PhD Winter School 2023:
ADVANCED METHODS for MATHEMATICAL IMAGE ANALYSIS

Computational Imaging Lab (A-B)

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Experimentation in Matlab: outline

- Presentation of the provided Matlab package:
  - provided scripts and functions, input and output folders,…
  - MAIN_2D script: short explanation of this main script
  - MAIN_2D script: activation/deactivation of all testing functionalities
- Experimental testing of unconstrained TIK-L_2, TV-L_2, TIK-L_1, TV-L_1 models, with TIK-L_2 solved directly (FFT or PCG), TV-L_2, TIK-L_1, TV-L_1 solved by ADMM:
  - For TV-L_2, TIK-L_1, TV-L_1 , inspection of ADMM iterations
  - Evaluation/comparison of the obtained restoration results
- Experimental testing of discrepancy-constrained TV-L_2 model:
  - Comparison with the unconstrained TV-L_2 model

Experimentation in Matlab: **outline**

- The performance of all considered variational models solved by some specific minimization algorithm will be evaluated in terms of accuracy and of efficiency. In particular, we adopt two scalar measures for both accuracy and efficiency:
  - ACCURACY: 1) Improved Signal-to-Noise-Ratio (**ISNR**)
    2) Improved Structural Similarity Index (**ISSIM**)

$$\mathrm{ISNR}\left(u^*, u_{true}, b\right) = \mathrm{SNR}\left(u^*, u_{true}\right) - \mathrm{SNR}\left(b, u_{true}\right), \quad \mathrm{SNR}\left(x, u_{true}\right) = 10\log_{10}\left(\frac{\left\| K u_{true} - E\left[K u_{true}\right]\right\|_2^2}{\left\| K u_{true} - x\right\|_2^2}\right) \text{ [dB]}$$

$$\mathrm{ISSIM}\left(u^*, u_{true}, b\right) = \mathrm{SSIM}\left(u^*, u_{true}\right) - \mathrm{SSIM}\left(b, u_{true}\right), \quad \mathrm{SSIM}\left(x, u_{true}\right) \text{ defined according to [1],}$$

type "help ssim" in Matlab

  - EFFICIENCY: 1) **number of performed iterations** (for iterative algorithms)
    2) **CPU time** spent by the algorithm for obtaining the restoration

Clearly, accuracy will also be evaluated by visually inspecting the obtained results!!

Experimentation in Matlab: outline

- By means of the provided Matlab software, we can test experimentally 5 models:

TIK - L$_2$ - U
$$u^*(\mu) = \arg\min_{u\in\mathbb{R}^d} \left\{ J(u;\mu) = \frac{1}{2}\| Du \|_2^2 + \frac{\mu}{2}\| Au - b \|_2^2 \right\}$$

TIK - L$_1$ - U
$$u^*(\mu) = \arg\min_{u\in\mathbb{R}^d} \left\{ J(u;\mu) = \frac{1}{2}\| Du \|_2^2 + \mu\| Au - b \|_1 \right\}$$

TV - L$_2$ - U
$$u^*(\mu) = \arg\min_{u\in\mathbb{R}^d} \left\{ J(u;\mu) = \sum_{i=1}^{d} \sqrt{(D_h u)_i^2 + (D_v u)_i^2} + \frac{\mu}{2}\| Au - b \|_2^2 \right\}$$

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TV - L$_2$ - DC
$$u^*(\delta) \in \arg\min_{u\in\mathbb{R}^d} \left\{ J(u;\delta) = \sum_{i=1}^{d} \sqrt{(D_h u)_i^2 + (D_v u)_i^2} + \iota_{B(\delta)}(Au - b) \right\},$$
$$B(\delta) = \left\{ x \in \mathbb{R}^d : \| x \|_2^2 \le \delta^2 \right\}, \;\; \delta = \delta(\tau) = \tau\sqrt{d}\,\hat{\sigma}_n$$

U → Unconstrained

DC → Discrepancy-Constrained

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$$\Downarrow$$

$$B(\tau) = \left\{ x\in\mathbb{R}^d : \|x\|_2^2 \le \tau^2\left(d\,\hat{\sigma}_n^2\right) \right\}$$

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When we speak about **«the accuracy»** of a variational model, we mean the **maximum** accuracy achievable by that model by letting its free scalar parameter (μ or τ) span its entire domain $\mathbb{R}_{++}$