

$$p = K^* w$$

$$J(u) = \frac{1}{2} \|u\|^2 \Rightarrow p = u$$

This is an abstract smoothness condition, determines essentially which solutions are preferred / artefacts



Folklore of Reconstruction

Use no / minimal prior knowledge

Every reconstruction method uses some prior knowledge, but often it is hidden

Example: fixed point iteration for $Ku = f$ / gradient method for least squares $\|Ku - f\|^2$

$$u^{k+1} = u^k - \tau^k K^*(Ku^k - f)$$

Compute

$$u^1 = u^0 + K^*(\tau^0 f - \tau^0 Ku^0) = u^0 + K^*w^0$$

And

$$u^2 = u^1 + K^*(\tau^1 f - \tau^1 Ku^1) = u^0 + K^*w^1 + K^*(\tau^1 f - \tau^1 Ku^1) = u^0 + K^*w^2$$

Inductively we see

$$u^k = u^0 + K^*w^k$$

Model based regularizations

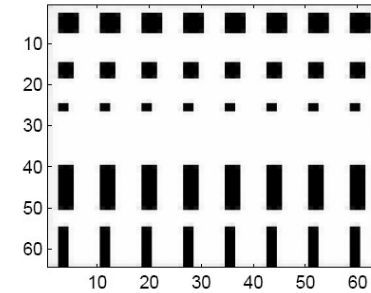
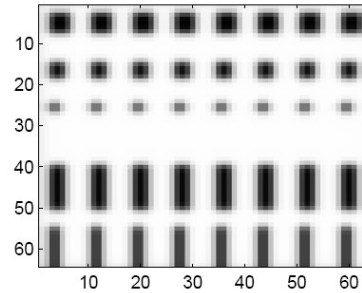
Images with sharp edges

Basic idea from denoising: want to smooth out random noise – local averaging

Simplest idea: Dirichlet energy - quadratic gradient regularization (Gaussian prior)

$$J(u) = \int |\nabla u|^2 dx$$

Leads to oversmoothing – no sharp edges



Regularity theory works against us: take $K : L^2 \rightarrow Y$

Optimality condition yields $p = -\Delta u = K^* w \in L^2$

Regularity at least $u \in H^2$ does not allow sharp edges

Model based regularizations

Images with sharp edges

Alternative idea: p-Laplacian energy

Similar regularity for $p > 1$

Limit: total variation $TV(u) = |u|_{BV} := \sup_{g \in C_0^\infty(\Omega)^d, g \in \mathcal{C}} \int_{\Omega} u \nabla \cdot g \, dx$

$$\mathcal{C} = \{g \in L^\infty(\Omega) \mid |g(x)| \leq 1 \text{ a.e. in } \Omega\}$$

Optimality condition $K^* \partial_x F(Ku, f) + \alpha \nabla \cdot g = 0$

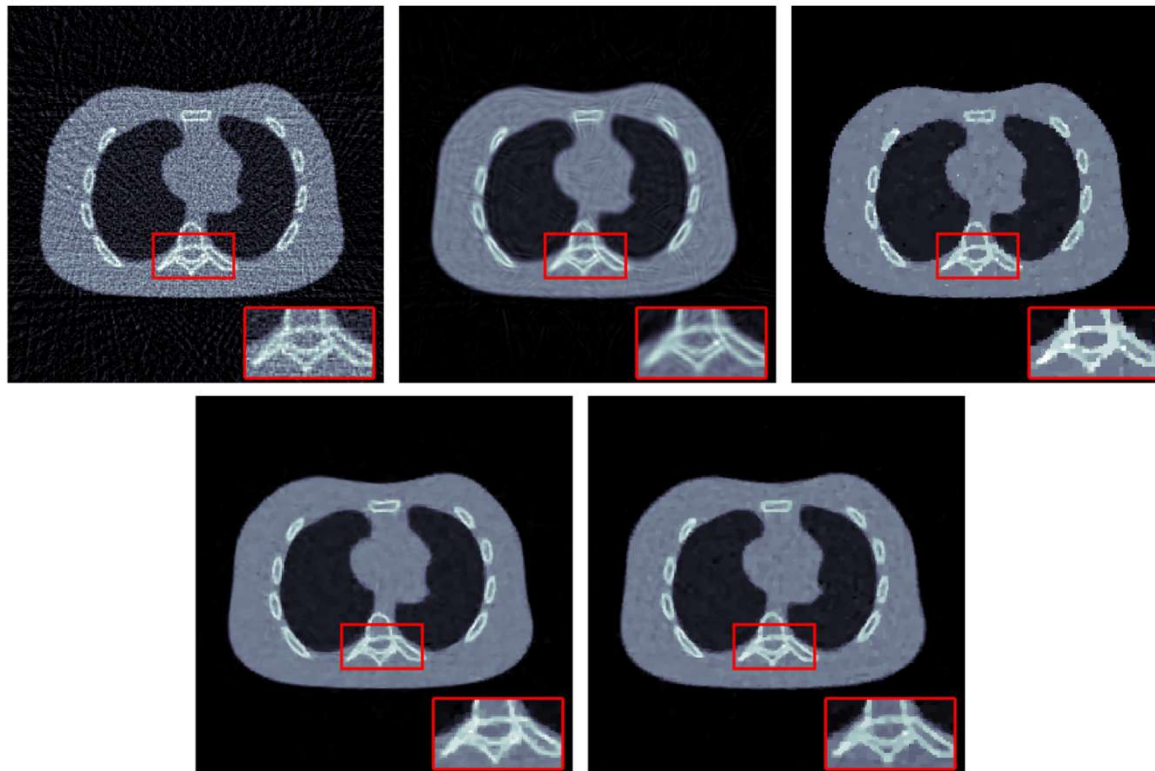
$$g \in \mathcal{C} \quad \int_{\Omega} g \cdot dDu = |u|_{BV}$$

Various extensions to cure bias (Bregman iterations) and to avoid staircasing (total generalized variation)

Model based regularizations

TV models on sparse angle tomography

Only 50 angles between -90 and 90 degrees measured (Göppel et al 2023)



Choice of regularization

Source condition

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$$\nabla \cdot g = K^* w$$

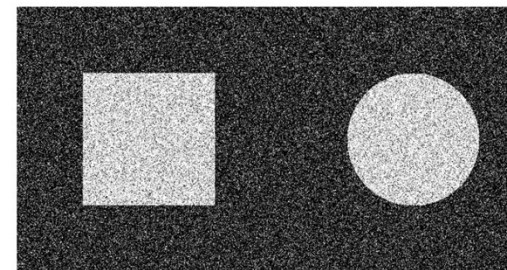
Note that g corresponds to (generalized) normal vector field on level sets (discontinuity sets) of u , its divergence equals mean curvature

Consequence: solutions of TV regularization can be discontinuous, but have nice discontinuity sets (smooth curvature)

Similar for sparsity and other one-homogeneous regularizations



(a) Test image (ground truth)



(b) Test image corrupted by additive Gaussian noise ($\mu = 0, \sigma^2 = 0.25$)



(c) Anisotropic TV denoising result ($\alpha = 10$)



(d) Isotropic TV denoising result ($\alpha = 1$)

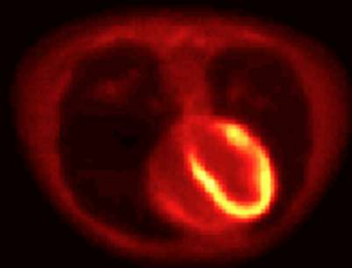
Total Variation Regularization

Example: PET reconstruction (inversion of Radon transform with Poisson noise)
[Müller et al 2013]

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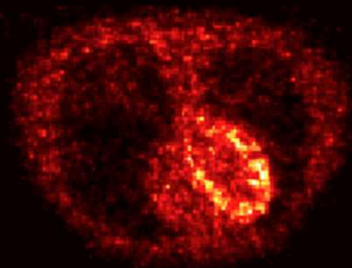
20min data
(low noise)

Standard EM



5s data
(high noise)

Standard EM



TV

