Heat kernel estimates for Brownian SDEs with low regularity coefficients and unbounded drift

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I - Non degenerate systems

Consider the d-dimensional diffusion

$$dX_s = b(s, X_s)ds + \sigma(s, X_s)dW_s, \quad s \geq 0, \; X_0 = x. \tag{1}$$

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Bounded case: For $\sigma\sigma^* > 0$, for *bounded* and *Hölder continuous* coefficients, it is well known that there exists a unique weak solution which admits a density $p(0, x, s, y)$,

$$C^{-1}\Gamma_{\lambda^{-1}}(s, x - y) \leq p(0, x; s, y) \leq C\Gamma_\lambda(s, x - y) \tag{2}$$

$$|\nabla_x^j p(0, x, s, y)| \leq Cs^{-\frac{j}{2}}\Gamma_\lambda(s, x - y), \quad j = 1, 2, \tag{3}$$

for any $x, y \in \mathbb{R}^d$ and $s \in [0, T]$, where

$$\Gamma_\lambda(s, y) := s^{-\frac{d}{2}} \exp\big(-y^2/(\lambda s)\big), \quad \lambda \geq 1, s > 0.$$

What can we expect for *unbounded* and possibly *irregular* drifts?

Example: Ornstein-Uhlenbeck (OU) process

$$dX_s = X_s ds + dW_s, \quad X_0 = x,$$

has explicit density

$$p_{\mathrm{OU}}(0,x,s,y) = (\pi(e^{2s}-1)/s)^{-d/2}\Gamma_{(e^{2s}-1)/s}(s, e^s x - y).$$

Primer: density estimates with smooth *Lipschitz* continuous drifts by *Delarue and Menozzi* (2010)

$$C^{-1}\Gamma_{\lambda^{-1}}(s,\gamma_s(x)-y) \le p(0,x;s,y) \le C\Gamma_\lambda(s,\gamma_s(x)-y), \qquad (4)$$

where C,λ depend both on the constants in the Assumptions and T, with

$$\dot{\gamma}_s(x) = b(s,\gamma_s(x)), \quad \gamma_0(x) = x.$$

If the drift b is bounded we recover the usual deviation $|x-y|$ (as *Friedman*): indeed

$$s^{-\frac{1}{2}}|x-y| - \|b\|_\infty s^{\frac{1}{2}} \le s^{-\frac{1}{2}}|\gamma_s(x)-y| \le s^{-\frac{1}{2}}|x-y| + \|b\|_\infty s^{\frac{1}{2}}.$$

Assumptions:

- (*Non degeneracy*). There exists a constant $\kappa_0 \geq 1$, such that

$$\kappa_0^{-1}|y|^2 \leq \langle \sigma\sigma^*(t,x)y, y\rangle \leq \kappa_0|y|^2, \quad x, y \in \mathbb{R}^d,\ t \geq 0,$$

  and for some $\alpha \in (0,1]$, $\sigma \in C^{\alpha}_{0,\infty}$, namely

$$|\sigma(t,x) - \sigma(t,y)| \leq \kappa_0|x-y|^{\alpha}, \ \ t \geq 0,\ x, y \in \mathbb{R}^d.$$

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- There exist positive constant $\kappa_1 > 0$ and $\beta \in [0,1]$ such that for all $x,y \in \mathbb{R}^d$ and $t \geq 0$,

$$|b(t,0)| \leq \kappa_1,\ |b(t,x) - b(t,y)| \leq \kappa_1(|x-y|^{\beta} \vee |x-y|).$$

Particular cases:

- For $\beta = 0$, b can possibly be an unbounded measurable function with linear growth.
- The drift $b(t,x) = c_1(t) + c_2(t)|x|^{\beta}$, $\beta \in [0,1]$, c_1, c_2 bounded measurable functions of time, enter this class.

Theorem *(Menozzi, P., Zhang 2020)*

For any $T > 0$, $0 \le t < s \le T$ and $x \in \mathbb{R}^d$, the unique weak solution $X_{t,s}(x)$ of (1) starting from x at time t admits a density $p(t, x; s, y)$ which is continuous in $x, y \in \mathbb{R}^d$. Moreover, $p(t, x; s, y)$ enjoys the following estimates:

- (*Two-sided density bounds*)

$$C_0^{-1}\Gamma_{\lambda_0^{-1}}(t - s, \gamma_{t,s}(x) - y) \le p(t, x; s, y) \le C_0\Gamma_{\lambda_0}(s - t, \gamma_{t,s}(x) - y);$$

- (*Gradient estimate in x*)

$$|\nabla_x p(t, x; s, y)| \le C_1(s - t)^{-\frac{1}{2}}\Gamma_{\lambda_1}(s - t, \gamma_{t,s}(x) - y);$$

- (*Second order derivative estimate in x*) If $\beta \in (0,1]$

$$\left|\nabla_x^2 p(t,x;s,y)\right| \le C_2(s-t)^{-1}\Gamma_{\lambda_2}(s-t,\gamma_{t,s}(x)-x);$$

- (*Gradient estimate in y*) If $\beta \in (0,1]$ and for some $\alpha \in (0,1)$ and $\kappa_2 > 0$,

$$\|\nabla\sigma\|_\infty \le \kappa_2, \quad |\nabla\sigma(t,x)-\nabla\sigma(t,y)| \le \kappa_2|x-y|^\alpha,$$

then

$$|\nabla_y p(t,x;s,y)| \le C_3(s-t)^{-\frac{1}{2}}\Gamma_{\lambda_3}(s-t,\gamma_{t,s}(x)-y).$$

- Note that "$\dot{\gamma}_t(x) = b(t, \gamma_t(x))$" is not generally well posed.

  Therefore we introduced the mollified (in space) drift

$$b_\varepsilon(t, x) = b(t, \cdot) * \varrho_\varepsilon(x),$$

  where $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(\varepsilon^{-1}x)$, and ρ is a test function supported in the unit ball with unit integral.

  Importantly we can prove the following *equivalence between mollified flows*: $\forall \varepsilon \in (0, 1]$, there exists $C = C(T, \kappa_1, d) \geq 1$ s.t

$$|\gamma^{(1)}_{t,s}(x) - y| + |s - t| \asymp_C |\gamma^{(\varepsilon)}_{t,s}(x) - y| + |s - t| \asymp_C |x - \gamma^{(\varepsilon)}_{s,t}(y)| + |s - t| \tag{5}$$

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- Similarly, if $\beta > 0$, $\gamma^{(\varepsilon)}_{t,s}(x)$ can be replaced as well by *any* Peano flow $\gamma_{t,s}(x)$.

- First work in a regularized setting: in this case there exists the transition density $p(t,x;s,y)$ which is C^∞- smooth in variables x, y for all $t < s$, by Hörmander's theorem, and is the fundamental solution of

$$\partial_t u + \mathcal{L}_{t,x} u = 0, \quad p(t,\cdot;s,y) \longrightarrow \delta_y(\cdot) \quad t \uparrow s, \tag{6}$$

  where

$$\mathcal{L}_{t,x} f(x) = \frac{1}{2}\mathrm{tr}\big(\sigma\sigma^*(t,x)\nabla_x^2 f(x)\big) + \langle b(t,x), \nabla_x f(x)\rangle.$$

- Derive bounds that are actually independent on the regularization procedure

Drift adapted parametrix method

Let $\Gamma^{\tau,\xi}(t,x;s,y)$ the Gaussian fundamental solution for the operator

$$\partial_t + \mathcal{L}^{\tau,\xi}_{t,x} := \partial_t + \frac{1}{2}\sum \sigma_i\sigma_j(t,\gamma_{t,\tau}(\xi))\partial_i\partial_j + \sum b_i(t,\gamma_{t,\tau}(\xi))\partial_i$$

- **parametrix** as: $Z(t,x;s,y) := \Gamma^{s,y}(t,x;s,y)$. In particular

$$Z(t,x;s,y) \le C_0\Gamma_\lambda(s-t,\gamma_{t,s}(x)-y) =: p_\lambda(t,x;s,y).$$

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By iterating the Duhamel formula N times

$$p = Z + \sum_{k=1}^{N-1} Z \otimes H^{\otimes k} + p \otimes H^{\otimes N}$$

where $$H(t,x;s,y) = (\mathcal{L}_{t,x} - \mathcal{L}^{\tau,\xi}_{t,x})_{(\tau,\xi)=(s,y)}Z(t,x;s,y)$$

- Classically $N \longrightarrow \infty$

First correction

The first term of the series reads

$$Z \otimes H(t,x;s,y) = \int_t^s \int_{\mathbb{R}^d} Z(t,x;r,z) \underbrace{(\mathcal{L}_{r,z} - \mathcal{L}_{r,z}^{s,y}) Z(r,z;s,y)}_{(*)} \, dzdr$$

where

$$(*) \sim \frac{|z - \gamma_{s,r}(y)|^\alpha}{s-r} p_\lambda(r,z;s,y) \sim (s-r)^{\frac{\alpha}{2}-1} p_\lambda(r,z;s,y).$$

$$\Downarrow$$

$$|Z \otimes H(t,x;s,y)| \lesssim \int_t^s \frac{1}{(s-r)^{1-\alpha/2}} \int_{\mathbb{R}^d} p_\lambda(t,x;r,z) p_\lambda(r,z;s,y) dzdr$$

However, because of the presence of the flow,

$$\int_{\mathbb{R}^d} p_\lambda(t,x;r,z)p_\lambda(r,z;s,y)dz \lesssim p_{\lambda(1+\varepsilon)}(t,x;s,y), \quad \varepsilon = \varepsilon(\lambda,T) > 0.$$

Therefore we fail to control the iterated kernels $H^{\otimes k}$ uniformly in k.

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Therefore we fail to control the iterated kernels $H^{\otimes k}$ uniformly in k.

Control of the remainder: By the Kernel estimate

$$H^{\otimes N}(t,x;s,y) \lesssim (s-t)^{\frac{N\alpha}{2}-1} p_\lambda(t,x;s,y)$$

where $\alpha \in (0, \frac{1}{2}]$ depends on the regularity of the coefficients, we have:

$$|(p \otimes H^{\otimes N})(t,x;s,y)| \lesssim \int_t^s (r-t)^{\frac{N\alpha}{2}-1} \mathbb{E}[p_\lambda(r, X_{r,t}(x); s,y)]dr.$$

- the expectation is controlled by a variational *representation formula by Boué and Dupuis*;

Gradient bound

Assuming the lower bound: there exists some δ depending on the general assumptions s.t. $p_\lambda \leq \bar{p}$, where $\bar{p}$ is the density of the SDE with $\sigma(t,x) \equiv \delta \mathbb{I}_{d\times d}$.

- Assume $t = 0$ and for $s \in (0, T]$, we define

$$f_1(s) := \sup_{x,y} |\nabla_x p(0,x,s,y)|/\bar{p}(0,x,s,y)$$

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$$f_1(s) := \sup_{x,y} |\nabla_x p(0,x,s,y)| / \bar{p}(0,x,s,y)$$

- By a 1-step Duhamel representation we have

$$|\nabla_x p(0,x,s,y)| \leq |\nabla_x Z(0,x,s,y)| + |\nabla_x p \otimes H(0,x,s,y)|$$

  Where

$$|\nabla_x Z(0,x,s,y)| \lesssim s^{-1/2} p_\lambda(0,x,s,y) \lesssim s^{-1/2} \bar{p}(0,x,s,y).$$

Moreover

$$
\begin{aligned}
|\nabla_x p \otimes H(0,x,s,y)| &\leq \int_0^s \int_{\mathbb{R}^d} f_1(r)\bar{p}(0,x,r,z)|H(r,z,s,y)|dzdr \\
&\lesssim \int_0^s f_1(r)(s-r)^{-1+\frac{\alpha}{2}} \int_{\mathbb{R}^d} \bar{p}(0,x,r,z)\bar{p}(r,z,s,y)dzdr \\
&= \left(\int_0^s f_1(r)(s-r)^{-1+\frac{\alpha}{2}} dr \right) \bar{p}(0,x,s,y),
\end{aligned}
$$

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&= \left(\int_0^s f_1(r)(s-r)^{-1+\frac{\alpha}{2}} dr\right) \bar{p}(0,x,s,y),
\end{aligned}
$$

$$
\Longrightarrow f_1(s) \lesssim s^{-\frac{1}{2}} + \int_0^s (s-r)^{-1+\frac{\alpha}{2}} f_1(r)dr
$$

and by the Volterra type Gronwall inequality, we obtain

$$
f_1(s) \lesssim s^{-\frac{1}{2}} \Rightarrow |\nabla_x p(0,x,s,y)| \lesssim s^{-\frac{1}{2}} \bar{p}(0,x,s,y). \tag{7}
$$

II - Kynetic type systems

Consider the 2d-dimensional system of SDEs

$$\begin{cases} dX_t^1 = F_1(t, X_t^1, X_t^2)dt + \sigma(t, X_t^1, X_t^2)dW_t, \\ dX_t^2 = F_2(t, X_t^1, X_t^2)dt, \end{cases} \tag{8}$$

Consider the $2d$-dimensional system of SDEs

$$\begin{cases} dX_t^1 = F_1(t, X_t^1, X_t^2)dt + \sigma(t, X_t^1, X_t^2)dW_t, \\ dX_t^2 = F_2(t, X_t^1, X_t^2)dt, \end{cases} \tag{8}$$

Assuming some kind of *weak* Hörmander condition:

- $\sigma\sigma^* > 0$ uniformly
- $\nabla_{x_1} F_2$ has full rank

Applications:

- physics: Hamiltonian systems

$$H(\mathbf{x}) = V(x_2) + |x_1|^2/2 \implies F_H(\mathbf{x}) = (-\nabla_{x_2} V(x_2), x_1)^*$$

- finance: path dependent contracts

Kinetic Gaussian case

$F_1 \equiv 0$ and $F_2(t, X_t^1, X_t^2) = X_t^1$

$$dX_t^1 = dW_t, \quad dX_t^2 = X_t^1 dt, \qquad t \geq 0.$$

is a Gaussian process with mean and covariance matrix given by

$$\gamma_t(\mathbf{x}) = (x_1, x_2 + x_1 t), \qquad \mathbf{K}_t = \begin{pmatrix} t & \frac{t^2}{2} \\ \frac{t^2}{2} & \frac{t^3}{3} \end{pmatrix} > 0 \quad \forall t > 0$$

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$\Rightarrow$ the process admits a density for every $t > 0$, explicitly given by

$$\mathbf{y} \mapsto \left(\frac{\sqrt{3}}{\lambda \pi t^2}\right)^d \exp\left(-\frac{1}{2}|\mathbf{K}_t^{-\frac{1}{2}}(\mathbf{y} - \gamma_t(\mathbf{x}))|^2\right)$$

$$\mathbf{K}_t \sim \begin{pmatrix} t & 0 \\ 0 & t^3 \end{pmatrix} =: \mathbb{T}_t.$$

- *Non-diffusive* time-scale

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- *Non-diffusive* time-scale
- Different growth-rates in the two components

$$\begin{aligned} |\nabla_{x_i} p(0,\mathbf{x};t,\mathbf{y})| &\leq \left| \left((\mathbf{K}_t^{-\frac{1}{2}} \nabla\gamma_t(\mathbf{x}))^* \mathbf{K}_t^{-\frac{1}{2}} (\gamma_t(\mathbf{x}) - \mathbf{y}) \right)_i \right| p(0,\mathbf{x};t,\mathbf{y}) \\ &\lesssim \frac{1}{t^{i-\frac{1}{2}+2d}} \exp\left(-\frac{1}{2\lambda} |\mathbb{T}_t^{-\frac{1}{2}} (\mathbf{y} - \gamma_t(\mathbf{x}))|^2 \right) \\ &=: \frac{1}{t^{i-\frac{1}{2}}} g_\lambda(t, \mathbf{y} - \gamma_t(\mathbf{x})) \end{aligned}$$

Linear drifts

- Weber (1951), Polidoro (1994), Di Francesco-Pascucci (2005), Lucertini-Pagliarani-Pascucci (2022)
- Pascucci-P. (2022)

Regular, nonlinear drift

- Delarue-Menozzi (2010)
- Pigato (2022)

Regularization by noise

- Fedrizzi-Flandoli-Priola-Vovelle (2017)
- X.Zhang (2018)
- Chaudru de Raynal (2018), CdR-Honoré-Menozzi (2021-2022)

Functional framework

- *Homogeneous norm*

$$|\mathbf{x}|_{\mathbf{d}} := |x_1| + |x_2|^{\frac{1}{3}} \quad \Longrightarrow \quad |\mathbb{T}_t \mathbf{x}|_{\mathbf{d}} = t|\mathbf{x}|_{\mathbf{d}} \tag{9}$$

- Corresponding *Anisotropic Hölder spaces*: $\mathcal{C}^{j+\alpha}_{\mathbf{d}}(\mathbb{R}^{2d};\mathbb{R}^l)$

$$\|f\|_{\mathcal{C}^{j+\alpha}_{\mathbf{d}}} := \sup_{x_2\in\mathbb{R}^d} \|f(\cdot,x_2)\|_{\mathcal{C}^{j+\alpha}} + \sup_{x_1\in\mathbb{R}^d} \|f(x_1,\cdot)\|_{\mathcal{C}^{(j+\alpha)/3}} < \infty.$$

$$f \in \mathcal{C}^{\alpha}_{\mathbf{d}} \implies |f(\mathbf{x}) - f(\mathbf{y})| \le c_\alpha \|f\|_{\mathcal{C}^{\alpha}_{\mathbf{d}}} |\mathbf{x}-\mathbf{y}|^{\alpha}_{\mathbf{d}}$$

$$f \in \mathcal{C}^{1+\alpha}_{\mathbf{d}} \implies |f(\mathbf{x}) - \underbrace{\left(f(\mathbf{y}) + \nabla_{x_1} f(\mathbf{y})(x_1 - y_1)\right)}_{=:\mathcal{T}_1 f(\mathbf{x},\mathbf{y})}| \le c_\alpha \|f\|_{\mathcal{C}^{1+\alpha}_{\mathbf{d}}} |\mathbf{x}-\mathbf{y}|^{1+\alpha}_{\mathbf{d}}$$

Assume

- For some $\alpha \in (0,1]$, $\kappa_0 \geq 1$

$$\kappa_0^{-1}|\xi| \leq \langle \sigma\sigma^*(t,\mathbf{x})\xi, \xi\rangle \leq \kappa_0|\xi|, \quad \xi \in \mathbb{R}^d$$

  and

$$\|\sigma(t,\cdot)\|_{\mathcal{C}^{\alpha}_{\mathbf{d}}(\mathbb{R}^{2d},\mathbb{R}^d)} < \kappa_0.$$

- F_1 is a measurable function with linear growth

$$|F_1(t,\mathbf{0})| \leq \kappa_1, \quad |F_1(t,\mathbf{x}) - F_1(t,\mathbf{y})| \leq \kappa_1(1 + |\mathbf{x}-\mathbf{y}|)$$

- For some $\alpha \in (0,1]$ and $\kappa_1, \kappa_2 > 0$, it holds that

$$\|F_2(t,\cdot)\|_{\mathcal{C}^{1+\alpha}_{\mathbf{d}}(\mathbb{R}^{2d},\mathbb{R}^d)} \leq \kappa_2.$$

  Moreover, there exists a closed convex subset $\mathcal{E} \subset GL_d(\mathbb{R})$ (the set of all invertible $d \times d$ matrices) such that $\nabla_{x_1} F_2(t,\mathbf{x}) \in \mathcal{E}$ for all $t \geq 0$ and $\mathbf{x} \in \mathbb{R}^{2d}$.

Theorem *(Chaudru de Raynal, Menozzi, P., Zhang 2022)*

$\exists!$ weak solution which admits a transition density $p(t, \mathbf{x}; s, \mathbf{y})$, $0 \le t < s \le T$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{2d}$. Moreover, $p(t, \mathbf{x}; s, \mathbf{y})$ enjoys the following estimates:

(i) (Two sides estimates)

$$C_0^{-1} g_{\lambda_0^{-1}} \left(s-t, \gamma_{t,s}(\mathbf{x}) - \mathbf{y}\right) \le p(t, \mathbf{x}; s, \mathbf{y}) \le C_0 g_{\lambda_0} \left(s-t, \gamma_{t,s}(\mathbf{x}) - \mathbf{y}\right).$$

(ii) (Gradient estimate in x_1)

$$|\nabla_{x_1} p(t, \mathbf{x}; s, \mathbf{y})| \lesssim_{C_1} (s-t)^{-\frac{1}{2}} g_{\lambda_1} \left(s-t, \gamma_{t,s}(\mathbf{x}) - \mathbf{y}\right).$$

(iii) (Hölder estimate in x) Let $\eta_0, \eta_1 \in (0,1)$, $j = 0, 1$

$$\begin{aligned} \left|\nabla_{x_1}^j p(t, \mathbf{x}; s, \mathbf{y}) - \nabla_{x_1}^j p(t, \mathbf{x}'; s, \mathbf{y})\right| &\lesssim_{C_j} |\mathbf{x} - \mathbf{x}'|_{\mathbf{d}}^{\eta_j} (s-t)^{-\frac{j+\eta_j}{2}} \\ &\times \Big(g_{\lambda_j} \left(s-t, \gamma_{t,s}(\mathbf{x}) - \mathbf{y}\right) + g_{\lambda_j} \left(s-t, \gamma_{t,s}(\mathbf{x}') - \mathbf{y}\right) \Big). \end{aligned}$$

About the flow:

- "$\dot{\gamma}_{s,t}(\mathbf{x}) = F(s, \gamma_{s,t}(\mathbf{x}))$" is not generally well posed.

$\gamma_{t,s}(x)$ can be replaced by any Peano flow associated solving:

$$\dot{\widetilde{\gamma}}_{t,s}(\mathbf{x}) = \widetilde{F}(s, \widetilde{\gamma}_{t,s}(\mathbf{x})), \quad \gamma_{t,t}(\mathbf{x}) = \mathbf{x},$$

where

$$\widetilde{F}(s, \mathbf{x}) = ([F_1(s, \cdot) * \rho_1](\mathbf{x}), F_2(t, \mathbf{x})).$$

Equivalently $\hat{\gamma}_{t,s}(\mathbf{x})$ associated with

$$\hat{F}(s, \mathbf{x}) = ([F_1(s, \cdot) * \rho_1](\mathbf{x}), [F_2(s, \cdot) * \rho_{|s-t|^{3/2}}](\mathbf{x})).$$

We have

$$|\mathbb{T}_{s-t}^{-1}(\widetilde{\gamma}_{t,s}(\mathbf{x}) - \hat{\gamma}_{t,s}(\mathbf{x}))| \leq C$$

Perturbative argument

- Consider a *linear* approximation of the associated Kolmogorov operator: for fixed $(\tau, \boldsymbol{\xi}) \in \mathbb{R}^+ \times \mathbb{R}^{2d}$

$$\widetilde{\mathcal{K}}_t^{\tau,\boldsymbol{\xi}} = \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^*(t, \gamma_{t,\tau}(\boldsymbol{\xi}))\nabla_{x_1}^2\right) + \langle \widetilde{F}^{\tau,\boldsymbol{\xi}}(t,x), \nabla\rangle + \partial_t$$

$$\widetilde{F}^{\tau,\boldsymbol{\xi}}(t,x) := F(t, \gamma_{t,\tau}(\boldsymbol{\xi})) + (DF)\,(t, \gamma_{t,\tau}(\boldsymbol{\xi}))\,(\mathbf{x} - \gamma_{t,\tau}(\boldsymbol{\xi}))\,,$$

where

$$DF := \begin{pmatrix} 0_{d\times d} & 0_{d\times d} \\ \nabla_{x_1} F_2 & 0_{d\times d} \end{pmatrix}.$$

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- Consider a *linear* approximation of the associated Kolmogorov operator: for fixed $(\tau, \boldsymbol{\xi}) \in \mathbb{R}^+ \times \mathbb{R}^{2d}$

$$\widetilde{\mathcal{K}}_t^{\tau,\boldsymbol{\xi}} = \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^*(t, \gamma_{t,\tau}(\boldsymbol{\xi}))\nabla^2_{x_1}\right) + \langle \widetilde{F}^{\tau,\boldsymbol{\xi}}(t,x), \nabla\rangle + \partial_t$$
$$\widetilde{F}^{\tau,\boldsymbol{\xi}}(t,x) := F(t, \gamma_{t,\tau}(\boldsymbol{\xi})) + (DF)\,(t, \gamma_{t,\tau}(\boldsymbol{\xi}))\,(\mathbf{x} - \gamma_{t,\tau}(\boldsymbol{\xi}))\,,$$

  where
$$DF := \begin{pmatrix} 0_{d\times d} & 0_{d\times d} \\ \nabla_{x_1}F_2 & 0_{d\times d} \end{pmatrix}.$$

- *First step approximation* of p: $Z(t,\mathbf{x};s,\mathbf{y}) := \widetilde{p}^{(s,\mathbf{y})}(t,\mathbf{x};s,\mathbf{y})$

$$Z(t,\mathbf{x};s,\mathbf{y}) \sim g_\lambda(s-t, \gamma_{t,s}(\mathbf{x}) - \mathbf{y})$$
$$|\nabla^j_{x_1} Z(t,\mathbf{x};s,\mathbf{y})| \lesssim (s-t)^{-\frac{j}{2}} g_\lambda(s-t, \gamma_{t,s}(\mathbf{x}) - \mathbf{y})$$
$$|\nabla_{x_2} Z(t,\mathbf{x};s,\mathbf{y})| \lesssim (s-t)^{-\frac{3}{2}} g_\lambda(s-t, \gamma_{t,s}(\mathbf{x}) - \mathbf{y})$$

- First step expansion (Duhamel)

$$p = Z + p \otimes (\mathcal{K} - \widetilde{\mathcal{K}})Z$$

In general

$$\left.\begin{array}{l} |x_1| \asymp t^{\frac{1}{2}} \\ |x_2| \asymp t^{\frac{3}{2}} \end{array}\right\} \implies |x|_{\mathbf{d}} \asymp t^{\frac{1}{2}}$$

Diffusion perturbation

$$\frac{|\mathbf{y} - \gamma_t(\mathbf{x})|_{\mathbf{d}}^{\alpha}}{t} \sim t^{\frac{\alpha}{2}-1}$$

Drift perturbation (second component)

$$\frac{|\mathbf{y} - \gamma_t(\mathbf{x})|_{\mathbf{d}}^{1+\alpha}}{t^{\frac{3}{2}}} \sim t^{\frac{\alpha}{2}-1}$$

Theorem *(Chaudru de Raynal, Menozzi, P., Zhang 2022)*

i) If, for some $\alpha > 0$,

$$\|F_1(t,\cdot)\|_{\mathcal{C}^\alpha_{\mathbf{d}}} \leq \kappa_1 \quad t \geq 0$$

$\Longrightarrow$ Existence and estimates for the second order derivative

$(\eta_2 < \alpha)$

ii) For gradient estimates in the degenerate component x_2, we need extra regularity, since for kinetic operators we only have 2/3-gain of regularity in x_2

If σ and F_1 also satisfy that

$$|\sigma(t,\mathbf{x}) - \sigma(t,\mathbf{y})| \leq \kappa_0(|x_1 - y_1| + |x_2 - y_2|^{\frac{1+\alpha}{3}})$$
$$|F_1(t,\mathbf{x}) - F_1(t,\mathbf{y})| \leq \kappa_1(|x_1 - y_1|^\alpha + |x_2 - y_2|^{\frac{1+\alpha}{3}}),$$

$\Longrightarrow$ Existence and estimates for the *first order derivative in* x_2

- (Gradient estimate in x_3)

$$|\nabla_{x_2} p(t, \mathbf{x}; s, \mathbf{y})| \lesssim_{C_3} (s-t)^{-\frac{3}{2}} g_{\lambda_3}\left(s-t, \gamma_{t,s}(\mathbf{x}) - \mathbf{y}\right).$$

- (Hölder estimate in x) For $\eta_3 < \alpha$

$$|\nabla_{x_2} p(t, \mathbf{x}; s, \mathbf{y}) - \nabla_{x_2} p(t, \mathbf{x}'; s, \mathbf{y})| \lesssim_{C_4} |\mathbf{x} - \mathbf{x}'|_{\mathbf{d}}^{\eta_3} (s-t)^{-\frac{3+\eta_3}{2}} \times \Big(g_{\lambda_4}\left(s-t, \gamma_{t,s}(\mathbf{x}) - \mathbf{y}\right) + g_{\lambda_4}\left(s-t, \gamma_{t,s}(\mathbf{x}') - \mathbf{y}\right) \Big).$$

References

- Menozzi - P. - Zhang. *Density and gradient estimates for non degenerate Brownian SDEs with unbounded measurable drift*, **J. Differ. Equ.** (2021)
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Thank you for your attention