Universidad de los Andes

TÉCNICO LISBOA

Stochastic Interacting Systems: Limiting Behavior, Evaluation, Regularity and Applications

LiBERA

Co-funded by the European Union

Large deviations for light-tailed Lévy bridges on short time scales

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joint work with

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Probability in Mathematical Physics Seminar IST Lisboa

26/06/2025

Overview

o. Lévy processes and LDPs

I. Motivation: Crámer’s and Schilder’s theorems

II. Fine density estimates for a CPP with Weibull increments $\alpha > 1$

III. A negative LDP result for rescaled Weibull CPP on short time scales

IV. A LDP for the bridges of the a class of rescaled Lévy processes on short time scales

o. What is a Lévy process?

From Random walks to Brownian Motion

Binomial distributions: For $X_1, \ldots, X_n$ an i.i.d. random variables with $X_i \sim \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ we know that

$$\sum_{i=1}^{n} X_i \sim \mathcal{B}_{n,\frac{1}{2}}, \qquad \mathcal{B}_{n,\frac{1}{2}}(\{k\}) = \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k}$$

Replacing

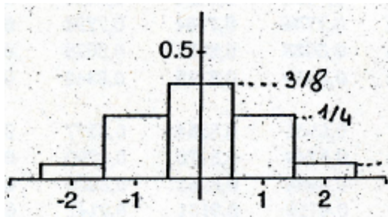
$$\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 \qquad \text{by} \qquad \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$$

we obtain that

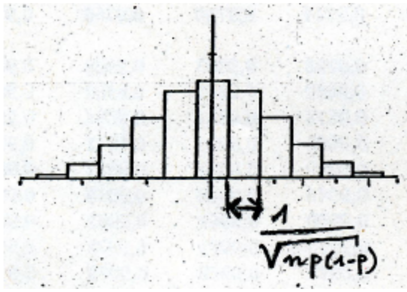
$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$$

has a **centered Binomial distribution** with values

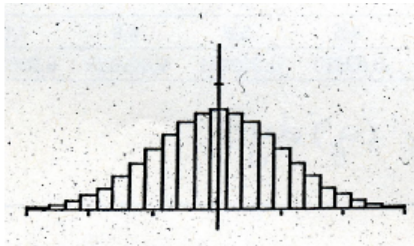
$$\{ -\sqrt{n} = \frac{n}{\sqrt{n}},\ -\frac{n-1}{\sqrt{n}}, \ldots,\ -\frac{1}{\sqrt{n}},\ 0,\ \frac{1}{\sqrt{n}}, \ldots\ -\frac{n-1}{\sqrt{n}}, \sqrt{n}\}$$



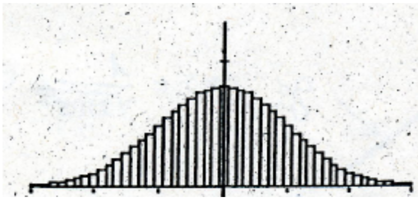
$n = 4$



$n = 16$



$n = 64$



$n = 256$

Central limit theorem (de Moivre/Laplace):

For an i.i.d. sequence $(\mathbf{X_i})_{\mathbf{i}\in\mathbb{N}}$ with $\mathbf{X_i} \sim \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ and

$$\mathbf{S_n} := \frac{1}{\sqrt{\mathbf{n}}}\sum_{\mathbf{i}=1}^{\mathbf{n}} \mathbf{X_i}$$

we have

$$\lim_{\mathbf{n}\to\infty} \mathbb{P}(\mathbf{S_n} \in [\mathbf{a},\mathbf{b}]) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{e}^{-\frac{1}{2}\mathbf{x}^2}\mathbf{dx} = \mathcal{N}_{0,1}([\mathbf{a},\mathbf{b}])$$

Donsker's theorem: Central limit theorem for random walks

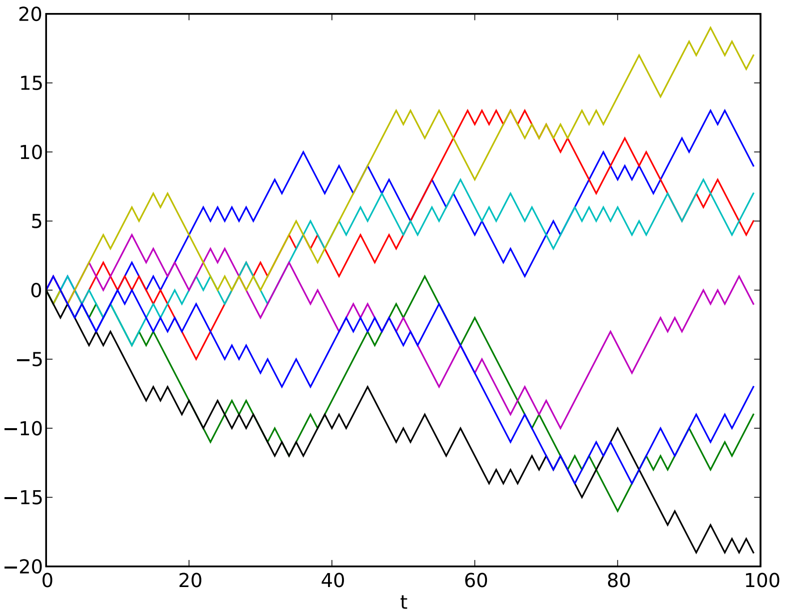
Define a sequence of continuous time random walks $\mathbf{B_n} : [\mathbf{0}, \infty) \to \mathbb{R}$

$$\mathbf{B_n}(\mathbf{t}) := \frac{\mathbf{1}}{\sqrt{\mathbf{n}}} \sum_{\mathbf{i}=\mathbf{1}}^{\lfloor \mathbf{nt} \rfloor} \mathbf{X_i} + \frac{(\mathbf{nt} - \lfloor \mathbf{nt} \rfloor)}{\sqrt{\mathbf{n}}} \mathbf{X}_{\lfloor \mathbf{nt} \rfloor + \mathbf{1}}$$

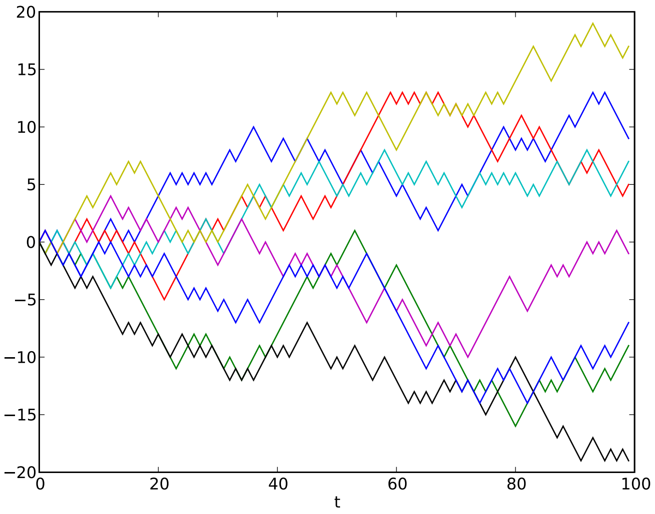
Then for any Borel set of paths $\mathbf{C} \subset \mathcal{C}_\mathbf{0}([\mathbf{0}, \infty), \mathbb{R})$ **we have the limit**

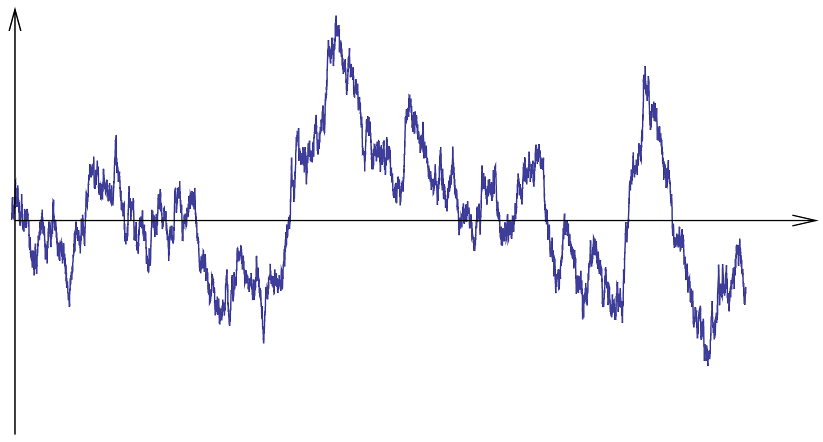
$$\lim_{\mathbf{n}\to\infty} \mathbb{P}(\mathbf{B_n} \in \mathbf{C}) = \mathcal{W}(\mathbf{C}),$$

defining a measure on $\mathcal{W}$ **on** $\mathcal{C}_\mathbf{0}([\mathbf{0}, \infty), \mathbb{R})$**, the Wiener measure.**



Donsker's theorem: Central limit theorem for random walks





[1]

[1] By Morn (talk) - I created this work entirely by myself., GFDL, https://commons.wikimedia.org/w/index.php?curid=9398546

Brownian Motion

A **Brownian Motion** $(B(t))_{t \geqslant 0}$ is a **random vector in** $\mathcal{C}_0([0,\infty),\mathbb{R}^d)$ which obeys the **Wiener measure** $\mathcal{W}_d$.

Or:

A **Brownian Motion** $(B(t))_{t \geqslant 0}$ is a **stochastic process with values in in** $\mathbb{R}^d$

- $B(0) = 0$
- **independent increments**

$$B(t_n) - B(t_{n-1}), \ldots, \; B(t_2) - B(t_1), B(t_1)$$

- **stationary increments**

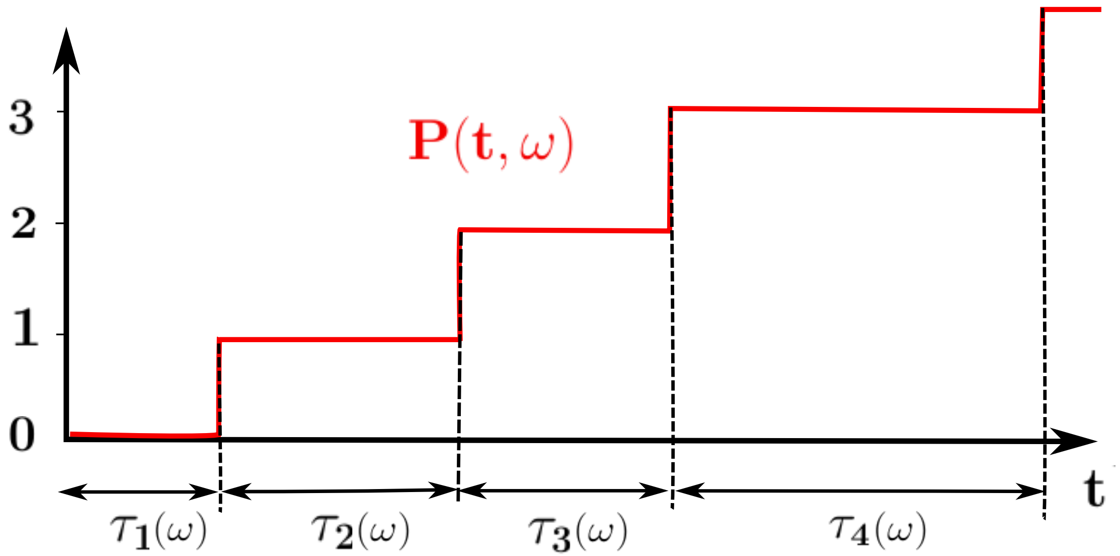
$$B(t) - B(s) \sim B(t-s) \sim N(0,(t-s)I_d) = N(0,I_d)^{*(t-s)}$$

- **Continuous paths** $t \mapsto B(t)$

Poisson Process

A Poisson Process $(\mathbf{P(t)})_{\mathbf{t}\geqslant 0}$ **is given by i.i.d. memoryless waiting times** $(\tau_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}}$**, and**

$$\mathbf{P(t)} = \max\{\mathbf{n}\in\mathbb{N} \mid \tau_1 + \cdots + \tau_{\mathbf{n}} \leqslant \mathbf{t}\}$$



- **Memorylessness of** τ_1**:** $\quad \mathbb{P}(\tau_1 > \mathbf{t}+\mathbf{h} \mid \tau_1 > \mathbf{t}) = \mathbb{P}(\tau_1 > \mathbf{h})$

  that is $\tau_1 \sim \mathcal{E}_\lambda$ **for an "intensity"** $\lambda > 0$**,** $\quad \mathbb{P}(\tau_1 > \mathbf{t}) = \mathbf{e}^{-\lambda \mathbf{t}}$.

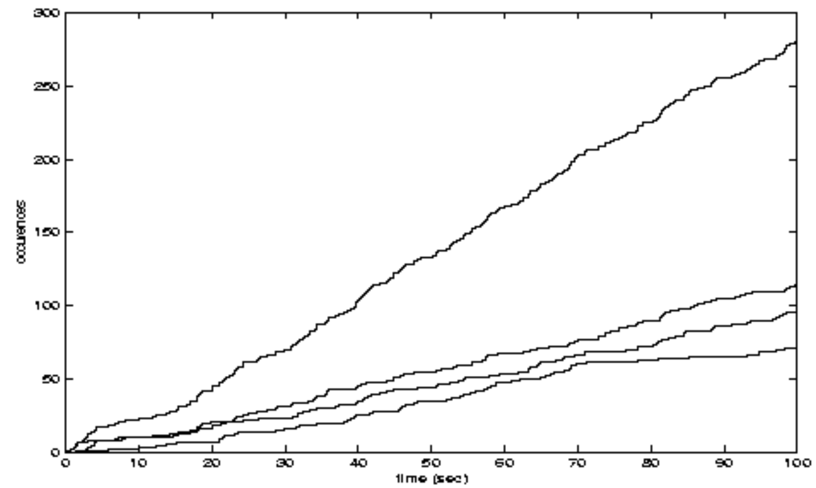
Properties:

1. $\mathbf{P(0) = 0}$

2. $\mathbf{P(t_n) - P(t_{n-1}), \dots, \; P(t_2) - P(t_1), P(t_1)}$

   independent increments

3. $\mathbf{P(t) - P(s) \sim P(t-s) \sim \mathcal{P}_{\lambda(t-s)} = \mathcal{P}_\lambda{}^{*(t-s)}}$ **stationary increments**

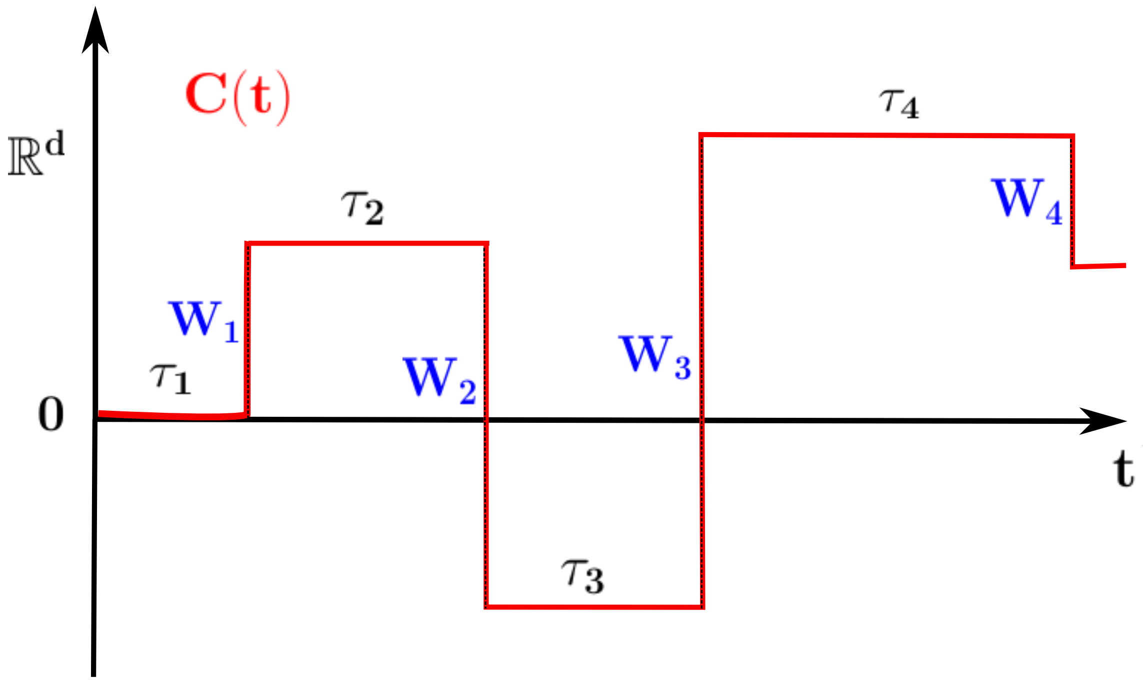


Extension:

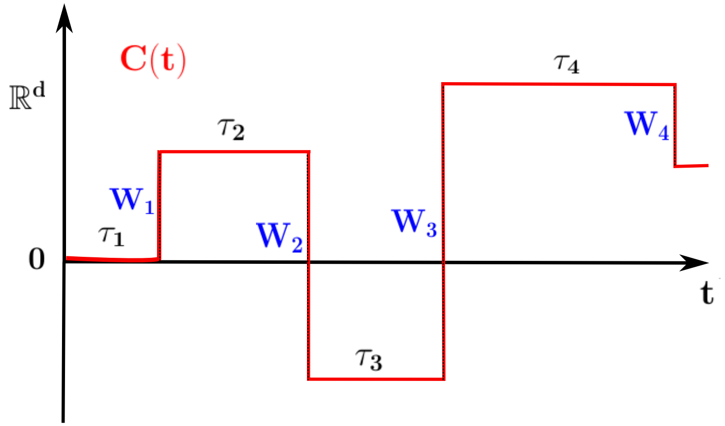
- **Compound Poisson process** $(\mathbf{C}(\mathbf{t}))_{\mathbf{t}\geq 0}$ **in** $\mathbb{R}^{\mathbf{d}}$.

$$\mathbf{C}(\mathbf{t}) := \sum_{\mathbf{i}=1}^{\mathbf{P}(\mathbf{t})} \mathbf{W}_{\mathbf{i}}$$

  - **Poisson process** $\mathbf{P} = (\mathbf{P}(\mathbf{t}))_{\mathbf{t}\geqslant 0}$ **with intensity** $\lambda > 0$
  - **Sequence of i.i.d. random increment vectors** $\mathbf{W} = (\mathbf{W}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}}$ **in** $\mathbb{R}^d$
  - $\mathbf{P} \perp \mathbf{W}$



Compound Poisson Marginals $\mathbf{C}(\mathbf{t})$ in general not in closed form!



Characteristic function:

$$\mathbf{u} \mapsto \mathbb{E}\left[\mathbf{e}^{\mathbf{i}\langle\mathbf{u},\mathbf{C}(\mathbf{t})\rangle}\right] = \exp\left(\mathbf{t}\int_{\mathbb{R}^{\mathrm{d}}}\left(\mathbf{e}^{\mathbf{i}\langle\mathbf{u},\mathbf{y}\rangle} - \mathbf{1}\right)\ \nu(\mathbf{dy})\right).$$

where

$\nu = \lambda\mu,$ **for** $\mathbf{W}_1 \sim \mu$ **and** **Poisson intensity** $\lambda > 0.$

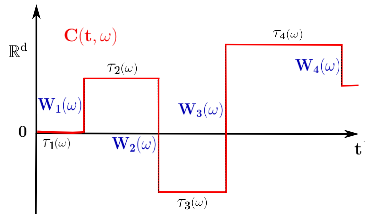
Properties:

1. $\mathbf{C(0) = 0}$

2. $\mathbf{C(t_n) - C(t_{n-1}), \ldots,\ C(t_2) - C(t_1), C(t_1)}$

   independent increments

3. $\mathbf{C(t) - C(s) \sim C(t-s) \sim \mathcal{L}(C(1))^{*(t-s)}}$ **stationary increments**



Lévy process $(\mathbf{L}(\mathbf{t}))_{\mathbf{t}\geqslant 0}$ in $\mathbb{R}^{\mathbf{d}}$

1. $\mathbf{L}(\mathbf{0}) = \mathbf{0}$.

2. $\mathrm{t} \mapsto \mathrm{X_t}$ is almost surely **right continuous with left limits** (càdlàg)

3. $\mathbf{L}(\mathbf{t_n}) - \mathbf{L}(\mathbf{t_{n-1}}), \ldots, \ \mathbf{L}(\mathbf{t_2}) - \mathbf{L}(\mathbf{t_1}), \mathbf{L}(\mathbf{t_1})$
   independent increments

4. $\mathbf{L}(\mathbf{t}) - \mathbf{L}(\mathbf{s}) \sim \mathbf{L}(\mathbf{t}-\mathbf{s}) \sim \mathcal{L}(\mathbf{L}(\mathbf{1}))^{*(\mathbf{t}-\mathbf{s})}$ **stationary increments**

Lévy-Chinchine representation of the law

Theorem:

Every Lévy process $(\mathrm{L}(\mathrm{t}))_{\mathrm{t}\geqslant 0}$ has a unique characteristic triplet $(\mathrm{a}, \mathrm{A}, \nu)$

- **vector $\mathbf{a} \in \mathbb{R}^{\mathrm{d}}$,**
- **covariance matrix $\mathbf{A} \in \mathbb{R}^{\mathrm{d}\times\mathrm{d}}$**
- **"Lévy measure" $\nu : \mathcal{B}(\mathbb{R}^{\mathrm{d}}) \to [0, \infty]$ measure**

$$\nu(\{|\mathbf{y}| \geqslant 1\}) + \int_{|\mathbf{y}|<1} |\mathbf{y}|^2 \nu(\mathbf{dy}) < \infty, \qquad \nu(\{\mathbf{0}\}) = \mathbf{0},$$

which determines the characteristic function (and hence the law)

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle \mathrm{u}, \mathrm{L}(\mathrm{t})\rangle}\right] = \mathbb{E}\left[\mathbf{e}^{\mathbf{i}\langle \mathbf{u}, \mathbf{L}(\mathbf{1})\rangle}\right]^{\mathbf{t}} = \exp\left(\mathbf{t}\Psi(\mathrm{u})\right), \qquad \mathbf{u} \in \mathbb{R}^{\mathrm{d}}$$

$$\Psi(\mathrm{u}) = \mathbf{i}\langle \mathbf{a}, \mathbf{u}\rangle - \frac{\mathbf{1}}{\mathbf{2}}\langle \mathbf{u}, \mathbf{A}\mathbf{u}\rangle + \int_{|\mathbf{y}|\geqslant 1} \left(\mathbf{e}^{\mathbf{i}\langle \mathbf{u}, \mathbf{y}\rangle} - \mathbf{1}\right)\nu(\mathbf{dy}) + \int_{|\mathbf{y}|<1} \left(\mathbf{e}^{\mathbf{i}\langle \mathbf{u}, \mathbf{y}\rangle} - \mathbf{1} - \mathbf{i}\langle \mathbf{u}, \mathbf{y}\rangle\right) \nu(\mathbf{dy}).$$

Lévy-Itô representation of the paths

Theorem:

For any Lévy process $(\mathrm{L}(\mathrm{t}))_{\mathrm{t}\geqslant 0}$ with characteristic triplet $(\mathrm{a}, \mathrm{A}, \nu)$

$$\mathrm{L}(\mathrm{t}) = \mathrm{at} + \mathrm{A}^{1/2}\mathrm{B}(\mathrm{t}) + \mathrm{J}^{\infty}(\mathrm{t}) + \mathrm{J}^{0}(\mathrm{t}) \qquad \text{a.s. for all } \mathrm{t} \geqslant 0,$$

where

- B is a standard Brownian motion in $\mathbb{R}^{\mathrm{d}}$

- J^{∞} is a compound Poisson Process in $\mathbb{R}^{\mathrm{d}}$
  - with intensity $\lambda = \nu(\mathrm{B}_1^{\mathrm{c}}(\mathbf{0}))$
  - and jump measure

$$\mu = \frac{\nu(\mathrm{B}_1^{\mathrm{c}}(\mathbf{0}) \cap \cdot)}{\lambda}$$

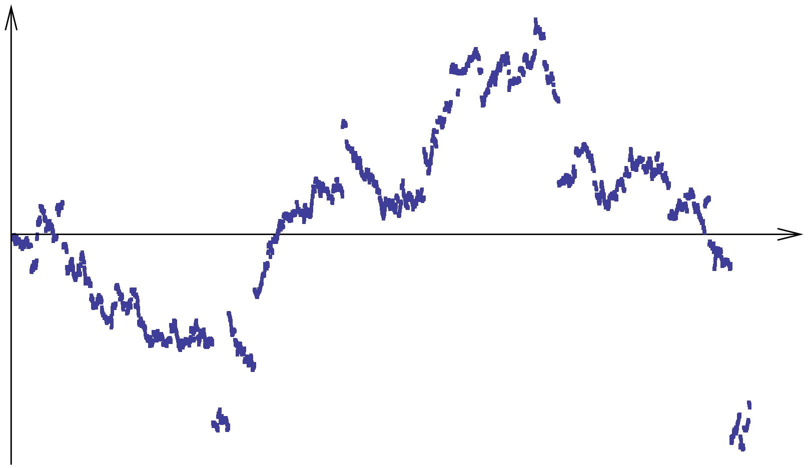
- J^{0} is a pure jump process
  - of possibly infinite intensity
  - with jumps bounded by 1.

A pure jump Lévy process in $\mathbb{R}^d$ is
a Lévy process $(\mathbf{L}(\mathbf{t}))_{\mathbf{t}\geqslant \mathbf{0}}$ in $\mathbb{R}^d$ with

$$\mathbf{L}(\mathbf{t}) = \mathbf{J}^{\infty}(\mathbf{t}) + \mathbf{J}^{\mathbf{0}}(\mathbf{t}),$$

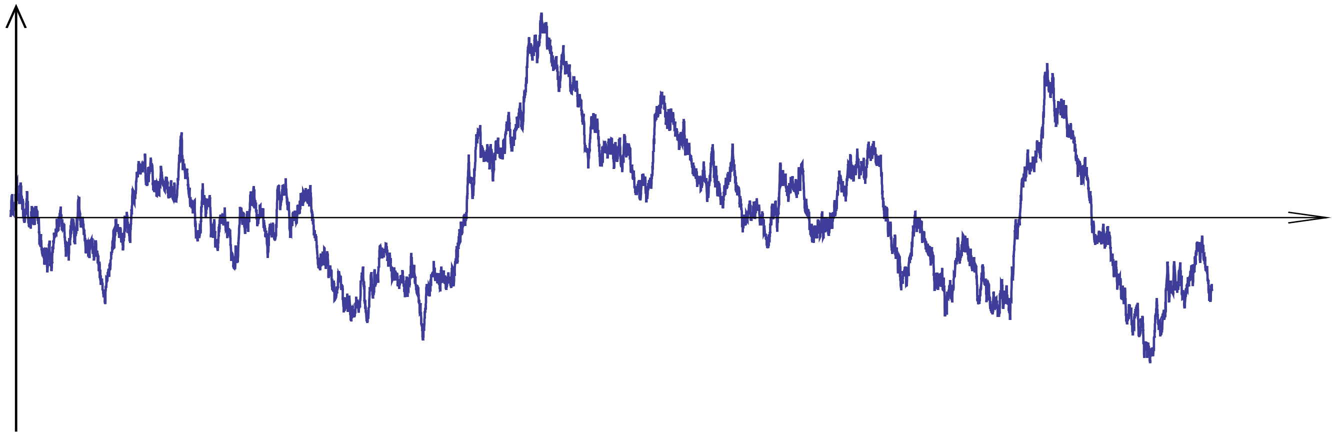
with Lévy measure ν

$$\nu(\{|\mathbf{y}| \geqslant \mathbf{1}\}) + \int_{|\mathbf{y}|<\mathbf{1}} |\mathbf{y}|^{\mathbf{2}}\nu(\mathbf{dy}) < \infty, \qquad \nu(\{\mathbf{0}\}) = \mathbf{0},$$

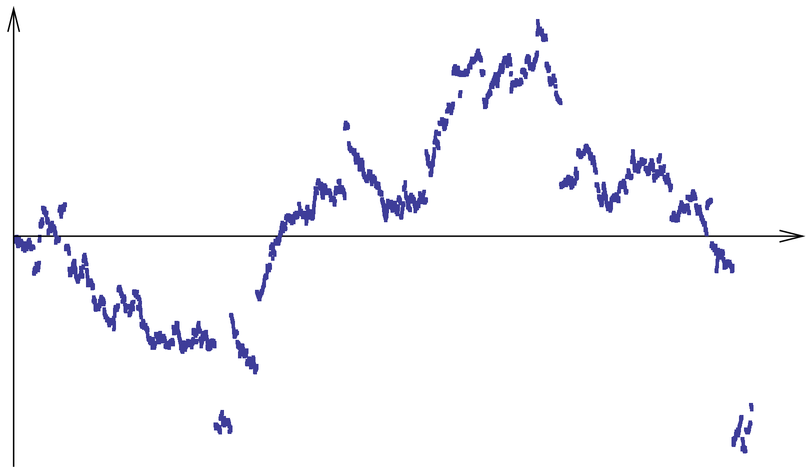


Comparison: Brownian motion vs. pure jump Lévy process

	Brownian motion
Representation	Gaussian densities, very wellknown
Paths	continuous, but rough “local variation”
Moments	all moments including exponential ones



	a pure-jump Lévy process with Gaussian increments
Representation	no densities in closed form, characteristic function with the Lévy measure instead distribution of jumps via Lévy measure
Paths	discontinuous, jumps occur (only right continuous with left limits)
Moments	exponential moments



Definition:

(i) A function $\mathbf{S} : (\mathbf{0}, \infty) \to (\mathbf{0}, \infty)$ is called a **speed function**, if $\lim_{\varepsilon \to \mathbf{0}+} \mathbf{S}(\varepsilon) = \mathbf{0}$ and there exists a continuous invertible function $\mathbf{S_o} : (\mathbf{0}, \infty) \to (\mathbf{0}, \infty)$, such that

$$\lim_{\varepsilon \to \mathbf{0}+} \frac{\mathbf{S_o}(\varepsilon)}{\mathbf{S}(\varepsilon)} = \mathbf{1}$$

(ii) Let $(\mathfrak{X}, \mathcal{T})$ be a topological space equipped with its Borel-σ-algebra $\mathcal{B}$, and $(\mathbf{X}^\varepsilon)_{\varepsilon > \mathbf{0}}$ be a family of random elements with values in $\mathfrak{X}$. Law$(X^\varepsilon)_{\varepsilon > 0}$ is said to satisfy a **large deviations principle (LDP) on** $(\mathfrak{X}, \mathcal{T})$ **with respect to a speed function** S **and a good rate function** I, if for every open subset $A \subset \mathfrak{X}$

$$\liminf_{\varepsilon \to 0} S(\varepsilon) \ln \mathbf{P}(X^\varepsilon \in A) \geq - \inf_{x \in A} I(x)$$

is valid and for every closed subset $A \subset \mathfrak{X}$

$$\limsup_{\varepsilon \to 0} S(\varepsilon) \ln \mathbf{P}(X^\varepsilon \in A) \leq - \inf_{x \in A} I(x).$$

I. The question: LDPs for rescaled Lévy processes

The setup:

Centered Lévy process $(\mathbf{L_t})_{\mathbf{t}\geqslant\mathbf{0}}$
with exponential moments

$$\mathbf{E}\mathbf{e}^{\lambda|\mathbf{L_1}|} < \infty$$

scaling $\varepsilon \to \mathbf{r}_\varepsilon > \mathbf{0}$ **such that** $\lim_{\varepsilon\to 0} \mathsf{r}_\varepsilon \in [0,\infty]$ **exists**

$$\mathbf{X}^\varepsilon = (\mathbf{X}^\varepsilon_\mathbf{t})_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]}, \qquad \mathbf{X}^\varepsilon_\mathbf{t} := \varepsilon\,\mathbf{L}_{\mathbf{t}\cdot\mathbf{r}_\varepsilon}$$

Motivation: Cramér's theorem

L **centered Lévy process with** $\mathbf{E}[\mathbf{e}^{\lambda|\mathbf{L_1}|}] < \infty$ **and** $\boxed{\mathbf{r}_\varepsilon = \varepsilon^{-1}}$

$$\mathbf{X}^\varepsilon_\mathbf{t} := \varepsilon\, \mathbf{L}_{\mathbf{t}\cdot\varepsilon^{-1}}$$

rewritten as a telescopic sum is a random walk

$$\mathbf{X}^\varepsilon_\mathbf{t} = \varepsilon \sum_{\mathbf{i}=\mathbf{1}}^{\varepsilon^{-1}} \underbrace{(\mathbf{L}_{\mathbf{it}} - \mathbf{L}_{(\mathbf{i}-\mathbf{1})\mathbf{t}})}_{\text{i.i.d.}}$$

Cramér's theorem yields a LDP in $\mathbb{R}^\mathbf{d}$, where the rate function is given as

$$\mathbf{I}(\mathbf{E}) = \inf_{\mathbf{z}\in\mathbf{E}} \mathbf{I}(\mathbf{z}), \qquad \mathbf{I}(\mathbf{z}) = \mathbf{\Lambda}^*(\mathbf{z}) \qquad \mathbf{\Lambda}(\lambda) = \mathbf{E}\mathbf{e}^{\mathbf{i}\langle\lambda,\mathbf{X_1}\rangle}$$

where $\mathbf{\Lambda}^*$ is the Fenchel-Legendre transform of the Log-Laplace $\mathbf{\Lambda}$ of $\mathbf{L_1}$.

$\longrightarrow$ LDP for the marginals $\mathbf{L_t}$ for and $\mathbf{t}$ in $\mathbb{R}^d$ ✓

$\longrightarrow$ LDP functional limit theorems in $\mathbf{D}_{[\mathbf{0},\mathbf{T}],\mathbb{R}^\mathbf{d}}$ ✓

Schilder's theorem:

$\mathrm{B} = (\mathrm{B_t})_{\mathrm{t}\in[0,\mathrm{T}]}$ **a Brownian motion,** $\boxed{\mathrm{r}_\varepsilon \equiv 1}$ **and** $\mathrm{X}^\varepsilon_{\mathbf{t}} = \varepsilon \mathrm{B}_{\mathbf{t}}$**.**

Then $\mathrm{P}^\varepsilon =$ **Law**(X^ε) **satisfies a LDP with good rate function**

$$\mathbf{I}(\mathbf{w}) = \begin{cases} \frac{1}{2}\int\limits_0^{\mathbf{T}} |\dot{\mathbf{w}}(\mathbf{s})|^2 \mathbf{ds} & \text{if } \mathbf{w} \in \mathbf{H}^1_0(\mathbf{0},\mathbf{T}) \\ \infty & \text{else} \end{cases}, \qquad \mathbf{w} \in \mathbf{C_0}([\mathbf{0},\mathbf{T}],\mathbb{R}^{\mathbf{n}})$$

$\longrightarrow$ **LDP on path space** $\mathbf{r}_\varepsilon = \mathbf{1} \ll \varepsilon^{-1}$

Schilder's theorem:

$B = (B_t)_{t \in [0,T]}$ **a Brownian motion,** $\boxed{r_\varepsilon \equiv 1}$ **and** $X_t^\varepsilon = \varepsilon B_t$.

Then $P^\varepsilon =$ **Law**(X^ε) **satisfies a LDP with good rate function**

$$\mathbf{I}(\mathbf{w}) = \begin{cases} \frac{1}{2}\int\limits_0^T |\dot{\mathbf{w}}(\mathbf{s})|^2 \mathbf{ds} & \text{if } \mathbf{w} \in \mathbf{H}_0^1(0,\mathbf{T}) \\ \infty & \text{else} \end{cases}, \qquad \mathbf{w} \in \mathbf{C_0}([\mathbf{0},\mathbf{T}], \mathbb{R}^{\mathbf{n}})$$

$\longrightarrow$ **LDP on path space** $\mathbf{r}_\varepsilon = \mathbf{1} \ll \varepsilon^{-1}$

Q: Is there an similar LDP for certain Lévy processes?
$\longrightarrow$ **What about a CPP with Gaussian increments?**

Centered Lévy process $(\mathbf{L_t})_{\mathbf{t}\geqslant 0}$
with exponential moments

$$\mathbf{E}\mathbf{e}^{\lambda|\mathbf{L_1}|} < \infty$$

scaling $\varepsilon \to \mathbf{r}_\varepsilon > 0$ **with** $\boxed{\mathbf{r}_\varepsilon \ll \varepsilon^{-1}}$

$$\mathbf{X}^\varepsilon = (\mathbf{X}^\varepsilon_\mathbf{t})_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]}, \qquad \mathbf{X}^\varepsilon_\mathbf{t} := \varepsilon\, \mathbf{L}_{\mathbf{t}\cdot\mathbf{r}_\varepsilon}$$

Main question:

Is there a LDP for $\mathrm{P}^\varepsilon =$ **Law**(X^ε) **on path space** $(D, \mathcal{B}(D))$**?**

$$-\inf_{z\in E^\circ} I(z) \leq \liminf_{\varepsilon\to 0+} S(\varepsilon)\ln \mathbf{P}^\varepsilon(E) \leq \limsup_{\varepsilon\to 0+} S(\varepsilon)\ln \mathbf{P}^\varepsilon(E) \leq -\inf_{z\in \bar{E}} I(z)$$

II. Fine density estimates for CPP with Weibull increments $\alpha > 1$

L **a centered compound Poisson process with jump measure** ν

$$\frac{\nu(\mathbf{dz})}{\mathbf{dz}} = \exp(-\mathbf{f}(|\mathbf{z}|))$$

where $\mathbf{f} : (\mathbf{0}, \infty) \to \mathbb{R}$ **is a smoothly regularly varying function of index** $\alpha > 1$

Example:

$$\mathbf{f}(\mathbf{r}) = \mathbf{r}^{\alpha}, \qquad \alpha > \mathbf{1}.$$

For $\alpha = \mathbf{2}$ think of

$$\frac{\nu(\mathbf{dz})}{\mathbf{dz}} \propto \exp(-\frac{\mathbf{1}}{\mathbf{2}}|\mathbf{z}|^{\mathbf{2}})$$

a CPP with Gaussian jumps.

Definition 2.1. *(i) Let $z \in \mathbb{R}$. A function $f : (z, \infty) \to \mathbb{R}$ is called* ***regular varying with index*** $\alpha \in \mathbb{R}$, *if* $\sup\{\Lambda \geq z \mid f(\Lambda) \leq 0\} < \infty$ *and* $\lim_{\Lambda \to \infty} \frac{f(\lambda\Lambda)}{f(\Lambda)} = \lambda^\alpha$ *holds for every* $\lambda > 0$*. We denote by* R_α *the class of regular varying functions with index* α*.*

(ii) Let $z \in \mathbb{R}$. A function $f : (z, \infty) \to \mathbb{R}$ is called ***smoothly regularly varying with index*** $\alpha \in \mathbb{R}$, *if f is infinitely often differentiable,* $z_o := \max\{1, \sup\{\Lambda \geq z \mid f(\Lambda) \leq 0\}\} < \infty$, *and* $h : (\ln z_o, \infty) \to \mathbb{R}$, $h(\cdot) := \ln f(\exp(\cdot))$ *satisfies* $\lim_{\Lambda \to \infty} h'(\Lambda) = \alpha$ *and* $\lim_{\Lambda \to \infty} h^{(m)}(\Lambda) = 0$ *for any* $m = 2, 3, \ldots$*. We denote by* SR_α *the class of smoothly regularly varying functions with index* α*.*

It is known that $\mathbf{f}(\mathbf{\Lambda}) = \mathbf{\Lambda}^\alpha \ell(\mathbf{\Lambda})$ (see Bingham, Goldie, Teugels).
Think of $\mathbf{f}(\mathbf{\Lambda}) = \mathbf{\Lambda}^\alpha$. Then

$$\lim_{\mathbf{\Lambda} \to \infty} \frac{\mathbf{d}}{\mathbf{d\Lambda}} \ln(\mathbf{f}(\exp(\mathbf{\Lambda})) = \lim_{\mathbf{\Lambda} \to \infty} \frac{\alpha(\exp(\mathbf{\Lambda})^{\alpha - \mathbf{1}})}{(\exp(\mathbf{\Lambda}))^\alpha} \exp(\mathbf{\Lambda}) = \alpha$$

Lemma 2.2 (Properties of smoothly regularly varying functions)**.** *Let* $\alpha, \beta \in \mathbb{R}$*,* $f \in SR_\alpha$ *and* $g \in SR_\beta$*. Then the following statements are valid:*

(i) $f \in R_\alpha$

(ii) $\lim_{\Lambda\to\infty} \frac{\Lambda f'(\Lambda)}{f(\Lambda)} = \alpha$

(iii) *If* $\alpha \geq \beta$ *then* $f + g \in SR_\alpha$.

(iv) *If* $\alpha > \beta$ *then* $f - g \in SR_\alpha$.

(v) $f \cdot g \in SR_{\alpha+\beta}$

(vi) *If* $\lim_{\Lambda\to\infty} g(\Lambda) = \infty$ *then* $f \circ g \in SR_{\alpha\beta}$.

(vii) $1/f \in SR_{-\alpha}$

$(viii)$ *If* $\alpha \neq 0$ *then* $|f'| \in SR_{\alpha-1}$.

(ix) *If* $\alpha > -1,$ *then* z *can be chosen sufficiently large, such that* $\Lambda \mapsto \int_z^\Lambda f(y)dy$ *exists. This function belongs to* $SR_{\alpha+1}$.

(x) *If* $\alpha > 0$*, then* z *can be chosen sufficiently large, such that* f *is invertible on* $[z, \infty)$*, and its inverse function* f^{-1} *belongs to* $SR_{\frac{1}{\alpha}}$.

Theorem: Asymptotic exponential density estimates

Let L given a CPP under the given hypotheses.

Set μ_{t}, $\mathrm{t} > 0$, the density of the marginal law L_{t}.

Then for any $\delta > 0$ and $\rho < \gamma < 1$ $\exists$ $\mathrm{k} > 0$ s.t. for all $|\mathbf{x}| > \mathrm{k}$ and $\mathrm{t} \in [|\mathbf{x}|^{\rho}, |\mathbf{x}|^{\gamma}]$:

$$-|\mathbf{x}|\Big(\mathbf{f}'(\mathbf{g}(\frac{|\mathbf{x}|}{\mathbf{t}})) - (\mathbf{1}-\delta)\mathbf{g}(\frac{|\mathbf{x}|}{\mathbf{t}})^{-1}\Big) \leq \ln \mu_{\mathbf{t}}(\mathbf{x}) \leq -|\mathbf{x}|\Big(\mathbf{f}'(\mathbf{g}(\frac{|\mathbf{x}|}{\mathbf{t}})) - (\mathbf{1}+\delta)\mathbf{g}(\frac{|\mathbf{x}|}{\mathbf{t}})^{-1}\Big)$$

where $\mathrm{g} : [\Lambda_{\mathbf{0}}, \infty) \to \mathbb{R}$ is the unique solution of the nonlinear functional equation

$$\mathbf{g}(\boldsymbol{\Lambda})\mathbf{f}'(\mathbf{g}(\boldsymbol{\Lambda})) - \mathbf{f}(\mathbf{g}(\boldsymbol{\Lambda})) + \ln \mathbf{g}(\boldsymbol{\Lambda}) - \frac{\mathbf{n}}{\mathbf{2}} \ln \mathbf{f}''(\mathbf{g}(\boldsymbol{\Lambda})) \;= \ln((\mathbf{2}\pi)^{-\frac{\mathbf{n}}{\mathbf{2}}}(\alpha - \mathbf{1})^{-\frac{\mathbf{n}-\mathbf{1}}{\mathbf{2}}}\boldsymbol{\Lambda})$$

Example: $\frac{\nu(\mathbf{dz})}{\mathbf{dz}} = \mathbf{c}_\alpha \exp(-|\mathbf{z}|^\alpha)$

For all $\mathbf{n} \in \mathbb{N}$, $\alpha > \mathbf{1}$, $\delta > \mathbf{0}$ and $\rho < \gamma < \mathbf{1}$
$\exists\ \mathbf{k} > \mathbf{0}$ s.th. for all $\mathbf{y} \in \mathbb{R}^{\mathbf{n}}$ with $|\mathbf{y}| > \mathbf{k}$ and $\mathbf{t} \in [|\mathbf{y}|^\rho, |\mathbf{y}|^\gamma]$ we have

$$-|\mathbf{y}|\left(\alpha\Big(\mathbf{g}(\frac{|\mathbf{y}|}{\mathbf{t}})\Big)^{\alpha-\mathbf{1}} - (\mathbf{1}-\delta)\mathbf{g}(\frac{|\mathbf{y}|}{\mathbf{t}})^{-\mathbf{1}}\right) \le \ln \mu_{\mathbf{t}}(\mathbf{y}) \le -|\mathbf{y}|\left(\alpha\Big(\mathbf{g}(\frac{|\mathbf{y}|}{\mathbf{t}})\Big)^{\alpha-\mathbf{1}} - (\mathbf{1}+\delta)\mathbf{g}(\frac{|\mathbf{y}|}{\mathbf{t}})^{-\mathbf{1}}\right)$$

where for some $\Lambda_{\mathbf{o}} > 0$ the function $\mathbf{g} : (\Lambda_{\mathbf{o}}, \infty) \to \mathbb{R}$ is given as the unique point-wise solution of the nonlinear functional equation

$$\big(\mathbf{g}(\mathbf{\Lambda})\big)^\alpha + \mathbf{C_1} \ln\big(\mathbf{g}(\mathbf{\Lambda})\big) = \mathbf{C_2} + \mathbf{C_3} \ln\big(\mathbf{\Lambda}\big), \qquad \mathbf{\Lambda} > \mathbf{\Lambda_0},$$

where $\mathbf{C_1} = \frac{\mathbf{2}-(\alpha-\mathbf{2})\mathbf{n}}{\mathbf{2}(\alpha-\mathbf{1})}, \quad \mathbf{C_2} = \frac{\ln(\alpha-\mathbf{1}) + \mathbf{n}\ln(\alpha) - \mathbf{n}\ln(\mathbf{2}\pi)}{\mathbf{2}(\alpha-\mathbf{1})}, \quad \mathbf{C_3} = \frac{\mathbf{1}}{\alpha-\mathbf{1}}.$

The function g is slowly varying and $\Lambda \to \mathrm{g}(\mathrm{e}^\Lambda)$ is a smoothly regularly varying function of order α^{-1}.

Lemma: [Key properties of g]
Let $\alpha > 1$ and $\mathbf{f} \in \mathbf{SR}_\alpha$. Let $\mathbf{b} < \alpha$ and $\mathbf{k} \in \mathbf{SR_b}$.

(i) **Existence, uniqueness and regularity:**
There is some $\mathbf{r_o} > \mathbf{0}$ such that for every $\mathbf{\Lambda} > \mathbf{r_o}$ the equation

$$\mathbf{g}(\mathbf{\Lambda})\mathbf{f}'(\mathbf{g}(\mathbf{\Lambda})) - \mathbf{f}(\mathbf{g}(\mathbf{\Lambda})) + \mathbf{k}(\mathbf{g}(\mathbf{\Lambda})) \;\; = \ln \mathbf{\Lambda} \tag{1}$$

has a unique solution $\mathbf{g} \in \mathbf{SR_o}$.

(ii) **Fine regularity:** Let $\mathbf{h} : (\ln \mathbf{r_o}, \infty) \to \mathbb{R}$ given by $\mathbf{h}(\mathbf{\Lambda}) = \mathbf{g}(\exp(\mathbf{\Lambda}))$. Then $\mathbf{h} \in \mathbf{SR}_{\frac{1}{\alpha}}$.

(iii) **The asymptotic cancellation relation:** For each $\gamma > \mathbf{0}, \delta > \mathbf{0}$ there is $\mathbf{z} > \mathbf{r_o}$ such that for every $\mathbf{\Lambda} > \mathbf{z}$ and $\mathbf{y} \in [(\ln \mathbf{\Lambda})^{-\gamma}, (\ln \mathbf{\Lambda})^{\gamma}]$ the following estimate is valid

$$\Big| \mathbf{g}(\mathbf{\Lambda})[\mathbf{f}'(\mathbf{g}(\mathbf{y\Lambda})) - \mathbf{f}'(\mathbf{g}(\mathbf{\Lambda}))] - \ln \mathbf{y} \Big| \leq \delta |\ln \mathbf{y}|.$$

(iv) **Asymptotic behavior:** The function $\mathbf{g}$ satisfies the following limits:

$$\lim_{\mathbf{\Lambda}\to\infty} \frac{\mathbf{g}'(\mathbf{\Lambda})\mathbf{\Lambda} \ln \mathbf{\Lambda}}{\mathbf{g}(\mathbf{\Lambda})} = \frac{\mathbf{1}}{\alpha}$$

$$\lim_{\mathbf{\Lambda}\to\infty} \mathbf{f}(\mathbf{g}(\mathbf{\Lambda}))(\ln \mathbf{\Lambda})^{-\mathbf{1}} = \frac{\mathbf{1}}{\alpha - \mathbf{1}}$$

By the CPP density representation

$$\mu_{\mathbf{t}}(\mathbf{z}) = \sum_{\mathbf{m}=\mathbf{1}}^{\infty} \mathbf{P}(\mathbf{N_t} = \mathbf{m})\, \frac{\nu^{*\mathbf{m}}(\mathbf{dz})}{\mathbf{dz}}, \qquad \mathbf{z} \neq \mathbf{0}$$

it is clear that estimates on $\ln \mu_{\mathbf{t}}(\mathbf{z})$ boil down to estimates on $\frac{\nu^{*\mathbf{m}}(\mathbf{dz})}{\mathbf{dz}}$

Proposition: The tails of the density of the m-th jump

Let L given a CPP under the given hypotheses and assume $\nu(\mathbb{R}^{\mathbf{d}}) = \mathbf{1}$.
For $m \in \mathbb{N}$ we denote the m-fold convolution of ν with itself by ν^{*m}.

Then for all $\delta > 0$ there is a $\mathrm{k} > 0$ such that for all $\mathrm{m} \in \mathbb{N}$ and $|\mathbf{x}| > \mathbf{km}$ it follows

$$\frac{\nu^{*\mathbf{m}}(\mathbf{dx})}{\mathbf{dx}} \leq (\alpha - \mathbf{1})^{\frac{(\mathbf{n}-\mathbf{1})(\mathbf{m}-\mathbf{1})}{\mathbf{2}}} (\mathbf{2}\pi \mathbf{f}''(\frac{|\mathbf{x}|}{\mathbf{m}})^{-\mathbf{1}})^{\frac{\mathbf{n}(\mathbf{m}-\mathbf{1})}{\mathbf{2}}} \cdot \exp\Big(-\mathbf{m}(\mathbf{f}(\frac{|\mathbf{x}|}{\mathbf{m}}) - \delta)\Big),$$

$$\frac{\nu^{*\mathbf{m}}(\mathbf{dx})}{\mathbf{dx}} \geq \frac{(\alpha - \mathbf{1})^{\frac{(\mathbf{n}-\mathbf{1})(\mathbf{m}-\mathbf{1})}{\mathbf{2}}}}{\mathbf{m}^{\frac{\mathbf{n}}{\mathbf{2}}}} (\mathbf{2}\pi \mathbf{f}''(\frac{|\mathbf{x}|}{\mathbf{m}})^{-\mathbf{1}})^{\frac{\mathbf{n}(\mathbf{m}-\mathbf{1})}{\mathbf{2}}} \cdot \exp\Big(-\mathbf{m}(\mathbf{f}(\frac{|\mathbf{x}|}{\mathbf{m}}) + \delta)\Big).$$

Proof:

- $\mathrm{m} = 1$ **the estimates are valid trivially.**
- $\mathrm{m} \geq 2$**, by rotational invariance it is enough to study** $\mathrm{x} = |\mathrm{x}|\mathrm{e}_1$**.**
  The convolution density reads

$$\frac{\nu^{*m}(dx)}{dx} = \int_{\mathbb{R}^n\setminus\{0\}} \cdots \int_{\mathbb{R}^n\setminus\{0\}} \exp\Big(- \sum_{i=1}^{m-1} f(|y_i|) - f\Big(\Big|x - \sum_{i=1}^{m-1} y_i\Big|\Big)\Big) dy_1 \ldots dy_{m-1}$$

$$= \int_{\mathbb{R}^n\setminus\{\frac{x}{m}\}} \cdots \int_{\mathbb{R}^n\setminus\{\frac{x}{m}\}} \exp\Big(- \sum_{i=1}^{m-1} f\Big(|\frac{x}{m} + y_i|\Big) - f\Big(|\frac{x}{m} - \sum_{i=1}^{m-1} y_i|\Big)\Big) dy_1 \ldots dy_{m-1}$$

$$= \exp\Big(- m f(|\frac{x}{m}|)\Big)$$

$$\cdot \int_{\mathbb{R}^n\setminus\{\frac{x}{m}\}} \cdots \int_{\mathbb{R}^n\setminus\{\frac{x}{m}\}} \exp\Big(- \sum_{i=1}^{m-1} f_{x,m}(y_i) - f_{x,m}\Big(- \sum_{i=1}^{m-1} y_i\Big)\Big) dy_1 \ldots dy_{m-1},$$

for the auxiliary function $f_{x,m} : \mathbb{R}^n \setminus \{-\frac{x}{m}\} \to \mathbb{R}$:

$$f_{x,m}(z) = f(|\frac{x}{m} + z|) - (f(\frac{|x|}{m}) + z_1 f'(\frac{|x|}{m}))$$

where the seemingly missing summands $\sum_{i=1}^{m-1}(-y_i)f'(|\frac{x}{m}|) - (-\sum_{i=1}^{m-1} y_i)f'(|\frac{x}{m}|)$ add up to zero.

For an estimate on the integral we need and estimate on $f_{x,m}$:

$$
\begin{aligned}
f_{x,m}(z) &= f(|\frac{x}{m}+z|) - f(\frac{|x|}{m}) - z_1 f'(\frac{|x|}{m})) \\
&= (f(|\frac{x}{m}+z|) - f(|\frac{x}{m}+z_1 e_1|)) + (f(\frac{|x|}{m}+z_1) - (f(\frac{|x|}{m}) + z_1 f'(\frac{|x|}{m})))
\end{aligned}
$$

Hypothesis I on f yields that for any $\delta > 0$, $c \in (1-\frac{\alpha}{2}, 1)$ there is $\frac{|x|}{m}$ sufficiently large such that for $|z| \leq (\frac{|x|}{m})^c$ it follows

$$
f(\frac{|x|}{m}+z_1) - (f(\frac{|x|}{m}) + z_1 f'(\frac{|x|}{m})) \quad \leq \sup_{|q| \leq (\frac{|x|}{m})^c} \frac{1}{2} f''(\frac{|x|}{m}+q) z_1^2 \quad \leq (1+\delta)\frac{1}{2} f''(\frac{|x|}{m}) z_1^2. \tag{3.3}
$$

For $\frac{|x|}{m}$ sufficiently large and $|z| < (\frac{|x|}{m})^c$ we have

$$
\begin{aligned}
f(|\frac{x}{m}+z|) - f(|\frac{x}{m}+z_1 e_1|) \quad &\leq \sup_{|q| \leq (\frac{|x|}{m})^c} \frac{1}{2} f'(\frac{|x|}{m}+q)(|\frac{x}{m}+z| - |\frac{x}{m}+z_1 e_1|) \\
&\leq (1+\delta) f'(\frac{|x|}{m}) \frac{1}{2} \frac{m}{|x|} |(0, z_2, \ldots, z_n)|^2 \quad \leq (1+2\delta) \frac{1}{\alpha-1} \frac{1}{2} f''(\frac{|x|}{m}) \sum_{i=2}^{n} z_i^2
\end{aligned}
$$

The corresponding lower bound of $f_{x,m}$ can be estimated similarly.

Hence for any $\delta > 0$, $c \in (1 - \frac{\alpha}{2}, 1)$ there is $k > 0$ such that for any $\frac{|x|}{m} > k$ and $|z| < (\frac{|x|}{m})^c$ we have

$$\Big| f_{x,m}(z) - \frac{1}{2} f''(\frac{|x|}{m}) \Big(z_1^2 + \frac{1}{\alpha - 1} \sum_{i=2}^{n} z_i^2 \Big) \Big| \leq \delta |z|^2 f''(\frac{|x|}{m}).$$

Upper bound of (2.10)**:**

$$\begin{aligned}\frac{\nu^{*m}(dx)}{dx} &\leq \exp(-mf(|\frac{x}{m}|))\int_{\mathbb{R}^n\setminus\{\frac{|x|}{m}\}}\cdots\int_{\mathbb{R}^n\setminus\{\frac{|x|}{m}\}}\exp\Big(-\sum_{i=1}^{m-1}f_{x,m}(y_i)\Big)dy_1\ldots dy_{m-1}\\ &= \exp(-mf(|\frac{x}{m}|))\Big(\int_{\mathbb{R}^n\setminus\{\frac{|x|}{m}\}}\exp(-f_{x,m}(y))dy\Big)^{m-1}\end{aligned}$$

We estimate the value of the integral for $y \in [-(\frac{|x|}{m})^c, (\frac{|x|}{m})^c]^n$

For $\frac{|x|}{m}$ large enough,
and $y \in [-(\frac{|x|}{m})^c, (\frac{|x|}{m})^c]$, we estimate $f_{x,m}(y) > (1-\delta)\frac{1}{2}f''(\frac{|x|}{m})(y_1^2 + \frac{1}{\alpha-1}\sum_{i=2}^n y_i^2)$

Gaussian renormalization $\sqrt{2\pi a} = \int_{\mathbb{R}}\exp(\frac{-s^2}{2a})ds$, $a > 0$,

$$\begin{aligned}\int_{\left[-(\frac{|x|}{m})^c,(\frac{|x|}{m})^c\right]^n}\exp(-f_{x,m}(y))dy &\leq \int_{\mathbb{R}^n}\exp\Big(-(1-\delta)\frac{1}{2}f''(\frac{|x|}{m})\Big(y_1^2+\frac{1}{\alpha-1}\sum_{i=2}^n y_i^2\Big)\Big)dy\\ &= (\alpha-1)^{\frac{n-1}{2}}(2\pi((1-\delta)f''(\frac{|x|}{m}))^{-1})^{\frac{n}{2}}.\end{aligned}$$

It remains to estimate the integral for $y \in \mathbb{R}^n \setminus [-(\frac{|x|}{m})^c, (\frac{|x|}{m})^c]^n$ for $|y| \geq (\frac{|x|}{m})^c$.

$$
\begin{aligned}
\frac{d^2}{ds^2} f_{x,m}(sy) &= \frac{d^2}{ds^2}\Big(f(|\frac{x}{m} + sy|) - f(|\frac{x}{m}|) - sy_1 f'(|\frac{x}{m}|)\Big) \\
&= \Big(\frac{d^2}{ds^2}|\frac{x}{m} + sy|\Big) f'(|\frac{x}{m} + sy|) + \Big(\frac{d}{ds}|\frac{x}{m} + sy|\Big)^2 f''(|\frac{x}{m} + sy|)
\end{aligned}
$$

for any $y \in \mathbb{R}^n \setminus \{0\}$, $s > 0$, such that $\frac{x}{m} + sy \neq 0$

In case of f being convex and nondecreasing hence the preceding RHS is eventually nonnegative.

Hence together with $f_{x,m}(0) = 0$ we have

$$
f_{x,m}(sz) \quad \geq s f_{x,m}(z) \qquad \text{for any } z \in \mathbb{R}^n \text{ and } s > 1.
$$

This condition can be removed in two steps at the cost of an asymptotic error, which tends to 0 fast enough, not to change the result.

Lower bound of (2.10):

Recall:

$$\exp(mf(|\frac{x}{m}|)) \cdot \frac{\nu^{*m}(dx)}{dx}$$
$$= \int\limits_{\mathbb{R}^n} \cdots \int\limits_{\mathbb{R}^n} \exp\Big(- \sum_{i=1}^{m-1} f_{x,m}(y_i) - f_{x,m}\Big(- \sum_{i=1}^{m-1} y_i\Big)\Big) d\mathbf{y}$$

Very rough outline:

1. Pass from the $m-1$ fold $\mathbb{R}^n$ integration $(y_1, \ldots, y_{m-1})$
   to $m-1$-fold $\mathbb{R}$ integration with variables $(\tilde{y}_1, \ldots, \tilde{y}_{m-1})$ to the power n.

2. Carry out an appropriate substitution over an $m-2$ dimensional subspace. Elementary, nontrival estimates

3. Finally, conclude with the help of an auxiliary function θ_{m-1}.

$$\theta_{m-1} : \mathbb{R}^{m-1} \to [0, \infty), \qquad \theta_{m-1}(z) := \sum_{i=1}^{m-1} z_i^2 + \Big(\sum_{i=1}^{m-1} z_i \Big)^2$$

Sketch of the lower bound of the density estimate $\ln \mu_t$

Consider the CPP case with $\nu(\mathbb{R}^n) = 1$.
Consider g_o the solution of the auxiliary nonlinear functional equation

$$\mathbf{g_o(\Lambda)f'(g_o(\Lambda)) - f(g_o(\Lambda)) + \ln g_o(\Lambda) - \frac{n}{2}\ln f''(g_o(\Lambda)) + \frac{n g_o(\Lambda) f'''(g_o(\Lambda))}{2f''(g_o(\Lambda))}}$$
$$\mathbf{+\frac{n}{2}\ln(2\pi) + \frac{n-1}{2}\ln(\alpha - 1) = \ln \Lambda.}$$

Claim 1 (lower bound):
For any $\delta > 0$ and $\rho < \gamma < 1$, there is $k > 0$, such that for all $|x| > k$ and $t \in [|x|^\rho, |x|^\gamma]$:

$$\mathbf{-|x|\Big(f'(g_o(\frac{|x|}{t})) + (n(\frac{\alpha}{2} - 1) - 1 + \delta) g_o(\frac{|x|}{t})^{-1}\Big) \ \leq \ln \mu_t(|x|)}$$

For $\mathbf{n} \in \mathbb{N}$, $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$ and $\mathbf{t} \in [|\mathbf{x}|^{\rho}, |\mathbf{x}|^{\gamma}]$ define the "maximum likelihood index"

$$\mathbf{m_{x,t}} := |\mathbf{x}|\mathbf{g_o}(\frac{|\mathbf{x}|}{\mathbf{t}})^{-1} \qquad \tilde{\mathbf{m}}_{\mathbf{x,t}} := \lfloor \mathbf{m_{x,t}} \rfloor + \mathbf{1}.$$

Since $\rho < \gamma < 1$ we have by the definition of $\mathbf{m_{x,t}}$ that

$$\lim_{|\mathbf{x}|\to\infty} \inf_{\mathbf{t}\in[|\mathbf{x}|^{\rho},|\mathbf{x}|^{\gamma}]} \frac{|\mathbf{x}|}{\tilde{\mathbf{m}}_{\mathbf{x,t}}} = \infty \qquad \text{and} \qquad \lim_{|\mathbf{x}|\to\infty} \sup_{\mathbf{t}\in[|\mathbf{x}|^{\rho},|\mathbf{x}|^{\gamma}]} \frac{\mathbf{t}}{\tilde{\mathbf{m}}_{\mathbf{x,t}}} = \mathbf{0}.$$

The first limit allows to apply the lower bound of the convolution density $\frac{\nu^{*\tilde{\mathbf{m}}_{\mathbf{x,t}}}(\mathbf{dx})}{\mathbf{dx}}$

The second limit is used below.

Recall the density

$$\mu_{\mathbf{t}}(\mathbf{z}) = \sum_{\mathbf{m}=1}^{\infty} \mathbf{P}(\mathbf{N_t} = \mathbf{m}) \, \frac{\nu^{*\mathbf{m}}(\mathbf{dz})}{\mathbf{dz}}, \qquad \mathbf{z} \neq \mathbf{0}$$

and that $\mathbf{N_t}$ has a Poisson distribution with expectation t.
Note: for large values of $|\mathbf{x}|$ and $\mathbf{t} \in [|\mathbf{x}|^{\rho}, |\mathbf{x}|^{\gamma}]$

$$\frac{|\mathbf{x}|}{\tilde{\mathbf{m}}_{\mathbf{x,t}}} = \mathbf{g_o}(\frac{|\mathbf{x}|}{\mathbf{t}})$$

Then for $\delta \in (\mathbf{0}, \mathbf{1})$, $|\mathbf{x}|$ can be chosen sufficiently large such that for all $\mathbf{t} \in [|\mathbf{x}|^{\rho}, |\mathbf{x}|^{\gamma}]$:

$$\begin{aligned}
&\ln \mu_{\mathbf{t}}(\mathbf{x}) \\
&\geq \ln \left(\mathbf{P}(\mathbf{N_t} = \tilde{\mathbf{m}}_{\mathbf{x},\mathbf{t}}) \frac{\nu^{*\tilde{\mathbf{m}}_{\mathbf{x},\mathbf{t}}}(\mathbf{dx})}{\mathbf{dx}} \right) \\
&= -\mathbf{t} + \ln \left(\frac{\mathbf{t}^{\tilde{\mathbf{m}}_{\mathbf{x},\mathbf{t}}}}{\tilde{\mathbf{m}}_{\mathbf{x},\mathbf{t}}!} \nu^{*\tilde{\mathbf{m}}_{\mathbf{x},\mathbf{t}}}(\mathbf{dx}) \right) \\
&\geq -\mathbf{m}_{\mathbf{x},\mathbf{t}} \left(\ln \frac{\mathbf{m}_{\mathbf{x},\mathbf{t}}}{\mathbf{t}} - \mathbf{1} + \mathbf{f}\left(\frac{|\mathbf{x}|}{\mathbf{m}_{\mathbf{x},\mathbf{t}}}\right) + \frac{\mathbf{n}}{\mathbf{2}} \ln \mathbf{f}''\left(\frac{|\mathbf{x}|}{\mathbf{m}_{\mathbf{x},\mathbf{t}}}\right) - \frac{\mathbf{n}-\mathbf{1}}{\mathbf{2}} \ln(\alpha - \mathbf{1}) - \frac{\mathbf{n}}{\mathbf{2}} \ln \mathbf{2}\pi + \frac{\delta}{\mathbf{2}} \right) \\
&= -|\mathbf{x}| \mathbf{g_o}\left(\frac{|\mathbf{x}|}{\mathbf{t}}\right)^{-1} \Big(\ln \frac{|\mathbf{x}|}{\mathbf{t}} - \ln \mathbf{g_o}\left(\frac{|\mathbf{x}|}{\mathbf{t}}\right) + \mathbf{f}\left(\mathbf{g_o}\left(\frac{|\mathbf{x}|}{\mathbf{t}}\right)\right) + \frac{\mathbf{n}}{\mathbf{2}} \ln \mathbf{f}''\left(\mathbf{g_o}\left(\frac{|\mathbf{x}|}{\mathbf{t}}\right)\right) \\
&\qquad - \frac{\mathbf{n}-\mathbf{1}}{\mathbf{2}} \ln(\alpha - \mathbf{1}) - \frac{\mathbf{n}}{\mathbf{2}} \ln \mathbf{2}\pi - \mathbf{1} + \frac{\delta}{\mathbf{2}} \Big) \\
&= -|\mathbf{x}| \mathbf{g_o}\left(\frac{|\mathbf{x}|}{\mathbf{t}}\right)^{-1} \left(\mathbf{g_o}\left(\frac{|\mathbf{x}|}{\mathbf{t}}\right) \mathbf{f}'\left(\mathbf{g_o}\left(\frac{|\mathbf{x}|}{\mathbf{t}}\right)\right) + \frac{\mathbf{n}}{\mathbf{2}} \frac{\mathbf{g_o}\left(\frac{|\mathbf{x}|}{\mathbf{t}}\right) \mathbf{f}'''\left(\mathbf{g_o}\left(\frac{|\mathbf{x}|}{\mathbf{t}}\right)\right)}{\mathbf{f}''\left(\mathbf{g_o}\left(\frac{|\mathbf{x}|}{\mathbf{t}}\right)\right)} - \mathbf{1} + \frac{\delta}{\mathbf{2}} \right).
\end{aligned} \tag{2}$$

NFE for $\boldsymbol{\Lambda} = \frac{|\mathbf{x}|}{\mathbf{t}}$

$$\begin{aligned}
&\ln \boldsymbol{\Lambda} - \ln \mathbf{g_o}(\boldsymbol{\Lambda}) + \mathbf{f}(\mathbf{g_o}(\boldsymbol{\Lambda})) + \frac{\mathbf{n}}{\mathbf{2}} \ln \mathbf{f}''(\mathbf{g_o}(\boldsymbol{\Lambda})) - \frac{\mathbf{n}-\mathbf{1}}{\mathbf{2}} \ln(\alpha - \mathbf{1}) - \frac{\mathbf{n}}{\mathbf{2}} \ln(\mathbf{2}\pi) \\
&= \mathbf{g_o}(\boldsymbol{\Lambda}) \mathbf{f}'(\mathbf{g_o}(\boldsymbol{\Lambda})) + \frac{\mathbf{n}}{\mathbf{2}} \frac{\mathbf{g_o}(\boldsymbol{\Lambda}) \mathbf{f}'''(\mathbf{g_o}(\boldsymbol{\Lambda}))}{\mathbf{f}''(\mathbf{g_o}(\boldsymbol{\Lambda}))}
\end{aligned}$$

III. No LDP for reparametrized Lévy processes on short time scale

1. For a **parametrized Lévy process** Z^ε such that Z_t^ε satisfies a **LDP with some speed function** $S(\varepsilon)$ **and some good rate function** I_t.
2. By the exponential density estimates, it is clear, that in case of $\mathbf{X}_\mathbf{t}^\varepsilon = \varepsilon \mathbf{L}_{\mathbf{tr}_\varepsilon}$ this is

$$\mathbf{I_t}(\mathbf{x}) = |\mathbf{x}|$$

since by the asymptotic density estimates we have

$$\mathbb{P}(\varepsilon \mathbf{L}_{\mathbf{tr}_\varepsilon} = \mathbf{y}) \quad {}'' \Leftrightarrow {}'' \quad \mu_{\mathbf{tr}_\varepsilon}(\frac{\mathbf{y}}{\varepsilon}) = \mu_{\mathbf{tr}_\varepsilon}(\frac{|\mathbf{y}|}{\varepsilon}\mathbf{e_1})$$

and

$$\ln \mu_{\mathbf{tr}_\varepsilon}(\frac{\mathbf{y}}{\varepsilon}) \approx -\frac{|\mathbf{y}|}{\varepsilon}\Big(\mathbf{f}'(\mathbf{g}(\frac{|\mathbf{y}|}{\mathbf{t}\varepsilon\mathbf{r}_\varepsilon})) \quad \underbrace{-\mathbf{g}(\frac{|\mathbf{y}|}{\mathbf{tr}_\varepsilon})^{-1}}_{\text{smaller order}} \quad \Big)$$

such that

$$\mathbf{S}(\varepsilon) \ln \mu_{\mathbf{tr}_\varepsilon}(\frac{\mathbf{y}}{\varepsilon}) = -|\mathbf{y}|$$

when $\mathbf{S}(\varepsilon) = \varepsilon \cdot \Big(\mathbf{f}'\big(\mathbf{g}\big((\varepsilon \mathbf{r}_\varepsilon)^{-1}\big)\big)\Big)^{-1}$.

3. Then by a result in Dembo-Zeitouni [Theorem 4.2.1] we have for $(Z_{t_1}, \dots, Z_{t_m})$ the LDP

$$\mathbf{I}_{(\mathbf{t_1},\dots,\mathbf{t_m})}(\mathbf{x_1}, \dots, \mathbf{x_m}) = \mathbf{I}_{\mathbf{t_1}}(\mathbf{x_1}) + \sum_{\mathbf{i=1}}^{\mathbf{m}} \mathbf{I}_{\mathbf{t_i}-\mathbf{t_{i-1}}}(\mathbf{x_i} - \mathbf{x_{i-1}})$$

4. By Feng Kurtz (Theorem 4.28) we have on the Skorokhod space with J_1-topology

$$\mathbf{I}(\varphi) := \sup_{\substack{m\in\mathbb{N} \\ 0\le t_1<\cdots<t_m \\ \phi \text{ cont. in } t_1,\dots,t_m}} \mathbf{I}_{(\mathbf{t_1},\dots,\mathbf{t_m})}(\varphi(\mathbf{t_1}), \dots, \varphi(\mathbf{t_m})), \qquad \varphi \in \mathbf{D}_{[\mathbf{0},\infty),\mathbb{R}^{\mathbf{n}}}$$

5. Assume that $\mathbf{I}$ defined that way is $\mathbf{S}(\varepsilon)$-exponentially tight in $\mathbf{D}_{[\mathbf{0},\mathbf{T}],\mathbb{R}^{\mathbf{n}}}$. Then we may take some $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$ with $|\mathbf{x}| = \mathbf{1}$ and

$$\varphi(\mathbf{t}) := \mathbf{x} \cdot \mathbf{1}_{[\mathbf{T},\infty)}(\mathbf{t})$$

Clearly $\mathbf{I}(\varphi) = \mathbf{1}$.

6. By construction φ is discontinuous in $\mathbf{T} = \mathbf{1}$. Since the uniform norm topology is strictly finer than the $\mathbf{J_1}$-topology on $\mathbf{D}$ there is a modulus of continuity, that is, there is a ball radius $\kappa_{\mathbf{1}} > \mathbf{0}$ such that for

$$\mathbf{A} := \{\vartheta \in \mathbf{D} \mid \mathbf{d}_{\mathbf{J_1}}(\vartheta, \varphi) < \kappa_{\mathbf{1}}\}$$

we have

$$\kappa_{\mathbf{2}} := \inf_{\vartheta \in \mathbf{A}} \sup_{\mathbf{t}} |\vartheta(\mathbf{t}) - \vartheta(\mathbf{t}-)| > \mathbf{0}.$$

7. But

$$\begin{aligned}
&\lim_{\kappa \to \mathbf{0}} \lim_{\varepsilon \to \mathbf{0}} \mathbf{S}(\varepsilon) \ln \mathbf{P}\big(\mathbf{d}(\varphi, \mathbf{X}^{\varepsilon}) < \kappa\big) \\
\leq &\lim_{\varepsilon \to \mathbf{0}} \mathbf{S}(\varepsilon) \ln \mathbf{P}\big(\mathbf{d}(\varphi, \mathbf{X}^{\varepsilon}\big) < \kappa_{\mathbf{1}}) \\
\leq &\lim_{\varepsilon \to \mathbf{0}} \mathbf{S}(\varepsilon) \ln \mathbf{P}\Big(\sup_{\mathbf{t} \leq \mathbf{2r}_{\varepsilon}\mathbf{T}} \varepsilon|\mathbf{L_t} - \mathbf{L_{t-}}| > \kappa_{\mathbf{2}}\Big) \\
= &-\infty \neq -\mathbf{1} = -\mathbf{I}(\varphi)
\end{aligned}$$

IV. A LDP for respective Lévy bridges on path space

- L a centered compound Poisson process with jump measure ν

$$\frac{\nu(\mathbf{dz})}{\mathbf{dz}} = \exp(-\mathbf{f}(|\mathbf{z}|))$$

  where $\mathrm{f} : (\mathbf{0}, \infty) \to \mathbb{R}$ is a smoothly regularly varying function of index $\alpha > 1$

- r_ε smoothly regularly varying scaling with index $\rho > -1$

$$\mathbf{X}_\mathbf{t}^\varepsilon := \varepsilon \mathbf{L}_{\mathbf{tr}_\varepsilon}$$

- Define the bridge $\mathrm{Y}^{\varepsilon,\bar{\mathrm{x}}}$ of X^ε conditioned to end at a given point $\bar{\mathrm{x}} \neq 0$ for given time $\mathrm{T} > 0$ (w.l.o.g. set $\mathrm{T} = 1$).

- **Denote the densitiy of $\mathbf{Y}^{\varepsilon,\bar{\mathbf{x}}}$ by $\bar{\mu}^{\varepsilon}_{\mathbf{t}}(\mathbf{y})$**

$$\bar{\mu}^{\varepsilon}_{\mathbf{t}}(\mathbf{y}) = \frac{\varepsilon^{-\mathbf{1}}\mu_{\mathbf{t}\mathbf{r}_\varepsilon}(\mathbf{y}\varepsilon^{-\mathbf{1}})\ \varepsilon^{-\mathbf{1}}\mu_{(\mathbf{1}-\mathbf{t})\mathbf{r}_\varepsilon}((\bar{\mathbf{x}}-\mathbf{y})\varepsilon^{-\mathbf{1}})}{\varepsilon^{-\mathbf{1}}\mu_{\mathbf{r}_\varepsilon}(\bar{\mathbf{x}}\varepsilon^{-\mathbf{1}})}$$

$$= \frac{\varepsilon^{-\mathbf{1}}\mu_{\mathbf{t}\mathbf{r}_\varepsilon}(\mathbf{y}\varepsilon^{-\mathbf{1}})\mu_{(\mathbf{1}-\mathbf{t})\mathbf{r}_\varepsilon}(\bar{\mathbf{x}}-\mathbf{y})}{\mu_{\mathbf{r}_\varepsilon}(\bar{\mathbf{x}}\varepsilon^{-\mathbf{1}})}$$

- **Note: It is given in terms of the densities μ_t, so we can use the previous upper and lower bounds.**

$$\ln \mu_{\mathbf{t}}(\mathbf{y}) \approx -|\mathbf{y}|\mathbf{f}'(\mathbf{g}(\frac{|\mathbf{x}|}{\mathbf{t}}))$$

- Hence

$$\begin{aligned}
&\ln \bar{\mu}^{\varepsilon}_{\mathbf{t}}(\mathbf{y}) \\
&= \mathbf{f}'(\mathbf{g}(\frac{|\mathbf{y}|\varepsilon^{-\mathbf{1}}}{\mathbf{t}\mathbf{r}_\varepsilon}))|\mathbf{y}|\varepsilon^{-\mathbf{1}} + \mathbf{f}'(\mathbf{g}(\frac{|\bar{\mathbf{x}}-\mathbf{y}|\varepsilon^{-\mathbf{1}}}{(\mathbf{1}-\mathbf{t})\mathbf{r}_\varepsilon}))|\bar{\mathbf{x}}-\mathbf{y}|\varepsilon^{-\mathbf{1}} - \mathbf{f}'(\mathbf{g}(\frac{|\bar{\mathbf{x}}|\varepsilon^{-\mathbf{1}}}{\mathbf{r}_\varepsilon}))|\bar{\mathbf{x}}|\varepsilon^{-\mathbf{1}} \\
&\qquad + \mathbf{o}\Big(\mathbf{g}(\frac{\varepsilon^{-\mathbf{1}}}{\mathbf{r}_\varepsilon})|\bar{\mathbf{x}}-\mathbf{y}|\varepsilon^{-\mathbf{1}}\Big).
\end{aligned}$$

On the segment $[[0, \bar{\mathbf{x}}]]$

For $\mathbf{y} \in [[\mathbf{0}, \bar{\mathbf{x}}]]$ we have $|\bar{\mathbf{x}}| = |\bar{\mathbf{x}} - \mathbf{y}| + |\mathbf{y}|$

$$\ln \bar{\mu}^{\varepsilon}_{\mathbf{t}}(\mathbf{y}) = \left(\mathbf{f}'(\mathbf{g}(\frac{|\mathbf{y}|\varepsilon^{-1}}{\mathbf{t}\mathbf{r}_{\varepsilon}})) - \mathbf{f}'(\mathbf{g}(\frac{|\bar{\mathbf{x}}|\varepsilon^{-1}}{\mathbf{r}_{\varepsilon}}))\right)|\mathbf{y}|\varepsilon^{-1}$$
$$+\left(\mathbf{f}'(\mathbf{g}(\frac{|\bar{\mathbf{x}} - \mathbf{y}|\varepsilon^{-1}}{(\mathbf{1}-\mathbf{t})\mathbf{r}_{\varepsilon}})) - \mathbf{f}'(\mathbf{g}(\frac{|\bar{\mathbf{x}}|\varepsilon^{-1}}{\mathbf{r}_{\varepsilon}}))\right)|\bar{\mathbf{x}} - \mathbf{y}|\varepsilon^{-1} + \mathbf{o}\Big(\mathbf{g}(\frac{\varepsilon^{-1}}{\mathbf{r}_{\varepsilon}})|\bar{\mathbf{x}} - \mathbf{y}|\varepsilon^{-1}\Big).$$

By the asymptotic cancellation relation of g we have for any $\gamma > 0$

$$\mathbf{f}'(\mathbf{g}(|\mathbf{u}|\mathbf{\Lambda})) - \mathbf{f}'(\mathbf{g}(\mathbf{\Lambda})) = \frac{\ln|\mathbf{u}|}{\mathbf{g}(\mathbf{\Lambda})} + \mathbf{o}(\frac{|\ln|\mathbf{u}||}{\mathbf{g}(\mathbf{\Lambda})})$$

uniformly for $|\mathbf{u}| \in [\ln(\mathbf{\Lambda})^{-\gamma}, \ln(\mathbf{\Lambda})^{\gamma}]$, as $\mathbf{\Lambda} \to \infty$.

For $\mathbf{\Lambda} = (\varepsilon\mathbf{r}_{\varepsilon})^{-1}$, $\mathbf{u_1} = \frac{\mathbf{y}}{\mathbf{t}}$, $\mathbf{u_2} = \bar{\mathbf{x}}$, $\mathbf{u_3} = \frac{\bar{\mathbf{x}}-\mathbf{y}}{\mathbf{1}-\mathbf{t}}$ and $\mathbf{u_4} = \bar{\mathbf{x}}$, and $|\mathbf{y}| \in [|\ln\varepsilon|^{-\gamma}, |\bar{\mathbf{x}}| - |\ln\varepsilon|^{-\gamma}]$

$$-\ln \bar{\mu}^{\varepsilon}_{\mathbf{t}}(\mathbf{y}) \approx_{\varepsilon} \left(\ln\frac{|\mathbf{y}|}{\mathbf{t}|\bar{\mathbf{x}}|}\right)\mathbf{g}((\varepsilon\mathbf{r}_{\varepsilon})^{-1})^{-1}|\mathbf{y}|\varepsilon^{-1} + \left(\ln\frac{|\bar{\mathbf{x}} - \mathbf{y}|}{(\mathbf{1}-\mathbf{t})|\bar{\mathbf{x}}|}\right)\mathbf{g}((\varepsilon\mathbf{r}_{\varepsilon})^{-1})^{-1}|\bar{\mathbf{x}} - \mathbf{y}|\varepsilon^{-1}$$
$$= \left(|\mathbf{y}|\ln\frac{|\mathbf{y}|}{\mathbf{t}} + |\bar{\mathbf{x}} - \mathbf{y}|\ln\frac{|\bar{\mathbf{x}} - \mathbf{y}|}{\mathbf{1}-\mathbf{t}} - (|\mathbf{y}| + |\bar{\mathbf{x}} - \mathbf{y}|)\ln|\bar{\mathbf{x}}|\right)\mathbf{g}((\varepsilon\mathbf{r}_{\varepsilon})^{-1})^{-1}\varepsilon^{-1}$$
$$= \left(|\mathbf{y}|\ln\frac{|\mathbf{y}|}{\mathbf{t}} + |\bar{\mathbf{x}} - \mathbf{y}|\ln\frac{|\bar{\mathbf{x}} - \mathbf{y}|}{\mathbf{1}-\mathbf{t}} - |\bar{\mathbf{x}}|\ln|\bar{\mathbf{x}}|\right)\mathbf{g}((\varepsilon\mathbf{r}_{\varepsilon})^{-1})^{-1}\varepsilon^{-1}.$$

That is

$$-\ln\bar{\mu}^{\varepsilon}_{\mathbf{t}}(\mathbf{y})=\left(|\mathbf{y}|\ln\frac{|\mathbf{y}|}{\mathbf{t}}+|\bar{\mathbf{x}}-\mathbf{y}|\ln\frac{|\bar{\mathbf{x}}-\mathbf{y}|}{\mathbf{1}-\mathbf{t}}-|\bar{\mathbf{x}}|\ln|\bar{\mathbf{x}}|\right)\mathbf{g}((\varepsilon\mathbf{r}_{\varepsilon})^{-\mathbf{1}})^{-\mathbf{1}}\varepsilon^{-\mathbf{1}}.$$

Consequently, with speed function $\mathrm{S}(\varepsilon)=\varepsilon\mathbf{g}((\varepsilon\mathbf{r}_{\varepsilon})^{-\mathbf{1}})$ we obtain the limit

$$\lim_{\varepsilon\to\mathbf{0}}-\mathbf{S}(\varepsilon)\ln\bar{\mu}^{\varepsilon}_{\mathbf{t}}(\mathbf{y})=|\mathbf{y}|\ln\frac{|\mathbf{y}|}{\mathbf{t}}+|\bar{\mathbf{x}}-\mathbf{y}|\ln\frac{|\bar{\mathbf{x}}-\mathbf{y}|}{\mathbf{1}-\mathbf{t}}-|\bar{\mathbf{x}}|\ln|\bar{\mathbf{x}}|.$$

The right-hand side has rudimentary Riemann sum structure

$$\lim_{\varepsilon \to 0} -\mathbf{S}(\varepsilon) \ln \bar{\mu}^{\varepsilon}_{\mathbf{t}}(\mathbf{y}) = \frac{|\mathbf{y}|}{\mathbf{t}-\mathbf{0}} \ln \left(\frac{|\mathbf{y}|}{\mathbf{t}-\mathbf{0}} \right) (\mathbf{t}-\mathbf{0}) + \frac{|\bar{\mathbf{x}}-\mathbf{y}|}{\mathbf{1}-\mathbf{t}} \ln \left(\frac{|\bar{\mathbf{x}}-\mathbf{y}|}{\mathbf{1}-\mathbf{t}} \right) (\mathbf{1}-\mathbf{t}) - |\bar{\mathbf{x}}| \ln |\bar{\mathbf{x}}|.$$

which reads for a m-dimensional distribution on $0 < t_1 < \dots t_m = 1$

$$\frac{|\mathbf{y_1}|}{\mathbf{t_1}} \ln \left(\frac{|\mathbf{y_1}|}{\mathbf{t_1}} \right) \mathbf{t_1} + \sum_{\mathbf{i=2}}^{\mathbf{m}} \frac{|\mathbf{y_i}-\mathbf{y_{i-1}}|}{\mathbf{t_i}-\mathbf{t_{i-1}}} \ln \left(\frac{|\mathbf{y_i}-\mathbf{y_{i-1}}|}{\mathbf{t_i}-\mathbf{t_{i-1}}} \right) (\mathbf{t_i}-\mathbf{t_{i-1}}) - |\bar{\mathbf{x}}| \ln |\bar{\mathbf{x}}|.$$

and which can be made rigorously satisfy an m-dimensional LDP with speed function $\mathbf{S}(\varepsilon) = \varepsilon \cdot \mathbf{g}((\varepsilon \mathbf{r}_{\varepsilon})^{-1}))^{-1}$ and good rate function

$$\mathbf{I}_{(\mathbf{t_1},\dots,\mathbf{t_m})}(\varphi) =$$
$$\frac{|\varphi(\mathbf{t_1})|}{\mathbf{t_1}} \ln \left(\frac{|\varphi(\mathbf{t_1})|}{\mathbf{t_1}} \right) \mathbf{t_1} + \sum_{\mathbf{i=2}}^{\mathbf{m}} \frac{|\varphi(\mathbf{t_i})-\varphi(\mathbf{t_{i-1}})|}{\mathbf{t_i}-\mathbf{t_{i-1}}} \ln \left(\frac{|\varphi(\mathbf{t_i})-\varphi(\mathbf{t_{i-1}})|}{\mathbf{t_i}-\mathbf{t_{i-1}}} \right) (\mathbf{t_i}-\mathbf{t_{i-1}}) - |\bar{\mathbf{x}}| \ln |\bar{\mathbf{x}}|$$

which tends (whenever the limit exists) to

$$\mathbf{I}_{\mathsf{x}}(\varphi) = \int_{\mathbf{0}}^{\mathbf{1}} |\varphi'(\mathbf{t})| \ln(|\varphi'(\mathbf{t})|) \mathbf{dt} - |\bar{\mathbf{x}}| \ln |\bar{\mathbf{x}}|.$$

Off the segment $[[0, \mathbf{x}]]$

For $\mathbf{y} \notin [[0, \bar{\mathbf{x}}]]$ then $|\bar{\mathbf{x}}| = |\bar{\mathbf{x}} - \mathbf{y}| + |\mathbf{y}| + (|\bar{\mathbf{x}}| - |\bar{\mathbf{x}} - \mathbf{y}| - |\mathbf{y}|)$ with $|\bar{\mathbf{x}}| - |\bar{\mathbf{x}} - \mathbf{y}| - |\mathbf{y}| < \mathbf{0}$.

Only an incomplete asymptotic cancellation, it remains a term proportional to

$$(|\bar{\mathbf{x}}| - |\bar{\mathbf{x}} - \mathbf{y}| - |\mathbf{y}|)\mathbf{f}'(\mathbf{g}(\frac{|\bar{\mathbf{x}}|\varepsilon^{-\mathbf{1}}}{\mathbf{t}})).$$

However, when we renormalize it with $S(\varepsilon)$ as before and obtain

$$\frac{\mathbf{S}(\varepsilon)}{\widetilde{\mathbf{S}}(\varepsilon)} \cdot \widetilde{\mathbf{S}}(\varepsilon)(|\bar{\mathbf{x}}| - |\bar{\mathbf{x}} - \mathbf{y}| - |\mathbf{y}|)\mathbf{f}'(\mathbf{g}(\frac{|\bar{\mathbf{x}}|\varepsilon^{-\mathbf{1}}}{\mathbf{t}})).$$

While the second factor converges to $-|y|$ as before, the first factor diverges

$$\frac{\mathbf{S}(\varepsilon)}{\widetilde{\mathbf{S}}(\varepsilon)} \approx \frac{\varepsilon \ln((\varepsilon \mathbf{r}_\varepsilon)^{-\mathbf{1}})^{\frac{\mathbf{1}}{\alpha}}}{\varepsilon(\ln((\varepsilon \mathbf{r}_\varepsilon)^{-\mathbf{1}})^{-\frac{\alpha-\mathbf{1}}{\alpha}}} = \ln((\varepsilon \mathbf{r}_\varepsilon)^{-\mathbf{1}}) \to \infty.$$

Therefore instead of I_t we obtain

$$\lim_{\varepsilon \to \mathbf{0}} -\mathbf{S}(\varepsilon) \ln \bar{\mu}^{\varepsilon}_{\mathbf{t}}(\mathbf{y}) = \infty.$$

The complete LDP for Weibull-type CPP

$\varepsilon \mapsto \mathbf{r}_\varepsilon$ to be a regular varying function with its index in $(-\mathbf{1}, \infty)$.

Hypotheses (H) on the Lévy measure:
The generating triplet $(\sigma^{\mathbf{2}}, \nu, \mathbf{\Gamma})$ of $\mathbf{L}$ satisfies:
The Lévy measure ν can be written as $\nu = \nu_\eta + \nu_\xi$.

1. **Weibull CPP component $\alpha > 1$:** The Lévy measure ν_η is finite, $\nu_\eta(\mathbb{R}^n) < \infty$, and has a density on $\mathbb{R}^n \setminus \{0\}$ of the form

$$\nu_\eta(\mathbf{dz})/\mathbf{dz} = \exp(-\mathbf{f}(|\mathbf{z}|)),$$

   where $\mathbf{f} \in \mathbf{SR}_\alpha$ for some $\alpha > \mathbf{1}$.

2. Let ξ denote a Lévy process with generating triplet $(\sigma^{\mathbf{2}}, \nu_\xi, \mathbf{\Gamma}_\xi)$ with

$$\mathbf{\Gamma}_\xi = \mathbf{\Gamma} + \int_{\{|\mathbf{y}| \leq \mathbf{1}\}} \mathbf{y} \nu_\eta(\mathbf{dy}).$$

   There is a $\tilde{\mathbf{s}} \in \mathbb{R}$, such that $\xi_\mathbf{t}$ has a density $\mu_{\xi,\mathbf{t}}$ on $\mathbb{R}^\mathbf{n} \setminus [-\tilde{\mathbf{s}}\mathbf{t}, \tilde{\mathbf{s}}\mathbf{t}]^\mathbf{n}$ for every $\mathbf{t} > \mathbf{0}$.

3. **Lighter-than-Weibull-tailed perturbations:** There is $\aleph > \mathbf{1} - \frac{\mathbf{1}}{\alpha}$, s.th. for all $\gamma < \mathbf{1}$

$$\lim_{\mathbf{\Lambda} \to \infty} \sup_{\substack{t < \Lambda^\gamma \\ |y| = \Lambda}} \frac{\ln \mu_{\xi,\mathbf{t}}(\mathbf{y})}{\mathbf{\Lambda}(\ln \mathbf{\Lambda})^\aleph} = -\infty.$$

Notation:

Given $\bar{\mathbf{x}} \in \mathbb{R}^{\mathbf{n}} \setminus \{\mathbf{0}\}$ and $\mathbf{T} > \mathbf{0}$.

$\mathbf{X}^{\varepsilon}$ with $\mathbf{X}^{\varepsilon}_{\mathbf{t}} = \varepsilon \mathbf{L}_{\mathbf{t}\mathbf{r}_{\varepsilon}}$, $\mathbf{L}$ with characteristics $(\sigma^{\mathbf{2}}, \nu, \mathbf{\Gamma})$ satisfying the Hypotheses (H).

$(\mathbf{X}^{\varepsilon})_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]}$ conditioned on the event $\{X^{\varepsilon}_{T} = \bar{\mathbf{x}}\}$ called $(\mathbf{Y}^{\bar{\mathbf{x}},\varepsilon}_{\mathbf{t}})_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]}$.

$\mathcal{D}_{[\mathbf{0},\mathbf{T}],\mathbb{R}^{\mathbf{n}}}$ the space of càdlàg functions $[\mathbf{0}, \mathbf{T}] \to \mathbb{R}^{\mathbf{n}}$ with the uniform norm $|| \cdot ||_{\infty}$.

$[[\mathbf{0}, \bar{\mathbf{x}}]] = \{\mathbf{s}\bar{\mathbf{x}} \mid \mathbf{s} \in [\mathbf{0}, \mathbf{1}]\}$ segment

$\mathcal{M}_{\bar{\mathbf{x}},\mathbf{T}} := \{\varphi \in \mathcal{D}_{[\mathbf{0},\mathbf{T}],\mathbb{R}^{\mathbf{n}}} \mid |\varphi(\cdot)|$ is continuous and nondecreasing with $\varphi([\mathbf{0}, \mathbf{T}]) = [[\mathbf{0}, \bar{\mathbf{x}}]]\}$

Theorem: [LDP with speed function S and rate function $\mathbf{I}_{\bar{\mathbf{x}}}$ for $\mathbf{Y}^{\bar{\mathbf{x}},\varepsilon}$]

The family $(\mathbf{P}^{\bar{\mathbf{x}},\varepsilon})_{\varepsilon>\mathbf{0}}$, $\mathbf{P}^{\bar{\mathbf{x}},\varepsilon} =$ Law$(\mathbf{Y}^{\bar{\mathbf{x}},\varepsilon})$ satisfies a **LDP on** $(\mathcal{D}_{[\mathbf{0},\mathbf{T}],\mathbb{R}^{\mathbf{n}}}, || \cdot ||_{\infty})$
with **speed function** $\mathrm{S}(\varepsilon) := \varepsilon \cdot \mathbf{g}(\varepsilon^{-\mathbf{1}}\mathbf{r}_{\varepsilon}^{-\mathbf{1}})$**, where** $\mathbf{g}$ **is defined by the NFE before**
and **the good rate function**

$$\mathbf{I}_{\bar{\mathbf{x}}}(\varphi) = \begin{cases} \displaystyle\int_0^T |\varphi|' \, \ln |\varphi|' \, dt - |\bar{\mathbf{x}}| \ln \frac{|\bar{\mathbf{x}}|}{T}, & \textbf{if } \varphi \in \mathcal{M}_{\bar{\mathbf{x}},T}, \\ \infty, & \textbf{otherwise}. \end{cases} \tag{3}$$

2

[2]Here we denote by $|\varphi|(t) = |\varphi(t)|$, $|\varphi|'(t) = \frac{d}{dt}|\varphi(t)|$ the total derivative, whenever it exists and set it equal to 0 otherwise. We set $r \ln r = 0$, whenever $r = 0$.

Sufficient conditions on the perturbation ξ:

There exists $\mathbf{\Lambda > 0}$, such that $\nu_\xi(\{\mathbf{y} \in \mathbb{R}^{\mathbf{n}} \mid |\mathbf{y}| > \mathbf{\Lambda}\}) = \mathbf{0}$.
Furthermore, one of the following conditions is satisfied:

1. det$\sigma^{\mathbf{2}} > \mathbf{0}$.

2. There is a parameter $\beta \in (\mathbf{0}, \mathbf{2})$ such that ν_ξ satisfies the following **Orey condition**

$$\lim_{\mathbf{r} \to \mathbf{0}+} \mathbf{r}^{-\beta} \int_{|\mathbf{y}|<\mathbf{r}} \langle \mathbf{v_i}, \mathbf{y} \rangle^{\mathbf{2}} \nu(\mathbf{dy}) > \mathbf{0}.$$

   where $(\mathbf{v_1}, \ldots, \mathbf{v_n})$ is an ONB of $\mathbb{R}^{\mathbf{n}}$

More comments:

- Corollary: Schilder's theorem for CPP with Weibull increments $\alpha > 1$
- Most probable paths
- Asymptotic normality
- LDP for the growth of the jumps
- Infinite energy parametrizations of the segment $[[0, \mathbf{x}]]$
- Degeneration of the rate function
- Symmetry break of exit locations (in comparison to Freidlin-Wentzell)

Most relevant literature:

1. Feng, J., and Kurtz, T. G.: *Large Deviations for Stochastic Processes.* Volume 131 of *Mathematical Surveys and Monographs.* American Mathematical Society, 2006.

2. Dembo, A., and Zeitouni, O.: *Large Deviations Techniques and Applications.* Springer Jones and Bartlett, Boston, 1993

3. Bingham, N. H., Goldie, C. M., and Teugels, J. L.: *Regular variation*, Volume 27 of *Encyclopedia of Mathematic and its Applications*, Cambridge University Press, Cambridge etc., 1987.

4. Wetzel, T.: LDPs für bedingte Lévyprozesse, Verfeinerte Analyse von First Exit Problemen. Ph.D. Thesis, Humboldt-Universität zu Berlin, 2021.

5. Högele, M.A. and Wetzel, T.: Large deviations for light-tailed Lévy bridges on short time scales `https://arxiv.org/abs/2505.23972`, 2025