PhD Winter School 2023:
ADVANCED METHODS for MATHEMATICAL IMAGE ANALYSIS

Computational Imaging Lab (B)

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Alternating Direction Method of Multipliers (ADMM)

(two- and three-blocks)

ADMM

From: Boyd et al., Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers, Foundations and Trends in Machine Learning, 2010

(file “01_ADMM_GENERAL.pdf” in “PAPERS_REPOSITORY” folder)

ADMM was originally proposed in the mid-1970s by Glowinski and Marrocco [86] and Gabay and Mercier [82]. There are a number of other important papers analyzing the properties of the algorithm, including [76, 81, 75, 87, 157, 80, 65, 33]. In particular, the convergence of ADMM has been explored by many authors, including Gabay [81] and Eckstein and Bertsekas [63].

ADMM has also been applied to a number of statistical problems, such as constrained sparse regression [18], sparse signal recovery [70], image restoration and denoising [72, 154, 134], trace norm regularized least squares minimization [174], sparse inverse covariance selection [178], the Dantzig selector [116], and support vector machines [74], among others. For examples in signal processing, see [42, 40, 41, 150, 149] and the references therein.

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Many papers analyzing ADMM do so from the perspective of *maximal monotone operators* [23, 141, 142, 63, 144]. Briefly, a wide variety of problems can be posed as finding a zero of a maximal monotone operator; for example, if f is closed, proper, and convex, then the subdifferential operator ∂f is maximal monotone, and finding a zero of ∂f is simply minimization of f; such a minimization may implicitly contain constraints if f is allowed to take the value $+\infty$. Rockafellar's *proximal point algorithm* [142] is a general method for finding a zero of a maximal monotone operator, and a wide variety of algorithms have been shown to be special cases, including proximal minimization (see §4.1), the method of multipliers, and ADMM. For a more detailed review of the older literature, see [57, §2].

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The method of multipliers was shown to be a special case of the proximal point algorithm by Rockafellar [141]. Gabay [81] showed that ADMM is a special case of a method called *Douglas-Rachford splitting* for monotone operators [53, 112], and Eckstein and Bertsekas [63] showed in turn that Douglas-Rachford splitting is a special case of the proximal point algorithm. (The variant of ADMM that performs an extra y-update between the x- and z-updates is equivalent to *Peaceman-Rachford splitting* [137, 112] instead, as shown by Glowinski and Le Tallec [87].)

ADMM (two-blocks)

Prototypical "two-blocks" optimization problem solved by **ADMM**

$$\boxed{\begin{array}{ll} \text{minimize} & f(x) + g(y) \\ \text{subject to} & Bx + Cy = c \end{array}}$$

$$x \in \mathbb{R}^n,\; y \in \mathbb{R}^m,\; B \in \mathbb{R}^{q \times n},\; C \in \mathbb{R}^{q \times m},\; c \in \mathbb{R}^q$$

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$$x\in\mathbb{R}^n,\ y\in\mathbb{R}^m,\ B\in\mathbb{R}^{q\times n},\ C\in\mathbb{R}^{q\times m},\ c\in\mathbb{R}^q$$

Main assumption:

$f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\},\ \ g:\mathbb{R}^m\to\mathbb{R}\cup\{+\infty\}$ closed, proper and convex

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- For non-convex problems, convergence is yet an open issue
- Convex constraints can be dealt with (indicator functions of closed convex sets)

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Main assumption:

$f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $g : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ closed, proper and convex

- For non-convex problems, convergence is yet an open issue
- Convex constraints can be dealt with (indicator functions of closed convex sets)
- Multi-blocks problems can also be dealt with (convergence guaranteed under more restrictive assumptions)

ADMM

"three-blocks" optimization problem solved by **ADMM**

$$\boxed{\begin{array}{ll} \text{minimize} & f(x)+g(y)+h(z) \\ \text{subject to} & Bx+Cy+Ez=c \end{array}}$$

$$x \in \mathbb{R}^n,\ y \in \mathbb{R}^m,\ z \in \mathbb{R}^p,\ B \in \mathbb{R}^{q\times n},\ C \in \mathbb{R}^{q\times m},\ E \in \mathbb{R}^{q\times p},\ c \in \mathbb{R}^q$$

ADMM (two-blocks)

"two-blocks"
problem solved
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$$x \in \mathbb{R}^n,\ y \in \mathbb{R}^m,\ \ c \in \mathbb{R}^q$$
$$B \in \mathbb{R}^{q\times n},\ C \in \mathbb{R}^{q\times m}$$

The associated **augmented Lagrangian** function:

$$L_\beta(x,y;\lambda) = f(x)+g(y)+\langle \lambda, Bx+Cy-c\rangle + \frac{\beta}{2}\|Bx+Cy-c\|_2^2$$

vector of Lagrange multipliers $\lambda \in \mathbb{R}^q$

penalty parameter $\beta \in \mathbb{R}_{++}$

ADMM (two-blocks)

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The associated **augmented Lagrangian** function:

$$L_\beta(x,y;\lambda) = f(x)+g(y)+\langle \lambda, Bx+Cy-c\rangle + \frac{\beta}{2}\|Bx+Cy-c\|_2^2$$

Solution of the original minimization problem by seeking for **saddle points** of L_β :

$$\text{find } \{x^*,y^*,\lambda^*\} \quad \text{s.t.} \quad L_\beta(x^*,y^*,\lambda) \le L_\beta(x^*,y^*,\lambda^*) \le L_\beta(x,y,\lambda^*)$$
$$\forall \{x^*,y^*,\lambda^*\} \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q$$

ADMM (two-blocks)

"two-blocks" problem solved by **ADMM**

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ADMM iterative algorithm (compute a saddle-point of L_β):

Primal descent (alternating)

$$x^{(k+1)} = \arg\min_{x\in\mathbb{R}^n} L_\beta(x, y^{(k)}; \lambda^{(k)})$$

$$y^{(k+1)} = \arg\min_{y\in\mathbb{R}^m} L_\beta(x^{(k+1)}, y; \lambda^{(k)})$$

Dual ascent

$$\lambda^{(k+1)} = \lambda^{(k)} + \beta\left(Bx^{(k+1)} + Cy^{(k+1)} - c\right)$$

ADMM (two-blocks)

"two-blocks" problem solved by **ADMM**

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$$B \in \mathbb{R}^{q \times n}, \; C \in \mathbb{R}^{q \times m}$$

The associated **augmented Lagrangian** function:

$$L_\beta(x,y;\lambda) = f(x)+g(y) - \langle \lambda, c - Bx - Cy) \rangle + \frac{\beta}{2}\| c - Bx - Cy \|_2^2$$

ADMM iterative algorithm (compute a saddle-point of L_β):

EQUIVALENT

(we use this)

Primal descent (alternating)

$$x^{(k+1)} = \arg\min_{x \in \mathbb{R}^n} L_\beta(x, y^{(k)}; \lambda^{(k)})$$

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Dual ascent

$$\lambda^{(k+1)} = \lambda^{(k)} - \beta(c - Bx^{(k+1)} - Cy^{(k+1)})$$

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$$\boxed{\begin{array}{ll} \text{minimize} & f(x)+g(y)+h(z) \\ \text{subject to} & Bx+Cy+Ez=c \end{array}}$$

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ADMM (three-blocks)

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$B \in \mathbb{R}^{q \times n},\ C \in \mathbb{R}^{q \times m},\ E \in \mathbb{R}^{q \times p}$

The associated **augmented Lagrangian** function:

$$L_\beta(x, y, z; \lambda) = f(x) + g(y) - \langle \lambda, c - Bx - Cy - Ez \rangle + \frac{\beta}{2} \| c - Bx - Cy - Ez \|_2^2$$

ADMM (three-blocks)

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Solution of the original minimization problem by seeking for **saddle points** of L_β :

$$\text{find } \{x^*, y^*, z^*, \lambda^*\} \quad \text{s.t.} \quad L_\beta\left(x^*, y^*, z^*, \lambda\right) \le L_\beta\left(x^*, y^*, z^*, \lambda^*\right) \le L_\beta\left(x, y, z, \lambda^*\right)$$

$$\forall \{x^*, y^*, z^*, \lambda^*\} \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^q$$

ADMM (three-blocks)

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Dual ascent

$$\lambda^{(k+1)} = \lambda^{(k)} - \beta\left(c - Bx^{(k+1)} - Cy^{(k+1)} - Ez^{(k+1)}\right)$$

ADMM convergence

- Convergence of ADMM in the two- and three-blocks cases has been studied in different manners
- For the two-blocks case, you can refer to the standard proof given in

  Boyd et al., Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers, Foundations and Trends in Machine Learning, 2010

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- For the three-blocks case (for a special situation, with a generalization on the usage of different penalty parameters β for different constraints), refer to the proof in

  C. Wu et al., Augmented Lagrangian Method for Total Variation Restoration with Non-quadratic Fidelity, Inverse Problems and Imaging, 2011

  (file “02_TV_NQ_FIDELITY_ADMM.pdf” in “PAPERS_REPOSITORY” folder)

Alternating Direction Method of Multipliers (ADMM)

for the numerical solution of the

(unconstrained)

TV-L_2 , TIK-L_1 , TV-L_1 models

ADMM (two-blocks)

for the numerical solution of the

(unconstrained)

TV-L_2 model

ADMM for TV-L_2 model

The **unconstrained convex non-smooth** model:

$$u^* = \arg\min_{u\in\mathbb{R}^d} \left\{ J(u) = \frac{\mu}{2}\|Au-b\|_2^2 + \sum_{i=1}^{d}\sqrt{(D_h u)_i^2 + (D_v u)_i^2} \right\}$$

Variables splitting: $t = Du \Leftrightarrow \begin{pmatrix} t_h \\ t_v \end{pmatrix} = \begin{pmatrix} D_h u \\ D_v u \end{pmatrix}, \quad t_i \triangleq \begin{pmatrix} t_{h,i} \\ t_{v,i} \end{pmatrix} = \begin{pmatrix} (D_h u)_i \\ (D_v u)_i \end{pmatrix} \in \mathbb{R}^2$

Split (linearly constrained) **model:**

$$\{u^*, t^*\} = \arg\min_{u\in\mathbb{R}^d,\, t\in\mathbb{R}^{2d}} \left\{ G(u,t) = \frac{\mu}{2}\|Au-b\|_2^2 + \sum_{i=1}^{d}\|t_i\|_2 \right\} \quad \text{subject to (s.t.)} \;\; t = Du$$

ADMM for TV-L_2 model

Split (linearly constrained) **model:**

$$\left\{u^*, t^*\right\} = \arg\min_{u \in \mathbb{R}^d,\, t \in \mathbb{R}^{2d}} \left\{ G(u,t) = \frac{\mu}{2}\left\|Au - b\right\|_2^2 + \sum_{i=1}^{d}\left\|t_i\right\|_2 \right\} \quad \text{s.t.} \quad t = Du$$

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It can be equivalently rewritten as:

$$\{u^*, t^*\} = \underset{u \in \mathbb{R}^d,\, t \in \mathbb{R}^{2d}}{\arg\min} \left\{ G(u,t) = f(u) + g(t) \right\} \quad \text{s.t.} \quad Du + (-I_{2d}\, t) = 0_{2d}$$

$$\text{with} \quad f(u) = \frac{\mu}{2}\|Au - b\|_2^2, \quad g(t) = \sum_{i=1}^{d} \|t_i\|_2, \quad \text{both closed, proper, convex}$$

ADMM for TV-L_2 model

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Is it a standard
"two-blocks" problem?

$$\begin{array}{ll} \text{minimize} & f(x) + g(y) \\ \text{subject to} & Bx + Cy = c \end{array}$$

ADMM for TV-L_2 model

Split (linearly constrained) **model:**

$$\left\{u^*, t^*\right\} = \underset{u \in \mathbb{R}^d,\, t \in \mathbb{R}^{2d}}{\arg\min} \left\{ G(u,t) = \frac{\mu}{2}\|Au - b\|_2^2 + \sum_{i=1}^{d} \|t_i\|_2 \right\} \quad \text{s.t.} \quad t = Du$$

It can be equivalently rewritten as:

$$\boxed{\begin{array}{l} \left\{u^*, t^*\right\} = \underset{u \in \mathbb{R}^d,\, t \in \mathbb{R}^{2d}}{\arg\min} \left\{ G(u,t) = f(u) + g(t) \right\} \quad \text{s.t.} \quad Du + (-I_{2d}\, t) = 0_{2d} \\ \text{with} \quad f(u) = \frac{\mu}{2}\|Au - b\|_2^2, \quad g(t) = \sum_{i=1}^{d} \|t_i\|_2, \quad \text{both closed, proper, convex} \end{array}}$$

Is it a standard "two-blocks" problem? **YES!**

$$\boxed{x = u,\; y = t,\; B = D,\; C = -I_{2d},\; c = 0_{2d}}$$

$$\boxed{\begin{array}{ll} \text{minimize} & f(x) + g(y) \\ \text{subject to} & Bx + Cy = c \end{array}}$$

ADMM for TV-L_2 model

Split (linearly constrained) **model:**

$$\left\{u^*,t^*\right\} = \underset{u\in\mathbb{R}^d,\, t\in\mathbb{R}^{2d}}{\arg\min} \left\{ G(u,t) = \frac{\mu}{2}\left\|Au-b\right\|_2^2 + \sum_{i=1}^{d}\left\|t_i\right\|_2 \right\} \quad \text{s.t.} \quad t = Du$$

$$\lambda = \begin{pmatrix}\lambda_h\\ \lambda_v\end{pmatrix} \in \mathbb{R}^{2d}$$

vector of Lagrange multipliers

The **augmented Lagrangian** function:

$$L(u,t;\lambda) = \frac{\mu}{2}\left\|Au-b\right\|_2^2 + \sum_{i=1}^{d}\left\|t_i\right\|_2 - \langle \lambda, t-Du\rangle + \frac{\beta}{2}\left\|t-Du\right\|_2^2$$

penalty parameter $\beta > 0$

Solution image as the **saddle point** of the above function, that is:

$$\text{find } \left\{u^*,t^*,\lambda^*\right\} \quad \text{s.t.} \quad L\left(u^*,t^*,\lambda\right) \le L\left(u^*,t^*,\lambda^*\right) \le L\left(u,t,\lambda^*\right)$$

$$\forall (u,t,\lambda) \in \mathbb{R}^d \times \mathbb{R}^{2d} \times \mathbb{R}^{2d}$$

ADMM for TV-L_2 model

Augmented Lagrangian function:

$$L(u,t;\lambda) = \frac{\mu}{2}\|Au-b\|_2^2 + \sum_{i=1}^{d}\|t_i\|_2 - \langle \lambda, t - Du\rangle + \frac{\beta}{2}\|t-Du\|_2^2$$ … saddle point ??

ADMM-based iterative algorithm (split optimization sub-problems):

Primal descent (alternating)

$$t^{(k+1)} = \arg\min_{t\in R^{2d}} L(u^{(k)}, t; \lambda^{(k)})$$ <------- closed-form proximal map

$$u^{(k+1)} = \arg\min_{u\in R^{d}} L(u, t^{(k+1)}; \lambda^{(k)})$$ <------ s.p.d. linear system

Dual ascent

$$\lambda^{(k+1)} = \lambda^{(k)} - \beta\,(t^{(k+1)} - Du^{(k+1)})$$ <--- closed-form

Primal descent: subproblem for t

Augmented Lagrangian function:

$$L(u,t;\lambda) = \frac{\mu}{2}\|Au-b\|_2^2 + \sum_{i=1}^{d}\|t_i\|_2 - \langle \lambda, t-Du \rangle + \frac{\beta}{2}\|t-Du\|_2^2$$

Subproblem for primal variable ***t*** :

$$t^{(k+1)} = \arg\min_{t \in \mathbb{R}^{2d}} L(u^{(k)}, t; \lambda^{(k)})$$ … drop the terms which do not depend on t …

$$= \arg\min_{t \in \mathbb{R}^{2d}} \left\{ \sum_{i=1}^{d}\|t_i\|_2 - \langle \lambda^{(k)}, t-Du^{(k)} \rangle + \frac{\beta}{2}\|t-Du^{(k)}\|_2^2 \right\}$$

$$= \arg\min_{t \in \mathbb{R}^{2d}} \left\{ \sum_{i=1}^{d}\|t_i\|_2 + \frac{\beta}{2}\|t-q^{(k)}\|_2^2 \right\}, \quad q^{(k)} = Du^{(k)} + \frac{1}{\beta}\lambda^{(k)} \in \mathbb{R}^{2d}$$

Primal descent: subproblem for t

… the cost function is the sum of *d* “separate” bivariate functions, in fact:

$$t^{(k+1)} = \arg\min_{t\in\mathbb{R}^{2d}} \sum_{i=1}^{d} \left\{ \left\| t_i \right\|_2 + \frac{\beta}{2} \left\| t_i - q_i^{(k)} \right\|_2^2 \right\}, \qquad q_i^{(k)} = \left(\left(Du^{(k)} \right)_i + \frac{1}{\beta} \lambda_i^{(k)} \right) \in \mathbb{R}^2, \quad i = 1, \ldots, d$$

… the problem thus reduces to *d* independent bivariate minimizations:

$$t_i^{(k+1)} = \arg\min_{t_i\in\mathbb{R}^2} \left\{ \left\| t_i \right\|_2 + \frac{\beta}{2} \left\| t_i - q_i^{(k)} \right\|_2^2 \right\}, \qquad i = 1, \ldots, d$$

… which all admit closed-form solution (see Proposition 1 in the next slide):

$$t_i^{(k+1)} = \max\left\{ \left\| q_i^{(k)} \right\|_2 - \frac{1}{\beta}, 0 \right\} \frac{q_i^{(k)}}{\left\| q_i^{(k)} \right\|_2}, \qquad i = 1, \ldots, d$$

linear computational complexity O(d)

Primal descent: subproblem for t

Proposition 1 (proximal map of $\|x\|_2$)

Let $\alpha \in \mathbb{R}_{++}$, $v \in \mathbb{R}^2$ be given constants. Then, the optimization problem

$$x^* = \operatorname{prox}^{\alpha}_{\|x\|_2}(v) = \arg\min_{x \in \mathbb{R}^2} \left\{ \theta(\mathrm{x}; v) = \|x\|_2 + \frac{\alpha}{2} \|x - v\|_2^2 \right\}$$

is strongly convex and admits the unique solution given by the following "shrinkage" (or soft-thresholding) operator:

$$x^* = \max\left\{ \|v\|_2 - \frac{1}{\alpha}, 0 \right\} \frac{v}{\|v\|_2}$$

(where $0 \cdot 0 / 0 = 0$ is assumed)

Primal descent: subproblem for u

Augmented Lagrangian function:

$$L(u,t;\lambda) = \frac{\mu}{2}\left\|Au-b\right\|_2^2 + \sum_{i=1}^{d}\left\|t_i\right\|_2 - \left\langle \lambda, t-Du \right\rangle + \frac{\beta}{2}\left\|t-Du\right\|_2^2$$

Subproblem for primal variable ***u***:

$$u^{(k+1)} = \arg\min_{u\in\mathbb{R}^d} \; L(u, t^{(k+1)}; \lambda^{(k)})$$ … drop the terms which do not depend on *u* …

$$= \arg\min_{u\in\mathbb{R}^d} \left\{ Z(u) = \frac{\mu}{2}\left\|Au-b\right\|_2^2 + \left\langle \lambda^{(k)}, Du \right\rangle + \frac{\beta}{2}\left\|t^{(k+1)} - Du\right\|_2^2 \right\}$$

The function *Z* is quadratic in the optimization variable *u*, hence its global minimizers (if there exists one) are to be sought among its stationary points:

$$u^{(k+1)} \in \text{ set of solutions of : } \quad \nabla Z(u) = 0_d$$

Primal descent: subproblem for u

… after some simple algebraic manipulations:

$$\nabla Z(u) = 0_d \quad \Leftrightarrow \quad \left(D^T D + \frac{\mu}{\beta} A^T A \right) u = D^T \left(t^{(k+1)} - \frac{1}{\beta} \lambda^{(k)} \right) + \frac{\mu}{\beta} A^T b$$

The coefficient matrix is almost identical to the one previously obtained for the TIK-L_2 model (see…): it is symmetric, positive definite, and has full rank, hence the linear system admits a unique solution giving the new iterate $u^{(k+1)}$

RESTORATION:

Assuming periodic/reflective/anti-reflective boundary conditions for *u*, the linear system can be solved (like for TIK-L_2 case) by 2D DFT/DCT/DST.

By using 2D FFT/FCT/FST implementations $\longrightarrow$ computational complexity O(d log d)

Primal descent: subproblem for u

… after some simple algebraic manipulations:

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INPAINTING:

the linear system can be solved (like for TIK-L_2 case) by iterative (P)CG

ADMM (two-blocks)

for the numerical solution of the

(unconstrained)

TIK-L_1 model

ADMM for TIK-L_1 model

The **unconstrained** **convex** **non-smooth** model:

$$u^* = \arg\min_{u \in \mathbb{R}^d} \left\{ J(u) = \mu \left\| Au - b \right\|_1 + \frac{1}{2} \left\| Du \right\|_2^2 \right\}$$

Variables splitting: $r = Au - b \in \mathbb{R}^d$ … residue image (noise image estimate)

Split (linearly constrained) model:

$$\left\{ u^*, r^* \right\} = \arg\min_{u \in R^d,\, r \in R^d} \left\{ G(u, r) = \mu \left\| r \right\|_1 + \frac{1}{2} \left\| Du \right\|_2^2 \right\} \quad \text{s.t.} \quad r = Au - b$$

ADMM for TIK-L_1 model

Split (linearly constrained) **model:**

$$\left\{u^*, r^*\right\} = \arg\min_{u \in R^d,\, r \in R^d} \left\{ G(u,r) = \mu \left\| r \right\|_1 + \frac{1}{2} \left\| Du \right\|_2^2 \right\} \quad \text{s.t.} \quad r = Au - b$$

ADMM for TIK-L_1 model

Split (linearly constrained) **model:**

$$\{u^*, r^*\} = \arg\min_{u\in R^d,\, r\in R^d} \left\{ G(u,r) = \mu \|r\|_1 + \frac{1}{2}\|Du\|_2^2 \right\} \quad \text{s.t.} \quad r = Au - b$$

It can be equivalently rewritten as:

$$\boxed{\begin{array}{l} \{u^*, r^*\} = \arg\min\limits_{u\in R^d,\, r\in R^d} \{ G(u,r) = f(u) + g(r) \} \quad \text{s.t.} \quad Au + (-I_d\, r) = b \\ \quad \text{with} \quad f(u) = \frac{1}{2}\|Du\|_2^2, \quad g(r) = \mu \|r\|_1, \quad \text{both closed, proper, convex} \end{array}}$$

ADMM for TIK-L_1 model

Split (linearly constrained) **model:**

$$\{u^*, r^*\} = \underset{u \in R^d,\, r \in R^d}{\arg\min} \left\{ G(u,r) = \mu \|r\|_1 + \frac{1}{2}\|Du\|_2^2 \right\} \quad \text{s.t.} \quad r = Au - b$$

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Is it a standard
"two-blocks" problem?

$$\boxed{\begin{array}{ll} \text{minimize} & f(x) + g(y) \\ \text{subject to} & Bx + Cy = c \end{array}}$$

ADMM for TIK-L_1 model

Split (linearly constrained) **model:**

$$\{u^*, r^*\} = \arg\min_{u \in R^d,\, r \in R^d} \left\{ G(u,r) = \mu \|r\|_1 + \frac{1}{2}\|Du\|_2^2 \right\} \quad \text{s.t.} \quad r = Au - b$$

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Is it a standard "two-blocks" problem?

YES!

$$\boxed{x = u,\ y = r,\ B = A,\ C = -I_d,\ c = b}$$

$$\boxed{\begin{array}{ll} \text{minimize} & f(x) + g(y) \\ \text{subject to} & Bx + Cy = c \end{array}}$$

ADMM for TIK-L_1 model

Split (linearly constrained) model:

$$\{u^*, r^*\} = \arg\min_{u \in R^d,\, r \in R^d} \left\{ G(u,r) = \mu \|r\|_1 + \frac{1}{2}\|Du\|_2^2 \right\} \quad \text{s.t.} \quad r = Au - b$$

vector of Lagrange multipliers $\lambda \in \mathbb{R}^d$

The **augmented Lagrangian** function:

$$L(u,r;\lambda) = \mu \|r\|_1 + \frac{1}{2}\|Du\|_2^2 - \langle \lambda, r - (Au - b) \rangle + \frac{\beta}{2}\|r - (Au - b)\|_2^2$$

penalty parameter $\beta > 0$

Solution image as the **saddle point** of the above function, that is:

$$\text{find } \{u^*, r^*, \lambda^*\} \quad \text{s.t.} \quad L(u^*, r^*, \lambda) \le L(u^*, r^*, \lambda^*) \le L(u, r, \lambda^*)$$
$$\forall (u, r, \lambda) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$$

ADMM for TIK-L_1 model

Augmented Lagrangian function: **… saddle point ??**

$$L(u,r;\lambda) = \mu \|r\|_1 + \frac{1}{2}\|Du\|_2^2 - \langle \lambda, r-(Au-b)\rangle + \frac{\beta}{2}\|r-(Au-b)\|_2^2$$

ADMM-based iterative algorithm (split optimization sub-problems):

Primal descent (alternating)

$$r^{(k+1)} = \arg\min_{r\in R^d} L(u^{(k)}, r; \lambda^{(k)})$$ ← closed-form proximal map

$$u^{(k+1)} = \arg\min_{u\in R^d} L(u, r^{(k+1)}; \lambda^{(k)})$$ ← s.p.d. linear system

Dual ascent

$$\lambda^{(k+1)} = \lambda^{(k)} - \beta\,(r^{(k+1)} - (Au^{(k+1)} - b))$$ ← closed-form

Primal descent: subproblem for r

Augmented Lagrangian function:

$$L(u,r;\lambda) = \mu \|r\|_1 + \frac{1}{2}\|Du\|_2^2 - \langle \lambda, r - (Au - b) \rangle + \frac{\beta}{2}\|r - (Au - b)\|_2^2$$

Subproblem for primal variable ***r*** :

$$r^{(k+1)} = \arg\min_{r \in \mathbb{R}^d} L(u^{(k)}, r; \lambda^{(k)})$$ … drop the terms which do not depend on *r* …

$$= \arg\min_{r \in \mathbb{R}^d} \left\{ \mu \|r\|_1 - \left\langle \lambda^{(k)}, r - \left(Au^{(k)} - b\right) \right\rangle + \frac{\beta}{2}\left\| r - \left(Au^{(k)} - b\right) \right\|_2^2 \right\}$$

$$= \arg\min_{r \in \mathbb{R}^d} \left\{ \|r\|_1 + \frac{\gamma}{2}\left\| r - q^{(k)} \right\|_2^2 \right\}, \quad q^{(k)} = Au^{(k)} - b + \frac{1}{\beta}\lambda^{(k)} \in \mathbb{R}^d$$

$$\gamma = \beta / \mu \in \mathbb{R}_{++}$$

Primal descent: subproblem for r

… the cost function is the sum of *d* “separate” univariate functions, in fact:

$$r^{(k+1)} = \arg\min_{r \in \mathbb{R}^d} \sum_{i=1}^{d} \left\{ |r_i| + \frac{\gamma}{2}\left(r_i - q_i^{(k)}\right)^2 \right\}, \qquad q_i^{(k)} = \left(\left(Au^{(k)} - b\right)_i + \frac{1}{\beta}\lambda_i^{(k)} \right) \in \mathbb{R},$$

$$i = 1, \ldots, d$$

… the problem thus reduces to *d* independent univariate minimizations:

$$r_i^{(k+1)} = \arg\min_{r_i \in \mathbb{R}} \left\{ |r_i| + \frac{\gamma}{2}\left(r_i - q_i^{(k)}\right)^2 \right\}, \qquad i = 1, \ldots, d$$

… which all admit closed-form solution (see Proposition 2 in the next slide):

$$r_i^{(k+1)} = \operatorname{sign}\left(q_i^{(k)}\right) \max\left\{ \left|q_i^{(k)}\right| - \frac{1}{\gamma}, 0 \right\}, \qquad i = 1, \ldots, d$$

linear computational complexity O(d)

Primal descent: subproblem for t

Proposition 2 (proximal map of |x|)

Let $\alpha \in \mathbb{R}_{++}$, $v \in \mathbb{R}$ be given constants. Then, the optimization problem

$$x^* = \operatorname{prox}_{|x|}^{\alpha}(v) = \arg\min_{x \in \mathbb{R}} \left\{ \theta(\mathrm{x}; v) = |x| + \frac{\alpha}{2}(x - v)^2 \right\}$$

is strongly convex and admits the unique solution given by the following "shrinkage" (or soft-thresholding) operator:

$$x^* = \operatorname{sign}(v) \cdot \max\left\{ |v| - \frac{1}{\alpha}, 0 \right\}$$

where $\operatorname{sign}(v) = -1$ for $v < 0$, $+1$ for $v > 0$, 0 for $v = 0$

Primal descent: subproblem for u

Augmented Lagrangian function:

$$L(u,r;\lambda) = \mu \|r\|_1 + \frac{1}{2}\|Du\|_2^2 - \langle \lambda, r-(Au-b)\rangle + \frac{\beta}{2}\|r-(Au-b)\|_2^2$$

Subproblem for primal variable ***u*****:**

$$u^{(k+1)} = \arg\min_{u\in\mathbb{R}^d} L(u, r^{(k+1)}; \lambda^{(k)})$$ … drop the terms which do not depend on *u* …

$$= \arg\min_{u\in\mathbb{R}^d} \left\{ Z(u) = \frac{1}{2}\|Du\|_2^2 + \langle \lambda^{(k)}, Au\rangle + \frac{\beta}{2}\left\|r^{(k+1)} - (Au-b)\right\|_2^2 \right\}$$

The function *Z* is quadratic in the optimization variable *u*, hence its global minimizers (if there exists one) are to be sought among its stationary points:

$$u^{(k+1)} \in \text{ set of solutions of : } \quad \nabla Z(u) = 0_d$$

Primal descent: subproblem for u

… after some simple algebraic manipulations:

$$\nabla Z(u) = 0_d \quad \Leftrightarrow \quad \left(\frac{1}{\beta} D^T D + A^T A \right) u = A^T \left(r^{(k+1)} - \frac{1}{\beta} \lambda^{(k)} + b \right)$$

The coefficient matrix is almost identical to the one previously obtained for the TIK-L_2 and TV-L_2 models (see…): it is s. p. d. and has full rank, hence the linear system admits a unique solution giving the new iterate $u^{(k+1)}$

RESTORATION:

$\Downarrow$

Assuming periodic/reflective/anti-reflective boundary conditions for *u*, the linear system can be solved (like for TIK-L_2 case) by 2D DFT/DCT/DST.

By using 2D FFT/FCT/FST implementations $\longrightarrow$ computational complexity O(d log d)

Primal descent: subproblem for u

… after some simple algebraic manipulations:

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INPAINTING:

$\Downarrow$

the linear system can be solved (like for TIK-L_2 case) by iterative (P)CG

ADMM (three-blocks → two-blocks)

for the numerical solution of the

(unconstrained)

TV-L_1 model

ADMM for TV-L_1 model

The **unconstrained convex non-smooth** model:

$$u^* \in \arg\min_{u \in \mathbb{R}^d} \left\{ J(u) = \mu \left\| Au - b \right\|_1 + \sum_{i=1}^{d} \sqrt{\left(D^{(h)}u\right)_i^2 + \left(D^{(v)}u\right)_i^2} \right\}$$

Variables splitting:

$$t = Du \Leftrightarrow \begin{pmatrix} t_h \\ t_v \end{pmatrix} = \begin{pmatrix} D_h u \\ D_v u \end{pmatrix}, \quad t_i \triangleq \begin{pmatrix} t_{h,i} \\ t_{v,i} \end{pmatrix} = \begin{pmatrix} (D_h u)_i \\ (D_v u)_i \end{pmatrix} \in \mathbb{R}^2$$

$$r = Au - b \in \mathbb{R}^d$$

Split (linearly constrained) model:

$$\left\{u^*, t^*, r^*\right\} \in \arg\min_{u, r \in \mathbb{R}^d, t \in \mathbb{R}^{2d}} \left\{ G(u,t,r) = \mu \left\| r \right\|_1 + \sum_{i=1}^{d} \left\| t_i \right\|_2 \right\} \text{ s.t. } t = Du, \ r = Au - b$$

ADMM for TV-L_1 model

Split (linearly constrained) **model:**

$$\{u^*, t^*, r^*\} \in \arg\min_{u,r\in\mathbb{R}^d, t\in\mathbb{R}^{2d}} \left\{ G(u,t,r) = \mu \|r\|_1 + \sum_{i=1}^{d} \|t_i\|_2 \right\} \text{ s.t. } t = Du, \; r = Au - b$$

ADMM for TV-L_1 model

Split (linearly constrained) **model:**

$$\left\{u^*,t^*,r^*\right\} \in \arg\min_{u,r\in\mathbb{R}^d,\, t\in\mathbb{R}^{2d}} \left\{ G(u,t,r) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 \right\} \text{ s.t. } t = Du,\; r = Au - b$$

It can be equivalently rewritten as:

$$\left\{u^*,t^*,r^*\right\} \in \arg\min_{u,r\in\mathbb{R}^d,\, t\in\mathbb{R}^{2d}} \left\{ G(u,t,r) = f(u) + g(t) + h(r) \right\}$$

$$\text{s.t. } \begin{pmatrix} D \\ A \end{pmatrix} u + \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t + \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r = \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} \text{ with } f(u)=0,\; g(t) = \sum_{i=1}^{d}\|t_i\|_2,\; h(r) = \mu\|r\|_1,$$

all closed, proper, convex

ADMM for TV-L_1 model

Split (linearly constrained) **model:**

$$\left\{u^*,t^*,r^*\right\} \in \underset{u,r\in\mathbb{R}^d,t\in\mathbb{R}^{2d}}{\arg\min} \left\{ G(u,t,r) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 \right\} \text{ s.t. } t = Du,\; r = Au - b$$

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$$\text{s.t. } \begin{pmatrix} D \\ A \end{pmatrix} u + \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t + \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r = \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} \text{ with } \begin{array}{c} f(u)=0,\; g(t) = \sum_{i=1}^{d}\|t_i\|_2,\; h(r) = \mu\|r\|_1, \\ \text{all closed, proper, convex} \end{array}$$

Is it a standard
"three-blocks" problem?

$$\begin{aligned} &\text{minimize} && f(x) + g(y) + h(z) \\ &\text{subject to} && Bx + Cy + Ez = c \end{aligned}$$

ADMM for TV-L_1 model

Split (linearly constrained) **model:**

$$\{u^*, t^*, r^*\} \in \arg\min_{u,r\in\mathbb{R}^d, t\in\mathbb{R}^{2d}} \left\{ G(u,t,r) = \mu \|r\|_1 + \sum_{i=1}^{d} \|t_i\|_2 \right\} \text{ s.t. } t = Du,\ r = Au - b$$

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$$\text{s.t. } \begin{pmatrix} D \\ A \end{pmatrix} u + \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t + \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r = \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} \text{ with } f(u) = 0,\ g(t) = \sum_{i=1}^{d} \|t_i\|_2,\ h(r) = \iota_{B(\delta)}(r),$$

all closed, proper, convex

$$x = u,\ y = t,\ z = r,$$

$$B = \begin{pmatrix} D \\ A \end{pmatrix},\ C = \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix},\ E = \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix},\ c = \begin{pmatrix} 0_{2d} \\ b \end{pmatrix}$$

Is it a standard "three-blocks" problem?

YES!

$$\begin{aligned} &\text{minimize} && f(x) + g(y) + h(z) \\ &\text{subject to} && Bx + Cy + Ez = c \end{aligned}$$

ADMM for TV-L_1 model

Split (linearly constrained) **model:**

$$\left\{u^*,t^*,r^*\right\} \in \arg\min_{u,r\in\mathbb{R}^d,\, t\in\mathbb{R}^{2d}} \left\{ G(u,t,r) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 \right\} \text{ s.t. } t = Du,\; r = Au - b$$

It can be equivalently rewritten as:

$$\left\{u^*,t^*,r^*\right\} \in \arg\min_{u,r\in\mathbb{R}^d,\, t\in\mathbb{R}^{2d}} \left\{ G(u,t,r) = f(u) + g(t) + h(r) \right\}$$

$$\text{s.t. } \begin{pmatrix} D \\ A \end{pmatrix} u + \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t + \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r = \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} \text{ with } \begin{array}{c} f(u)=0,\; g(t)=\sum_{i=1}^{d}\|t_i\|_2,\; h(r)=\iota_{B(\delta)}(r), \\ \text{all closed, proper, convex} \end{array}$$

Can it also be seen as a standard “two-blocks” problem?

YES!

$$x = u,\; y = (t;r),$$
$$g(y) = \ldots,\; B = \ldots,\; C = \ldots,\; c = \ldots$$

$$\begin{array}{ll} \text{minimize} & f(x) + g(y) \\ \text{subject to} & Bx + Cy = c \end{array}$$

ADMM for TV-L_1 model

Split (linearly constrained) **model:**

$$\left\{u^*,t^*,r^*\right\} \in \arg\min_{u,r\in\mathbb{R}^d,t\in\mathbb{R}^{2d}} \left\{G(u,t,r) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2\right\} \text{ s.t. } t = Du,\ r = Au - b$$

The augmented Lagrangian function:

$$L(u,t,r;\lambda) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \langle\lambda, c - Bu - Ct - Er\rangle + \frac{\beta}{2}\|c - Bu - Ct - Er\|_2^2$$

$$= \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \left\langle \lambda, \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} - \begin{pmatrix} D \\ A \end{pmatrix} u - \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t - \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r \right\rangle + \frac{\beta}{2}\left\| \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} - \begin{pmatrix} D \\ A \end{pmatrix} u - \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t - \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r \right\|_2^2$$

ADMM for TV-L_1 model

Split (linearly constrained) **model:**

$$\{u^*,t^*,r^*\} \in \arg\min_{u,r\in\mathbb{R}^d,\, t\in\mathbb{R}^{2d}} \left\{ G(u,t,r) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 \right\} \ \text{s.t.}\ \ t = Du,\ r = Au - b$$

The augmented Lagrangian function:

$$L(u,t,r;\lambda) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \langle \lambda, c - Bu - Ct - Er\rangle + \frac{\beta}{2}\|c - Bu - Ct - Er\|_2^2$$

$$= \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \left\langle \lambda, \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} - \begin{pmatrix} D \\ A \end{pmatrix} u - \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t - \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r \right\rangle + \frac{\beta}{2}\left\| \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} - \begin{pmatrix} D \\ A \end{pmatrix} u - \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t - \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r \right\|_2^2$$

$$\lambda \triangleq \begin{pmatrix} \lambda_t \\ \lambda_r \end{pmatrix},\ \ \lambda_t \in \mathbb{R}^{2d},\ \lambda_r \in \mathbb{R}^{d},\ \lambda \in \mathbb{R}^{3d}$$

ADMM for TV-L_1 model

Split (linearly constrained) **model:**

$$\left\{u^*,t^*,r^*\right\} \in \underset{u,r\in\mathbb{R}^d,t\in\mathbb{R}^{2d}}{\arg\min} \left\{ G(u,t,r) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 \right\} \text{ s.t. } t = Du,\ r = Au - b$$

The augmented Lagrangian function:

$$L(u,t,r;\lambda) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \langle \lambda, c - Bu - Ct - Er\rangle + \frac{\beta}{2}\|c - Bu - Ct - Er\|_2^2$$

$$= \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \left\langle \lambda, \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} - \begin{pmatrix} D \\ A \end{pmatrix} u - \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t - \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r \right\rangle + \frac{\beta}{2}\left\| \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} - \begin{pmatrix} D \\ A \end{pmatrix} u - \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t - \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r \right\|_2^2$$

$$L(u,t,r;\lambda_t,\lambda_r) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \langle \lambda_t, t - Du\rangle + \frac{\beta}{2}\|t - Du\|_2^2 - \langle \lambda_r, r - (Au - b)\rangle + \frac{\beta}{2}\|r - (Au - b)\|_2^2$$

ADMM for TV-L_1 model

Split (linearly constrained) **model:**

$$\left\{u^*,t^*,r^*\right\} \in \arg\min_{u,r\in\mathbb{R}^d,\, t\in\mathbb{R}^{2d}} \left\{ G(u,t,r) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 \right\} \text{ s.t. } t = Du,\ r = Au - b$$

The augmented Lagrangian function:

$$L(u,t,r;\lambda) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \langle \lambda, c - Bu - Ct - Er\rangle + \frac{\beta}{2}\|c - Bu - Ct - Er\|_2^2$$

$$= \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \left\langle \lambda, \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} - \begin{pmatrix} D \\ A \end{pmatrix} u - \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t - \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r \right\rangle + \frac{\beta}{2}\left\| \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} - \begin{pmatrix} D \\ A \end{pmatrix} u - \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t - \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r \right\|_2^2$$

$$L(u,t,r;\lambda_t,\lambda_r) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \langle \lambda_t, t - Du\rangle + \frac{\beta}{2}\|t - Du\|_2^2 - \langle \lambda_r, r - (Au - b)\rangle + \frac{\beta}{2}\|r - (Au - b)\|_2^2$$

- It is like if we “Lagrange-augment the two constraints t = Du, r = Au-b separately”

ADMM for TV-L_1 model

Split (linearly constrained) **model:**

$$\{u^*,t^*,r^*\} \in \arg\min_{u,r\in\mathbb{R}^d,\, t\in\mathbb{R}^{2d}} \left\{ G(u,t,r) = \mu \|r\|_1 + \sum_{i=1}^{d} \|t_i\|_2 \right\} \ \text{s.t.}\ \ t = Du,\ r = Au - b$$

The augmented Lagrangian function:

$$L(u,t,r;\lambda) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \langle \lambda, c - Bu - Ct - Er\rangle + \frac{\beta}{2}\|c - Bu - Ct - Er\|_2^2$$

$$= \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \left\langle \lambda, \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} - \begin{pmatrix} D \\ A \end{pmatrix} u - \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t - \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r \right\rangle + \frac{\beta}{2}\left\| \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} - \begin{pmatrix} D \\ A \end{pmatrix} u - \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t - \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r \right\|_2^2$$

$$L(u,t,r;\lambda_t,\lambda_r) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \langle \lambda_t, t - Du\rangle + \frac{\beta_t}{2}\|t - Du\|_2^2 - \langle \lambda_r, r - (Au - b)\rangle + \frac{\beta_r}{2}\|r - (Au - b)\|_2^2$$

- It is like if we “Lagrange-augment the two constraints t = Du, r = Au-b separately”
- It is possible (and we do it) to use two different β values for the two constraints

ADMM for TV-L_1 model

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$$= \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \left\langle \lambda, \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} - \begin{pmatrix} D \\ A \end{pmatrix} u - \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t - \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r \right\rangle + \frac{\beta}{2}\left\| \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} - \begin{pmatrix} D \\ A \end{pmatrix} u - \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t - \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r \right\|_2^2$$

$$L(u,t,r;\lambda_t,\lambda_r) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \langle \lambda_t, t - Du\rangle + \frac{\beta_t}{2}\|t - Du\|_2^2 - \langle \lambda_r, r - (Au - b)\rangle + \frac{\beta_r}{2}\|r - (Au - b)\|_2^2$$

- It is like if we “Lagrange-augment the two constraints t = Du, r = Au-b separately”
- It is possible (and we do it) to use two different β values for the two constraints
- Convergence of this “three-blocks ADMM” with different β values has been proved in

C. Wu et al., Augmented Lagrangian Method for Total Variation Restoration with Non-quadratic Fidelity, Inverse Problems and Imaging, 2011 (file “02_TV_NQ_FIDELITY_ADMM” in REPOSITORY)

ADMM for TV-L_1 model

Split (linearly constrained) model:

$$\left\{u^*, t^*, r^*\right\} \in \arg\min_{u, r \in \mathbb{R}^d, t \in \mathbb{R}^{2d}} \left\{ G(u,t,r) = \mu \|r\|_1 + \sum_{i=1}^{d} \|t_i\|_2 \right\} \quad \text{s.t. } \; t = Du, \; r = Au - b$$

like for TV-L_2

The augmented Lagrangian function:

like for TIK-L_1

$$L(u,t,r;\lambda_t,\lambda_r) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \left\langle \lambda_t, t - Du \right\rangle + \frac{\beta_t}{2}\|t - Du\|_2^2 - \left\langle \lambda_r, r - (Au - b) \right\rangle + \frac{\beta_r}{2}\|r - (Au - b)\|_2^2$$

Solution image as the saddle point of the above function, that is:

$$\text{find } \left\{u^*, t^*, r^*, \lambda_t^*, \lambda_r^*\right\} \quad \text{s.t.} \quad L\left(u^*, t^*, r^*, \lambda_t, \lambda_r\right) \le L\left(u^*, t^*, r^*, \lambda_t^*, \lambda_r^*\right) \le L\left(u, t, r, \lambda_t^*, \lambda_r^*\right)$$

$$\forall (u, t, r, \lambda_t, \lambda_r) \in \mathbb{R}^d \times \mathbb{R}^{2d} \times \mathbb{R}^d \times \mathbb{R}^{2d} \times \mathbb{R}^d$$

ADMM for TV-L_1 model

Saddle point of the Augmented Lagrangian function?

ADMM-based iterative algorithm (split optimization sub-problems):

Primal descent (alternating)

$$t^{(k+1)} = \arg\min_{t \in \mathbb{R}^{2d}} L(u^{(k)}, t, r^{(k)}; \lambda_t^{(k)}, \lambda_r^{(k)})$$

<--- closed-form proximal map

$$r^{(k+1)} = \arg\min_{r \in \mathbb{R}^{d}} L(u^{(k)}, t^{(k+1)}, r; \lambda_t^{(k)}, \lambda_r^{(k)})$$

<--- closed-form proximal map

$$u^{(k+1)} = \arg\min_{u \in \mathbb{R}^{d}} L(u, t^{(k+1)}, r^{(k+1)}; \lambda_t^{(k)}, \lambda_r^{(k)})$$

<--- s.p.d. linear system

Dual ascent

$$\lambda_t^{(k+1)} = \lambda_t^{(k)} - \beta_t \,(t^{(k+1)} - Du^{(k+1)})$$

$$\lambda_r^{(k+1)} = \lambda_r^{(k)} - \beta_r \,(r^{(k+1)} - (Au^{(k+1)} - b))$$

<--- closed-form

Primal descent: subproblem for t

Augmented Lagrangian function:

$$L(u,t,r;\lambda_t,\lambda_r) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \langle \lambda_t, t - Du\rangle + \frac{\beta_t}{2}\|t - Du\|_2^2 - \langle \lambda_r, r - (Au - b)\rangle + \frac{\beta_r}{2}\|r - (Au - b)\|_2^2$$

Subproblem for primal variable ***t*** :

$$t^{(k+1)} = \arg\min_{t \in \mathbb{R}^{2d}} L(u^{(k)}, t, r^{(k)}; \lambda_t^{(k)}, \lambda_r^{(k)})$$

… exactly the same as for TV-L_2 …

… closed-form solution (see Proposition 1) of *d* independent 2D problems:

$$t_i^{(k+1)} = \arg\min_{t_i \in \mathbb{R}^2} \left\{ \|t_i\|_2 + \frac{\beta_t}{2}\|t_i - q_i^{(k)}\|_2^2 \right\}, \quad q_i^{(k)} = \left(\left(Du^{(k)}\right)_i + \frac{1}{\beta_t}\lambda_{t,i}^{(k)} \right) \in \mathbb{R}^2,$$

$$= \max\left\{ \|q_i^{(k)}\|_2 - \frac{1}{\beta_t}, 0 \right\} \frac{q_i^{(k)}}{\|q_i^{(k)}\|_2}, \quad i = 1,\dots,d$$

linear computational complexity O(d)

Primal descent: subproblem for r

Augmented Lagrangian function:

$$L(u,t,r;\lambda_t,\lambda_r) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \langle\lambda_t, t-Du\rangle + \frac{\beta_t}{2}\|t-Du\|_2^2 - \langle\lambda_r, r-(Au-b)\rangle + \frac{\beta_r}{2}\|r-(Au-b)\|_2^2$$

Subproblem for primal variable ***r*** :

$$r^{(k+1)} = \arg\min_{r\in\mathbb{R}^d} L(u^{(k)}, t^{(k+1)}, r; \lambda_t^{(k)}, \lambda_r^{(k)})$$

… exactly the same as for TIK-L_1 …

… closed-form solution (see Proposition 2) of *d* independent 1D problems:

$$\gamma = \beta_r / \mu$$

$$r_i^{(k+1)} = \arg\min_{r_i\in\mathbb{R}}\left\{|r_i| + \frac{\gamma}{2}\left(r_i - w_i^{(k)}\right)^2\right\}, \quad w_i^{(k)} = \left(\left(Au^{(k)}-b\right)_i + \frac{1}{\beta_r}\lambda_{r,i}^{(k)}\right) \in \mathbb{R},$$

$$= \operatorname{sign}\left(w_i^{(k)}\right)\max\left\{\left|w_i^{(k)}\right| - \frac{1}{\gamma}, 0\right\}, \quad i = 1,\ldots,d$$

linear computational complexity O(d)

Primal descent: subproblem for u

Augmented Lagrangian function:

$$L(u,t,r;\lambda_t,\lambda_r) = \mu\|r\|_1 + \sum_{i=1}^{d}\|t_i\|_2 - \langle \lambda_t, t - Du\rangle + \frac{\beta_t}{2}\|t - Du\|_2^2 - \langle \lambda_r, r - (Au - b)\rangle + \frac{\beta_r}{2}\|r - (Au - b)\|_2^2$$

Subproblem for primal variable ***u***:

$$u^{(k+1)} = \arg\min_{u\in\mathbb{R}^d} L(u, t^{(k+1)}, r^{(k+1)}; \lambda_t^{(k)}, \lambda_r^{(k)})$$ … drop the terms independent of u …

$$= \arg\min_{u\in\mathbb{R}^d} \left\{ Z(u) = \langle \lambda_t^{(k)}, Du\rangle + \frac{\beta_t}{2}\|t^{(k+1)} - Du\|_2^2 + \langle \lambda_r^{(k)}, Au\rangle + \frac{\beta_r}{2}\|r^{(k+1)} - (Au - b)\|_2^2 \right\}$$

The function Z is quadratic in the optimization variable u, hence its global minimizers (if there exists one) are to be sought among its stationary points:

$$u^{(k+1)} \in \text{ set of solutions of : } \quad \nabla Z(u) = 0_d$$

Primal descent: subproblem for u

… after some simple algebraic manipulations:

$$\nabla Z(u) = 0_d \quad \Leftrightarrow \quad \left(D^T D + \frac{\beta_r}{\beta_t} A^T A \right) u = D^T \left(t^{(k+1)} - \frac{1}{\beta_t} \lambda_t^{(k)} \right) + \frac{\beta_r}{\beta_t} A^T \left(r^{(k+1)} - \frac{1}{\beta_r} \lambda_r^{(k)} + b \right)$$

The coefficient matrix is almost identical to the one previously obtained for the TIK-L_2 , TV-L_2 and TIK-L_1 models (see…): it is s. p. d. and has full rank, hence the linear system admits a unique solution giving the new iterate $u^{(k+1)}$

RESTORATION:

Assuming periodic/reflective/anti-reflective boundary conditions for *u*,

the linear system can be solved (like for TIK-L_2 case) by 2D DFT/DCT/DST.

By using 2D FFT/FCT/FST implementations $\longrightarrow$ computational complexity O(d log d)

Primal descent: subproblem for u

… after some simple algebraic manipulations:

$$\nabla Z(u) = 0_d \quad \Leftrightarrow \quad \left(D^T D + \frac{\beta_r}{\beta_t} A^T A \right) u = D^T \left(t^{(k+1)} - \frac{1}{\beta_t} \lambda_t^{(k)} \right) + \frac{\beta_r}{\beta_t} A^T \left(r^{(k+1)} - \frac{1}{\beta_r} \lambda_r^{(k)} + b \right)$$

The coefficient matrix is almost identical to the one previously obtained for the TIK-L_2 , TV-L_2 and TIK-L_1 models (see…): it is s. p. d. and has full rank, hence the linear system admits a unique solution giving the new iterate $u^{(k+1)}$

INPAINTING:

↓

the linear system can be solved (like for TIK-L_2 case) by iterative (P)CG

ADMM (three-blocks → two-blocks)

for the numerical solution of the

("discrepancy"-constrained)

TV-L_2 model

Discrepancy-constrained Variational Models
(for L_2 fidelity terms: additive white Gaussian noise)

- Discrepancy Principle (**DP**) definition: for any variational model of the form:

$$u^*(\mu) = \arg\min_{u \in \mathbb{R}^d} \left\{ J(u;\mu) = R(u) + \frac{\mu}{2} \| Au - b \|_2^2 \right\}, \quad \text{with } R \text{ any regularizer,}$$

and with:

$A = K \in \mathbb{R}^{d \times d}$ blurring matrix (for image restoration)

$A = S \in \mathbb{R}^{d \times d}$ inpainting matrix (for image inpainting)

Discrepancy-constrained Variational Models
(for L_2 fidelity terms: additive white Gaussian noise)

- Discrepancy Principle (**DP**) definition: for any variational model of the form:

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choose μ such that the solution $u^*(\mu)$ satisfies the dicrepancy constraint:

$$u^*(\mu) \in \mathcal{D}(\delta) \triangleq \left\{ u\in\mathbb{R}^d : \|Au-b\|_2^2 \le \delta^2 \right\}, \quad \delta = \delta(\tau) = \tau\sqrt{d}\,\hat{\sigma}_n, \quad \tau \simeq 1, \quad \hat{\sigma}_n \text{ est. of noise stdv}$$

- DP rationale: we aim at solutions $u^*(\mu)$ which are as near as possible to the sought clean image, u_{true}, which, according to the linear degradation model, satisfies:

$$b = Au_{true} + n \;\Rightarrow\; Au_{true} - b = n \;\Rightarrow\; \|Au_{true} - b\|_2^2 = \|n\|_2^2 \cong d\sigma_n^2 = \delta^2(\tau) \text{ with } \tau = 1$$

Hence, by DP we impose that the residual of $u^*(\mu)$ has the same variance of noise

- DP usefulness: tuning (by hand) the best (in terms of obtained restoration results) regularization parameter μ of unconstrained models can be a long and tedious task. If we are able to compute a good estimate $\hat{\sigma}_n$ of noise standard deviation, then imposing directly the discrepancy constraint – i.e., using discrepancy-constrained variational models – allows to obtain in one shot a good restoration!

ADMM for discrepancy-constrained TV-L_2 model

The **Unconstrained (U)** and **(discrepancy) Constrained (C)** **TV-L_2** models:

$$u_U^*(\mu) = \arg\min_{u\in\mathbb{R}^d} \left\{ J_U(u;\mu) = \mathrm{TV}(u) + \frac{\mu}{2}\|Au-b\|_2^2 \right\}$$ … seen in previous slides

$$u_C^*(\delta) = \arg\min_{u\in\mathbb{R}^d} \left\{ J_C(u;\delta) = \mathrm{TV}(u) + \iota_{\mathcal{D}(\delta)}(u) \right\}, \quad \mathcal{D}(\delta) = \left\{ u \in \mathbb{R}^d : \|Au-b\|_2^2 \le \delta^2 \right\}$$

Discrepancy set / constraint

hyper-ellipsoidal shape

ADMM for discrepancy-constrained TV-L_2 model

The **Unconstrained (U)** and **(discrepancy) Constrained (C)** **TV**-$\mathbf{L_2}$ models:

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$$u_C^*(\delta) = \arg\min_{u\in\mathbb{R}^d} \left\{ J_C(u;\delta) = \mathrm{TV}(u) + \imath_{\mathcal{D}(\delta)}(u) \right\}, \quad \mathcal{D}(\delta) = \left\{ u \in \mathbb{R}^d : \|Au-b\|_2^2 \le \delta^2 \right\}$$

where the indicator function $\imath_S(x)$ of a set S is defined as $\imath_S(x) = \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{if } x \notin S \end{cases}$,

such that the constrained model above can also be equivalently written as

$$u_C^*(\delta) = \arg\min_{u\in\mathcal{D}(\delta)} \mathrm{TV}(u), \quad \mathcal{D}(\delta) = \left\{ u \in \mathbb{R}^d : \|Au-b\|_2^2 \le \delta^2 \right\}$$ ← discrepancy set

discrepancy constraint

For our purposes, it is convenient to consider the form with the indicator function!

ADMM for discrepancy-constrained TV-L_2 model

The **Unconstrained (U)** and **(discrepancy) Constrained (C)** **TV-L_2** models:

$$u_U^*(\mu) = \arg\min_{u\in\mathbb{R}^d} \left\{ J_U(u;\mu) = \mathrm{TV}(u) + \frac{\mu}{2}\|Au-b\|_2^2 \right\}$$ … seen in previous slides

$$u_C^*(\delta) = \arg\min_{u\in\mathbb{R}^d} \left\{ J_C(u;\delta) = \mathrm{TV}(u) + \iota_{\mathcal{D}(\delta)}(u) \right\}, \quad \mathcal{D}(\delta) = \left\{ u\in\mathbb{R}^d : \|Au-b\|_2^2 \le \delta^2 \right\}$$

The constrained model can be equivalently and usefully rewritten (by rewriting in an equivalent form the discrepancy constraint) as follows:

$$u_C^*(\delta) = \arg\min_{u\in\mathbb{R}^d} \left\{ J_C(u;\delta) = \mathrm{TV}(u) + \iota_{B(\delta)}(Au-b) \right\}, \quad B(\delta) = \left\{ x\in\mathbb{R}^d : \|x\|_2^2 \le \delta^2 \right\}$$

where $B(\delta)$ is the l_2 ball of $\mathbb{R}^d$ with center the origin and radius δ

ADMM for discrepancy-constrained TV-L_2 model

The **Unconstrained (U)** and **(discrepancy) Constrained (C)** **TV-L_2** models:

$$u_U^*(\mu) = \arg\min_{u \in \mathbb{R}^d} \left\{ J_U(u;\mu) = \mathrm{TV}(u) + \frac{\mu}{2} \| Au - b \|_2^2 \right\}$$ … seen in previous slides

$$u_C^*(\delta) = \arg\min_{u \in \mathbb{R}^d} \left\{ J_C(u;\delta) = \mathrm{TV}(u) + \iota_{B(\delta)}(Au - b) \right\}, \quad B(\delta) = \left\{ x \in \mathbb{R}^d : \| x \|_2^2 \le \delta^2 \right\}$$

Motivation for the constrained model (numerically more challenging):

If we know (or we are able to estimate) the noise standard deviation σ_n, then the above constrained model (unlike the unconstrained one) allows us to automatically obtain a good-quality solution $u_C^*(\delta)$ by selecting $\delta = \tau \sqrt{d}\, \sigma_n$, with $\tau \simeq 1$, as in this way we are imposing that the standard deviation of the solution residual image $r_C^*(\delta) = K u_C^*(\delta) - b$ is approximately equal to the noise standard deviation σ_n ...

... this is called the DISCREPANCY PRINCIPLE

ADMM for discrepancy-constrained TV-L_2 model

The **Unconstrained (U)** and **(discrepancy) Constrained (C)** **TV-L_2** models:

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$$u_C^*(\delta) = \arg\min_{u \in \mathbb{R}^d} \left\{ J_C(u;\delta) = \mathrm{TV}(u) + \iota_{B(\delta)}(Au - b) \right\}, \quad B(\delta) = \left\{ x \in \mathbb{R}^d : \left\| x \right\|_2^2 \le \delta^2 \right\}$$

"Equivalence" of the unconstrained and constrained models

ADMM for discrepancy-constrained TV-L_2 model

The (discrepancy) constrained convex non-smooth model:

$$u^* = \arg\min_{u\in\mathbb{R}^d} \left\{ J(u;\delta) = \iota_{B(\delta)}\left(Au-b\right) + \sum_{i=1}^{d} \sqrt{\left(D_h u\right)_i^2 + \left(D_v u\right)_i^2} \right\}, \quad \text{with :}$$

$$B(\delta) = \left\{ x \in \mathbb{R}^d : \ \|x\|_2^2 \le \delta^2 \right\}, \quad l^2 \text{ ball in } \mathbb{R}^d \text{ with center the origin and radius } \delta$$

Variables splitting:

$$t = Du \Leftrightarrow \begin{pmatrix} t_h \\ t_v \end{pmatrix} = \begin{pmatrix} D_h u \\ D_v u \end{pmatrix}, \quad t_i \triangleq \begin{pmatrix} t_{h,i} \\ t_{v,i} \end{pmatrix} = \begin{pmatrix} \left(D_h u\right)_i \\ \left(D_v u\right)_i \end{pmatrix} \in \mathbb{R}^2$$

$$r = Au - b \ \in \mathbb{R}^d$$

Split (linearly constrained) model:

$$\left\{u^*, t^*, r^*\right\} \in \arg\min_{u,r\in\mathbb{R}^d,\, t\in\mathbb{R}^{2d}} \left\{ G(u,t,r) = \iota_{B(\delta)}(r) + \sum_{i=1}^{d} \left\| t_i \right\|_2 \right\} \ \text{s.t.} \ \begin{aligned} t &= Du, \\ r &= Au - b \end{aligned}$$

ADMM for discrepancy-constrained TV-L_2 model

Split (linearly constrained) **model:**

$$\left\{u^*, t^*, r^*\right\} \in \arg\min_{u, r \in \mathbb{R}^d, t \in \mathbb{R}^{2d}} \left\{ G(u,t,r) = \iota_{B(\delta)}(r) + \sum_{i=1}^{d} \left\| t_i \right\|_2 \right\} \text{ s.t. } t = Du, \; r = Au - b$$

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It can be equivalently rewritten as:

$$\left\{u^*,t^*,r^*\right\} \in \mathop{\arg\min}_{u,r\in\mathbb{R}^d,\, t\in\mathbb{R}^{2d}} \left\{ G(u,t,r) = f(u) + g(t) + h(r) \right\}$$

$$\text{s.t. } \begin{pmatrix} D \\ A \end{pmatrix} u + \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t + \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r = \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} \text{ with } f(u)=0,\ g(t)=\sum_{i=1}^{d}\left\|t_i\right\|_2,\ h(r)=\iota_{B(\delta)}(r),$$

all closed, proper, convex

ADMM for discrepancy-constrained TV-L_2 model

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all closed, proper, convex

Is it a standard
"three-blocks" problem?

$$\begin{aligned} &\text{minimize} && f(x) + g(y) + h(z) \\ &\text{subject to} && Bx + Cy + Ez = c \end{aligned}$$

ADMM for discrepancy-constrained TV-L_2 model

Split (linearly constrained) **model:**

$$\left\{u^*,t^*,r^*\right\} \in \underset{u,r\in\mathbb{R}^d,\, t\in\mathbb{R}^{2d}}{\arg\min} \left\{ G(u,t,r) = \iota_{B(\delta)}(r) + \sum_{i=1}^{d} \left\| t_i \right\|_2 \right\} \quad \text{s.t. } \; t = Du, \; r = Au - b$$

It can be equivalently rewritten as:

$$\left\{u^*,t^*,r^*\right\} \in \underset{u,r\in\mathbb{R}^d,\, t\in\mathbb{R}^{2d}}{\arg\min} \left\{ G(u,t,r) = f(u) + g(t) + h(r) \right\}$$

$$\text{s.t.} \quad \begin{pmatrix} D \\ A \end{pmatrix} u + \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t + \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r = \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} \quad \text{with} \quad \begin{array}{c} f(u) = 0, \; g(t) = \displaystyle\sum_{i=1}^{d} \left\| t_i \right\|_2, \; h(r) = \iota_{B(\delta)}(r), \\ \text{all closed, proper, convex} \end{array}$$

$$x = u, \; y = t, \; z = r,$$

$$B = \begin{pmatrix} D \\ A \end{pmatrix}, \; C = \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix}, \; E = \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix}, \; c = \begin{pmatrix} 0_{2d} \\ b \end{pmatrix}$$

Is it a standard "three-blocks" problem?

YES!

$$\begin{array}{ll} \text{minimize} & f(x) + g(y) + h(z) \\ \text{subject to} & Bx + Cy + Ez = c \end{array}$$

ADMM for discrepancy-constrained TV-L_2 model

Split (linearly constrained) **model:**

$$\left\{u^*, t^*, r^*\right\} \in \underset{u,r\in\mathbb{R}^d,\, t\in\mathbb{R}^{2d}}{\arg\min} \left\{ G(u,t,r) = \iota_{B(\delta)}(r) + \sum_{i=1}^{d} \left\| t_i \right\|_2 \right\} \text{ s.t. } t = Du,\ r = Au - b$$

It can be equivalently rewritten as:

$$\left\{u^*, t^*, r^*\right\} \in \underset{u,r\in\mathbb{R}^d,\, t\in\mathbb{R}^{2d}}{\arg\min} \left\{ G(u,t,r) = f(u) + g(t) + h(r) \right\}$$

$$\text{s.t. } \begin{pmatrix} D \\ A \end{pmatrix} u + \begin{pmatrix} -I_{2d} \\ 0_{d\times 2d} \end{pmatrix} t + \begin{pmatrix} 0_{2d\times d} \\ -I_d \end{pmatrix} r = \begin{pmatrix} 0_{2d} \\ b \end{pmatrix} \text{ with } \begin{array}{c} f(u) = 0,\ g(t) = \sum_{i=1}^{d} \left\| t_i \right\|_2,\ h(r) = \iota_{B(\delta)}(r), \\ \text{all closed, proper, convex} \end{array}$$

Can it also be seen as a standard "two-blocks" problem?

YES!

$$x = u,\ y = (t; r),$$
$$g(y) = \ldots,\ B = \ldots,\ C = \ldots,\ c = \ldots$$

$$\begin{array}{ll} \text{minimize} & f(x) + g(y) \\ \text{subject to} & Bx + Cy = c \end{array}$$

ADMM for discrepancy-constrained TV-L_2 model

Split (linearly constrained) model:

$$\{u^*, t^*, r^*\} \in \arg\min_{u,r \in \mathbb{R}^d, t \in \mathbb{R}^{2d}} \left\{ G(u,t,r) = \iota_{B(\delta)}(r) + \sum_{i=1}^{d} \|t_i\|_2 \right\} \text{ s.t. } \begin{array}{l} t = Du, \\ r = Au - b \end{array}$$

like for TV-L_1

like for TV-L_2

The **augmented Lagrangian** function:

like for TIK-L_1

$$L(u,t,r;\lambda_t,\lambda_r) = \iota_{B(\delta)}(r) + \sum_{i=1}^{d} \|t_i\|_2 - \langle \lambda_t, t - Du \rangle + \frac{\beta_t}{2} \|t - Du\|_2^2 - \langle \lambda_r, r - (Au - b) \rangle + \frac{\beta_r}{2} \|r - (Au - b)\|_2^2$$

Restored image as the saddle point of the above function, that is:

$$\text{find } \{u^*, t^*, r^*, \lambda^*, \lambda_r^*\} \quad \text{s.t.} \quad L(u^*, t^*, r^*, \lambda_t, \lambda_r) \le L(u^*, t^*, r^*, \lambda^*, \lambda_r^*) \le L(u, t, r, \lambda^*, \lambda_r^*)$$

$$\forall (u, t, r, \lambda_t, \lambda_r) \in \mathbb{R}^d \times \mathbb{R}^{2d} \times \mathbb{R}^d \times \mathbb{R}^{2d} \times \mathbb{R}^d$$

ADMM for discrepancy-constrained TV-L_2 model

Saddle point of the **Augmented Lagrangian** function?

ADMM-based iterative algorithm (split optimization sub-problems):

Primal descent (alternating)

$$t^{(k+1)} = \arg\min_{t \in \mathbb{R}^{2d}} L(u^{(k)}, t, r^{(k)}; \lambda_t^{(k)}, \lambda_r^{(k)})$$

<--- closed-form proximal map

$$r^{(k+1)} = \arg\min_{r \in \mathbb{R}^{d}} L(u^{(k)}, t^{(k+1)}, r; \lambda_t^{(k)}, \lambda_r^{(k)})$$

<--- Euclidean projection

$$u^{(k+1)} = \arg\min_{u \in \mathbb{R}^{d}} L(u, t^{(k+1)}, r^{(k+1)}; \lambda_t^{(k)}, \lambda_r^{(k)})$$

<--- s.p.d. linear system

Dual ascent

$$\lambda_t^{(k+1)} = \lambda_t^{(k)} - \beta_t \left(t^{(k+1)} - D u^{(k+1)}\right)$$

$$\lambda_r^{(k+1)} = \lambda_r^{(k)} - \beta_r \left(r^{(k+1)} - (A u^{(k+1)} - b)\right)$$

<--- closed-form

Primal descent: subproblem for t

Augmented Lagrangian function:

$$L(u,t,r;\lambda_t,\lambda_r) = \iota_{B(\delta)}(r) + \sum_{i=1}^{d} \|t_i\|_2 - \langle \lambda_t, t - Du \rangle + \frac{\beta_t}{2}\|t - Du\|_2^2 - \langle \lambda_r, r - (Au - b) \rangle + \frac{\beta_r}{2}\|r - (Au - b)\|_2^2$$

Subproblem for primal variable ***t*** :

$$t^{(k+1)} = \arg\min_{t \in \mathbb{R}^{2d}} L(u^{(k)}, t, r^{(k)}; \lambda_t^{(k)}, \lambda_r^{(k)})$$

… exactly the same as for TV-$L_{1,2}$ …

… closed-form solution (see Proposition 1) of *d* independent 2D problems:

$$t_i^{(k+1)} = \arg\min_{t_i \in \mathbb{R}^2} \left\{ \|t_i\|_2 + \frac{\beta_t}{2}\|t_i - q_i^{(k)}\|_2^2 \right\}, \quad q_i^{(k)} = \left(\left(Du^{(k)}\right)_i + \frac{1}{\beta_t}\lambda_{t,i}^{(k)} \right) \in \mathbb{R}^2,$$

$$= \max\left\{ \|q_i^{(k)}\|_2 - \frac{1}{\beta_t}, 0 \right\} \frac{q_i^{(k)}}{\|q_i^{(k)}\|_2}, \quad i = 1, \ldots, d$$

linear computational complexity O(d)

Primal descent: subproblem for r

Augmented Lagrangian function:

$$L(u,t,r;\lambda_t,\lambda_r) = \iota_{B(\delta)}(r) + \sum_{i=1}^{d} \|t_i\|_2 - \langle \lambda_t, t - Du \rangle + \frac{\beta_t}{2}\|t - Du\|_2^2 - \langle \lambda_r, r-(Au-b) \rangle + \frac{\beta_r}{2}\|r-(Au-b)\|_2^2$$

Subproblem for primal variable ***r*** :

$$r^{(k+1)} = \arg\min_{r\in\mathbb{R}^d} L(u^{(k)}, t^{(k+1)}, r; \lambda_t^{(k)}, \lambda_r^{(k)})$$ … drop the terms independent of *r* …

$$= \arg\min_{r\in\mathbb{R}^d} \left\{ \iota_{B(\delta)}(r) - \left\langle \lambda_r^{(k)}, r-\left(Au^{(k)}-b\right) \right\rangle + \frac{\beta_r}{2}\left\| r-\left(Au^{(k)}-b\right) \right\|_2^2 \right\}$$

$$= \arg\min_{r\in\mathbb{R}^d} \left\{ \iota_{B(\delta)}(r) + \frac{\beta_r}{2}\left\| r - w^{(k)} \right\|_2^2 \right\}, \quad w^{(k)} = KAu^{(k)} - b + \frac{1}{\beta_r}\lambda_r^{(k)} \in \mathbb{R}^d$$

$$= \arg\min_{r\in B(\delta)} \left\| r - w^{(k)} \right\|_2 = \prod\nolimits_{B(\delta)}\left(w^{(k)}\right)$$

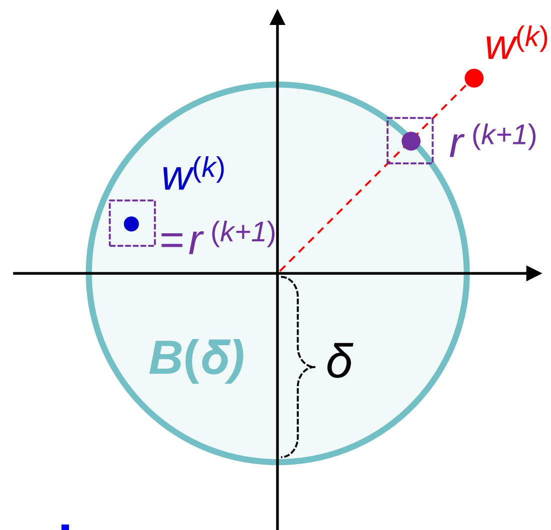
Euclidean (orthogonal) projection of vector $w^{(k)}$ onto the l_2 ball of radius δ

Primal descent: subproblem for r

Euclidean (orthogonal) projection of vector $w^{(k)}$ onto the l_2 ball of radius δ :

$$\boxed{r^{(k+1)}} = \arg\min_{r \in B(\delta)} \left\| r - w^{(k)} \right\|_2 = \Pi_{B(\delta)}\left(w^{(k)}\right), \quad \text{with} \quad \boxed{B(\delta) = \left\{ x \in \mathbb{R}^d : \ \|x\|_2 \le \delta \right\}}$$

The l_2 ball is a convex (compact) set, hence the projection $r^{(k+1)}$ exists and is unique, and admits a simple closed-form expression :



$$\boxed{r^{(k+1)}} = \begin{cases} \boxed{w^{(k)}} & \text{if } \left\|w^{(k)}\right\|_2 \le \delta \quad \textbf{(1)} \\ \dfrac{\delta}{\left\|w^{(k)}\right\|_2} \boxed{w^{(k)}} & \text{if } \left\|w^{(k)}\right\|_2 > \delta \quad \textbf{(2)} \end{cases}$$

$$r^{(k+1)} = \min\left\{ \frac{\delta}{\left\|w^{(k)}\right\|_2}, 1 \right\} w^{(k)}$$

linear computational complexity O(d)

Primal descent: subproblem for u

Augmented Lagrangian function:

$$L(u,t,r;\lambda_t,\lambda_r) = \iota_{B(\delta)}(r) + \sum_{i=1}^{d} \|t_i\|_2 - \langle \lambda_t, t - Du \rangle + \frac{\beta_t}{2} \|t - Du\|_2^2 - \langle \lambda_r, r - (Au - b) \rangle + \frac{\beta_r}{2} \|r - (Au - b)\|_2^2$$

Subproblem for primal variable ***u***:

$$u^{(k+1)} = \arg\min_{u \in \mathbb{R}^d} L(u, t^{(k+1)}, r^{(k+1)}; \lambda_t^{(k)}, \lambda_r^{(k)})$$

… exactly the same as for TV-L_1 …

$$= \arg\min_{u \in \mathbb{R}^d} \left\{ Z(u) = \langle \lambda_t^{(k)}, Du \rangle + \frac{\beta_t}{2} \left\| t^{(k+1)} - Du \right\|_2^2 + \langle \lambda_r^{(k)}, Au \rangle + \frac{\beta_r}{2} \left\| r^{(k+1)} - (Au - b) \right\|_2^2 \right\}$$

The function *Z* is quadratic in the optimization variable *u*, hence its global minimizers (if there exists one) are to be sought among its stationary points:

$$u^{(k+1)} \in \text{ set of solutions of : } \quad \nabla Z(u) = 0_d$$

Primal descent: subproblem for u

… after some simple algebraic manipulations:

$$\nabla Z(u) = 0_d \quad \Leftrightarrow \quad \left(D^T D + \frac{\beta_r}{\beta_t} A^T A \right) u = D^T \left(t^{(k+1)} - \frac{1}{\beta_t} \lambda_t^{(k)} \right) + \frac{\beta_r}{\beta_t} A^T \left(r^{(k+1)} - \frac{1}{\beta_r} \lambda_r^{(k)} + b \right)$$

The linear system is exactly the same as the one in the subproblem for *u* for model TV-L_1 : it admits a unique solution giving the new iterate $u^{(k+1)}$

RESTORATION:

Assuming periodic/reflective/anti-reflective boundary conditions for *u*, the linear system can be solved (like for TIK-L_2 case) by 2D DFT/DCT/DST.

By using 2D FFT/FCT/FST implementations $\longrightarrow$ computational complexity O(d log d)

Primal descent: subproblem for u

… after some simple algebraic manipulations:

$$\nabla Z(u) = 0_d \quad \Leftrightarrow \quad \left(D^T D + \frac{\beta_r}{\beta_t} A^T A \right) u = D^T \left(t^{(k+1)} - \frac{1}{\beta_t} \lambda_t^{(k)} \right) + \frac{\beta_r}{\beta_t} A^T \left(r^{(k+1)} - \frac{1}{\beta_r} \lambda_r^{(k)} + b \right)$$

The linear system is exactly the same as the one in the subproblem for *u* for model TV-L_1 : it admits a unique solution giving the new iterate $u^{(k+1)}$

⇩

INPAINTING:

the linear system can be solved (like for TIK-L_2 case) by iterative (P)CG