Regularisation Theory

Topic: Error Estimates for Variational Regularisation

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Assessment: Error Estimates for Variational Regularisation with Fenchel-Young Fidelities

Consider the variational regularisation problem of finding an approximate solution u_α^δ to the inverse problem $Ku = f$ from noisy data f^δ by solving the following minimisation problem

$$u_\alpha^\delta \in \arg\min_{u \in \mathcal{U}} \left\{ H_\Phi^{f^\delta}(Ku) + \alpha J(u) \right\},$$

where $\mathcal{U}$ and $\mathcal{V}$ are Hilbert spaces, $K : \mathcal{U} \to \mathcal{V}$ is a linear and bounded operator, $J : \mathcal{U} \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, and lower semi-continuous functional, and $\alpha > 0$ is the regularisation parameter.

The data fidelity term $H_\Phi^{f^\delta} : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$ is given by the Fenchel-Young fidelity,

$$H_\Phi^{f^\delta}(v) = \Phi(v) + \Phi^*(f^\delta) - \langle v, f^\delta \rangle$$

where $\Phi : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, lower semi-continuous functional, and Φ^* is its convex conjugate.

Your task is to derive an error estimate for the Bregman distance $D_J^{K^* v^\dagger}(u_\alpha^\delta, u^\dagger)$ between the regularised solution u_α^δ and a desired "true" solution $u^\dagger$.

Assumptions

1. Let $u^\dagger \in \mathcal{U}$ be a solution corresponding to exact data $f \in \mathcal{V}$, i.e., $Ku^\dagger = f$.
2. Assume a **source condition** holds for $u^\dagger$: there exists an element $v^\dagger \in \mathcal{V}$ such that

   $$K^* v^\dagger \in \partial J(u^\dagger)$$

   where $\partial J(u^\dagger)$ denotes the subdifferential of J at $u^\dagger$.
3. Assume the noise in the data is bounded in the sense of the data fidelity term, i.e.

   $$H_\Phi^{f^\delta}(f) \leq \delta^2$$

   for some noise level $\delta > 0$.

Guidance

1. Start from the optimality of u_α^δ for the minimisation problem.
2. Introduce the Bregman distance with respect to the functional J to obtain an initial inequality.
3. Add and subtract $H_\Phi^{f^\delta}$ to (and from) your right-hand-side.
4. Apply the source condition and the noise model assumption ($H_\Phi^{f^\delta}(f) \leq \delta^2$).

5. Expand the remaining fidelity terms ($H_{\Phi}^{f^\delta}$) using their definition and collect terms into logical groups.
6. Apply the Fenchel-Young **inequality**, $\langle a, b\rangle \leq \Phi(a) + \Phi^*(b)$, to simplify the expression.
7. Based on your final estimate, briefly discuss the conditions on the parameter choice rule $\alpha(\delta)$ that would ensure convergence.