An optimal multi-barrier strategy for a singular stochastic control problem with state-dependent reward

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Outline

Outline

1. Problem formulation
2. Barrier strategies
3. Multi-barrier strategies for Brownian motion with drift

Problem formulation

Let us consider the controlled process X^π given by

$$X_t^\pi := X_t - L_t^\pi, \qquad \text{for } t \geq 0,$$

where:

- $X = \{X_t : t \geq 0\}$ is a spectrally negative Lévy process starting at $x \in \mathbb{R}_+$.
- $\pi := \{L_t^\pi : t \geq 0\}$ is a non-decreasing, right-continuous, and adapted process such that $L_{0-}^\pi = 0$.
- We say that a strategy is admissible if $L_t^\pi - L_{t-}^\pi \leq X_{t-}^\pi$. The set of admissible strategies is denoted by $\mathcal{S}$.

Problem formulation

The expected reward functional for each admissible strategy $\pi \in \mathcal{S}$ is given by

$$V_\pi(x) := \mathbb{E}_x\left[\int_0^{\tau^\pi} \mathrm{e}^{-qs} g(X^\pi_{s-}) \circ \mathrm{d}L^\pi_s\right], \qquad x \in \mathbb{R}_+,$$

where $\tau^\pi = \inf\{t > 0 : X^\pi_t < 0\}$, $q > 0$, $g \in C^2((0,\infty))$ such that $g > 0$ on $(0,\infty)$ and

$$\int_0^{\tau^\pi} \mathrm{e}^{-qs} g(X^\pi_{s-}) \circ \mathrm{d}L^\pi_s = \int_0^{\tau^\pi} \mathrm{e}^{-qs} g(X^\pi_s)\mathrm{d}L^{\pi,c}_s + \sum_{0 \le s \le \tau^\pi} \mathrm{e}^{-qs} \Delta L^\pi_s \int_0^1 g(X^\pi_{s-} + \Delta X_s - \lambda \Delta L^\pi_s)\mathrm{d}\lambda,$$

where $L^{\pi,c}$ denotes the continuous part of L^π.

We want to maximize the performance criterion over the set of all admissible strategies $\mathcal{S}$ and find the value function of the problem given by:

$$V(x) := \sup_{\pi \in \mathcal{S}} V_\pi(x) = V_{\pi^*}(x), \qquad x \in \mathbb{R}_+,$$

for some $\pi^* \in \mathcal{S}$.

Verification Lemma

Suppose that $\hat{\pi} \in \mathcal{S}$ is such that $V_{\hat{\pi}}$ is sufficiently smooth on $(0, \infty)$, right-continuous at 0, and, for all $x > 0$,

$$\max\{(\mathcal{L} - q)V_{\hat{\pi}}(x), g(x) - V'_{\hat{\pi}}(x)\} \leq 0, \tag{HJB}$$

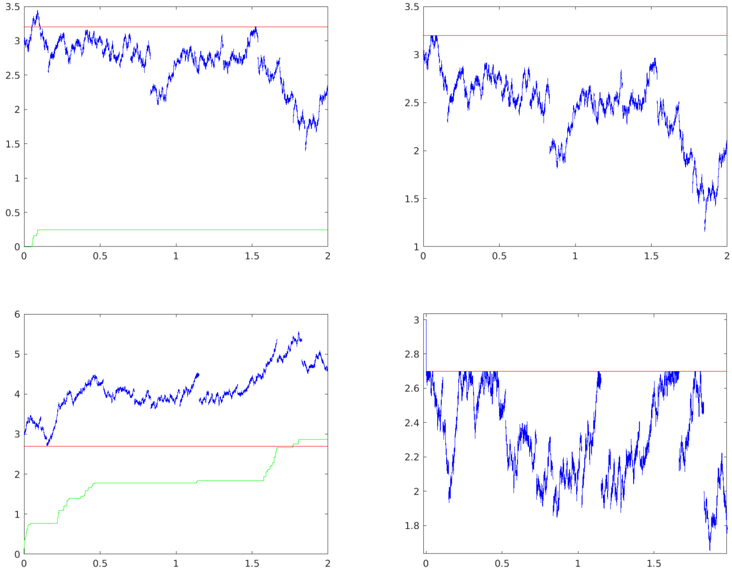
where $\mathcal{L}$ is the infinitesimal generator of the process X. Then $V_{\hat{\pi}}(x) = V(x)$ for all $x \geq 0$ and, hence, $\hat{\pi}$ is an optimal strategy.

This problem has been studied when:

- The process X is a spectrally negative Lévy process and $g = 1$.
  - F. Avram, Z. Palmowski, and M. Pistorius. On the optimal dividend problem for a spectrally negative Lévy process. Ann. Appl. Probab., 17(1):156–180, 2007.
  - R. Loeffen. On the optimality of the barrier strategy in de Finetti's problem for spectrally negative Lévy processes. Ann. Appl. Probab., 18(5):1669–1680, 2008.
- The process X is a diffusion and $g > 0$ is C^1 and non-increasing on $(0, \infty)$.
  - L. H. R. Alvarez. Singular stochastic control in the presence of a state-dependent yield structure. Stochastic Process. Appl., 86(2):323– 343, 2000.

In both cases some conditions are given for the optimality of barrier strategies.

Barrier strategies $\pi_{\mathfrak{b}}$



Spectrally negative Lévy process

- No monotone trajectories and no positive jumps.
- The Laplace exponent of X is given by

$$\begin{aligned}\psi(\theta) &:= \log \mathbb{E}[\theta X_1] \\ &= \mu\theta + \frac{\sigma^2}{2}\theta^2 - \int_{(0,\infty)} \big(1 - \mathrm{e}^{-\theta z} - \theta z \mathbf{1}_{\{0<z\leq 1\}}\big)\Pi(\mathrm{d}z), \quad \theta \geq 0,\end{aligned}$$

where $\mu \in \mathbb{R}$, $\sigma \geq 0$, and the Lévy measure of X, Π, is a measure defined on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} (1 \wedge z^2)\Pi(\mathrm{d}z) < \infty.$$

- The Lévy measure Π of the process X has a completely monotone density. That is, Π has a density ν whose n^{th} derivative $\nu^{(n)}$ exists for all $n \geq 1$ and satisfies

$$(-1)^n \nu^{(n)}(x) \geq 0, \quad x > 0.$$

Scale functions

- Let $W^{(q)} : \mathbb{R} \longrightarrow [0, \infty)$ be scale function of X, which takes the value zero on $(-\infty, 0)$, is strictly increasing functions on $[0, \infty)$ and is defined by its Laplace transform,

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q),$$

where $\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$.

- We also have $Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(z)\mathrm{d}z$.

Why are they important?

- $(\mathcal{L}-q)W^{(q)}(x)=0 \quad \text{and} \quad (\mathcal{L}-q)Z^{(q)}(x)=0, \quad x>0.$
- We define the stopping times τ_{a^-} and τ_{a^+},

$$\tau_a^- := \inf\{t>0 : X_t < a\} \quad \text{and} \quad \tau_a^+ := \inf\{t>0 : X_t > a\}, \quad a\in\mathbb{R};$$

here and further on, let $\inf\varnothing=\infty$. Then, for $a>b$ and $x\leq a$,

$$\mathbb{E}_x\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_b^-\}}\right]=\frac{W^{(q)}(x-b)}{W^{(q)}(a-b)},$$

$$\mathbb{E}_x\left[e^{-q\tau_b^-}\mathbf{1}_{\{\tau_a^+>\tau_b^-\}}\right]=Z^{(q)}(x-b)-Z^{(q)}(a-b)\frac{W^{(q)}(x-b)}{W^{(q)}(a-b)}.$$

- When $\Pi \equiv 0$, i.e.

$$X = x + \mu t + \sigma B_t.$$

where $B = \{B_t : t \geq 0\}$ is a Brownian motion, we have

$$W^{(q)}(x) = \frac{1}{\sqrt{\mu^2 + 2q\sigma^2}} \left[e^{\Phi(q)x} - e^{-\zeta_1 x} \right]$$

$$Z^{(q)}(x) = \frac{q}{\sqrt{\mu^2 + 2q\sigma^2}} \left[\frac{e^{\Phi(q)x}}{\Phi(q)} + \frac{e^{-\zeta_1 x}}{\zeta_1} \right],$$

where

$$\zeta_1 = \frac{1}{\sigma^2} \left(\sqrt{\mu^2 + 2q\sigma^2} + \mu \right) \text{ and } \Phi(q) = \frac{1}{\sigma^2} \left(\sqrt{\mu^2 + 2q\sigma^2} - \mu \right).$$

Outline

One barrier strategies

A barrier strategy $\pi_b = \{L_t^b : t \geq 0\}$ at level $b \geq 0$, is defined by

$$L_t^b = \left(\sup_{0 \leq s \leq t} \{X_{s-} - b\}\right) \vee 0, \quad \text{for } t \geq 0.$$

Observe that it is continuous if $x \leq b$, and has a unique jump of size $x - b$ at time zero if $x > b$, where x is the starting value of X.

Proposition

Let $b \geq 0$. Then

$$V_b(x) := V_{\pi_b}(x) = \begin{cases} \dfrac{g(b)}{W^{(q)\prime}(b)} W^{(q)}(x) & \text{if } \; x \leq b, \\ H(x; b) & \text{if } \; x > b, \end{cases}$$

with $H(x; u) = G(x) - G(u) + g(u)\dfrac{W^{(q)}(u)}{W^{(q)\prime}(u)}$ and $G(x) = \displaystyle\int_0^x g(y)\mathrm{d}y$.

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with $H(x; u) = G(x) - G(u) + g(u) \frac{W^{(q)}(u)}{W^{(q)\prime}(u)}$ and $G(x) = \int_0^x g(y) \mathrm{d}y$.

Optimal strategy

Consider the map $F : (0, \infty) \mapsto (0, \infty)$ by

$$F(u) = \frac{g(u)}{W^{(q)\prime}(u)} \quad \text{for } u > 0.$$

We then define our choice of optimal threshold $\mathscr{b}^*$ by

$$\mathscr{b}^* = \sup\{u \geq 0 : F(u) \geq F(x), \ \text{ for all } x \geq 0\}.$$

Let us assume that $\mathscr{b}^* < \infty$.

Lemma

For any $\mathscr{b} \geq 0$, $V_{\mathscr{b}} \in C^1((0, \infty)) \cap C^2((0, \infty)) \setminus \{\mathscr{b}\})$ and $V_{\mathscr{b}} > 0$ is increasing on $(0, \infty)$. Additionally, $V_{\mathscr{b}^*} \in C^2((0, \infty))$.

Optimal strategy

By the Verification Lemma we only need to show that $V_{\mathfrak{b}^*}$ satisfies (HJB) for $x > 0$.

- Since $W^{(q)}$ is harmonic, then

$$(\mathcal{L} - q)V_{\mathfrak{b}^*}(x) = (\mathcal{L} - q)\left(\frac{g(\mathfrak{b}^*)}{W^{(q)}(\mathfrak{b}^*)}W^{(q)}(x)\right) = 0, \quad 0 \leq x \leq \mathfrak{b}^*.$$

- The definition of $\mathfrak{b}^*$ gives $\dfrac{g(x)}{W^{(q)\prime}(x)} \leq \dfrac{g(\mathfrak{b}^*)}{W^{(q)\prime}(\mathfrak{b}^*)}$ for $0 \leq x \leq \mathfrak{b}^*$, hence

$$g(x) - V'_{\mathfrak{b}^*}(x) = g(x) - g(\mathfrak{b}^*)\frac{W^{(q)\prime}(x)}{W^{(q)\prime}(\mathfrak{b}^*)} \leq 0, \quad 0 \leq x \leq \mathfrak{b}^*.$$

- $V'(x) = g(x) \quad$ for $x \geq \mathfrak{b}^*$.

Optimal strategy

Theorem

If $F' \leq 0$ on $[b^*, \infty)$, then $(\mathcal{L} - q)V_{b^*} \leq 0$ on (b^*, ∞). Therefore, the barrier strategy π_{b^*} is optimal and $V(x) = V_{b^*}(x)$ for all $x \geq 0$.

Examples

- $g(x) = x^\alpha$ with $\alpha > 0$.
- $g(x) = e^{-\beta x}$ with $\beta > 0$.

$F' \leq 0$ is only sufficient

$F' \not\leq 0$ and $\flat^*$ still optimal...

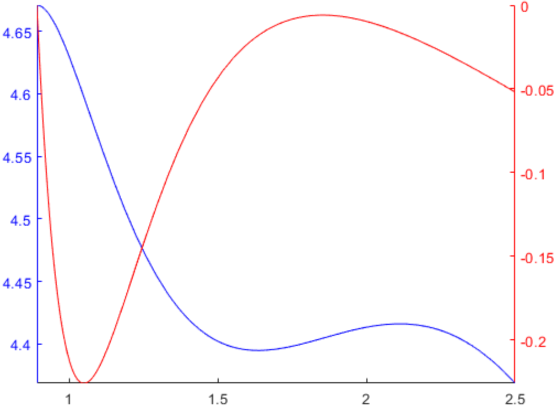


Figure: Plots of F (in blue) and $(\mathcal{L} - q)V_{\flat^*}$ (in red). We see that $\flat^*$ is where F attains its global maximum, but F' eventually becomes positive. $(\mathcal{L} - q)V_{\flat^*}(x) \leq 0$ for $x \geq \flat^*$.

$F' \leq 0$ is only sufficient

... or $\mathscr{b}^*$ not optimal

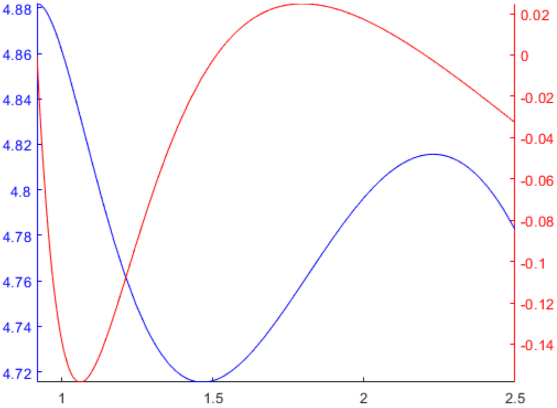


Figure: Plots of F (in blue) and $(\mathcal{L} - q)V_{\mathscr{b}^*}$ (in red). We see that $\mathscr{b}^*$ is where F attains its global maximum, but F' eventually becomes positive. $(\mathcal{L} - q)V_{\mathscr{b}^*}(\bar{x}) > 0$ for some $\bar{x} \geq \mathscr{b}^*$.

Outline

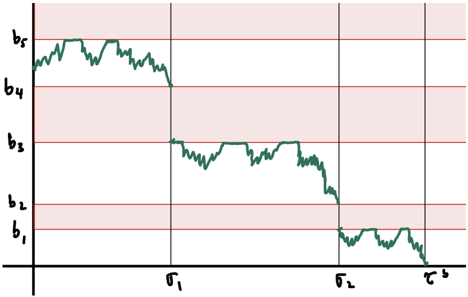
Multi-barrier strategies for Brownian motion with drift

Let us assume that there are $(2n+1)$-barriers $\mathbb{b}$ with

$$0 =: b_0 \leq b_1 \leq b_2 \leq \cdots \leq b_{2n+1},$$

and consider the following strategy $\pi_{\mathbb{b}}$:

- If the process is above b_{2n+1}, push the process to b_{2n+1};
- if it lies between (b_{2k}, b_{2k+1}) with $k \in \{0, 1, \ldots, n\}$ do nothing;
- and if it lies between $[b_{2k+1}, b_{2k+2}]$, with $k \in \{0, 1, \ldots, n-1\}$, push the process in b_{2k+1}.



Proposition

$$V_{\mathbb{b}}(x) := V_{\pi_{\mathbb{b}}}(x) = \begin{cases} \dfrac{g(b_1)}{W^{(q)\prime}(b_1)} W^{(q)}(x) & \text{if } x < b_1, \\ H(x; b_1) & \text{if } x \in [b_1, b_2], \\ \quad \vdots & \quad \vdots \\ \phi(x; \mathbb{b}_{2k+1}) & \text{if } x \in (b_{2k}, b_{2k+1}), \\ H(x; \mathbb{b}_{2k+1}) & \text{if } x \in [b_{2k+1}, b_{2k+2}], \\ \quad \vdots & \quad \vdots \\ \phi(x; \mathbb{b}) & \text{if } x \in (b_{2n}, b_{2n+1}), \\ H(x; \mathbb{b}) & \text{if } x \in [b_{2n+1}, \infty), \end{cases}$$

where for each $k \in \{0, 1, 2, \dots, n\}$, $\mathbb{b}_{2k+1} := \{b_i\}_{i=1}^{2k+1}$ and

$$\begin{aligned} \phi(x; \mathbb{b}_{2k+1}) &:= H(b_{2k}; \mathbb{b}_{2k-1}) Z^{(q)}(x - b_{2k}) \\ &\quad + W^{(q)}(x - b_{2k}) \left(\frac{g(b_{2k+1}) - qH(b_{2k}; \mathbb{b}_{2k-1}) W^{(q)}(b_{2k+1} - b_{2k})}{W^{(q)\prime}(b_{2k+1} - b_{2k})} \right), \\ H(x; \mathbb{b}_{2k+1}) &:= G(x) - G(b_{2k+1}) + \phi(b_{2k+1}; \mathbb{b}_{2k+1}). \end{aligned}$$

Proposition

$$V_{\mathbb{b}}(x) := V_{\pi_{\mathbb{b}}}(x) = \begin{cases} \frac{g(\mathscr{b}_1)}{W^{(q)\prime}(\mathscr{b}_1)} W^{(q)}(x) & \text{if } x < \mathscr{b}_1, \\ H(x; \mathscr{b}_1) & \text{if } x \in [\mathscr{b}_1, \mathscr{b}_2], \\ \vdots & \vdots \\ \phi(x; \mathbb{b}_{2k+1}) & \text{if } x \in (\mathscr{b}_{2k}, \mathscr{b}_{2k+1}), \\ H(x; \mathbb{b}_{2k+1}) & \text{if } x \in [\mathscr{b}_{2k+1}, \mathscr{b}_{2k+2}], \\ \vdots & \vdots \\ \phi(x; \mathbb{b}) & \text{if } x \in (\mathscr{b}_{2n}, \mathscr{b}_{2n+1}), \\ H(x; \mathbb{b}) & \text{if } x \in [\mathscr{b}_{2n+1}, \infty), \end{cases}$$

where for each $k \in \{0, 1, 2, \ldots, n\}$, $\mathbb{b}_{2k+1} := \{\mathscr{b}_i\}_{i=1}^{2k+1}$ and

$$\phi(x; \mathbb{b}_{2k+1}) := H(\mathscr{b}_{2k}; \mathbb{b}_{2k-1}) Z^{(q)}(x - \mathscr{b}_{2k}) + W^{(q)}(x - \mathscr{b}_{2k}) \left(\frac{g(\mathscr{b}_{2k+1}) - qH(\mathscr{b}_{2k}; \mathbb{b}_{2k-1}) W^{(q)}(\mathscr{b}_{2k+1} - \mathscr{b}_{2k})}{W^{(q)\prime}(\mathscr{b}_{2k+1} - \mathscr{b}_{2k})} \right),$$

$$H(x; \mathbb{b}_{2k+1}) := G(x) - G(\mathscr{b}_{2k+1}) + \phi(\mathscr{b}_{2k+1}; \mathbb{b}_{2k+1}).$$

Optimal multi-barrier strategy

$V_{\mathbb{b}} \in C((0,\infty)) \cap C^1((0,\infty) \setminus \{b_{2k}\}_{k=1}^{n}) \cap C^2((0,\infty) \setminus \mathbb{b}))$ for any $\mathbb{b}$. The smooth fit principle requires finding barriers that increase the regularity of this function. Consider the function

$$F(v,z;\mathbb{b}_{2k+1}) := \frac{g(z) - qH(v;\mathbb{b}_{2k+1})W^{(q)}(z-v)}{W^{(q)\prime}(z-v)}, \quad \text{for } (v,z) \in \mathcal{A}_{2k+1},$$

where

$$\mathcal{A}_{2k+1} := \{(v,z) \in [b_{2k+1},\infty) \times [b_{2k+1},\infty) : v < z\},$$

with $k \in \{0,1,\ldots,n\}$ and $n \geq 1$.

Lemma

For $k \in \{0,1,\ldots,n\}$

$$\partial_z F(v, v+; \mathbb{b}_{2k+1}) = (\mathcal{L} - q)H(v;\mathbb{b}_{2k+1}) \quad \text{for } v \geq b_{2k+1}.$$

Optimal multi-barrier strategy

We now present an algorithm that returns a sequence of optimal barriers $\mathbb{b}^* = \{b_i^*\}_{i=1}^{2n+1}$, for some $n \geq 0$.

Assumption

$W^{(q)}$ grows faster that g.

Algorithm

(1) Choose b_1^* as

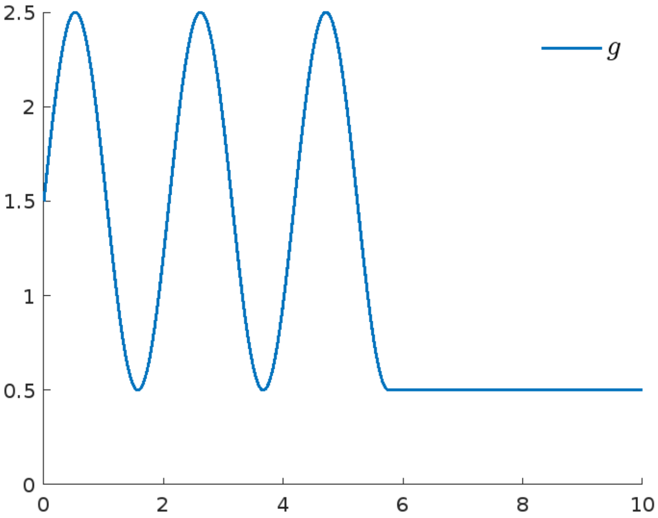
$$b_1^* = \sup\{u \geq 0 : F(u) \geq F(x), \quad \text{for all } x \geq 0\}.$$

(2) If

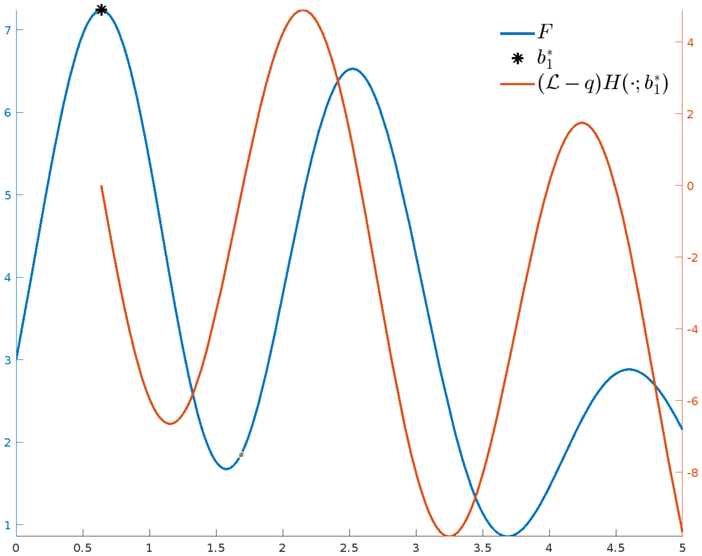
$$(\mathcal{L} - q)H(x; b_1^*) \leq 0 \quad \text{for } x \geq b_1^*,$$

choose the barrier b_1^*, let $n = 0$ and stop. Otherwise let $k = 1$.

Example



Example



Optimal multi-barrier strategy

Algorithm

(3) Let $\mathcal{P}_{2k-1}$ be the set of points $\tilde{v} > b^*_{2k-1}$ such that for some $\varepsilon > 0$ small enough,

$$\partial_z F(v, v+; \mathbb{b}^*_{2k-1}) = (\mathcal{L} - q)H(v; \mathbb{b}^*_{2k-1}) \begin{cases} < 0, & \text{if } v \in (\tilde{v} - \varepsilon, \tilde{v}), \\ = 0, & \text{if } v = \tilde{v}, \\ > 0, & \text{if } v \in (\tilde{v}, \tilde{v} + \varepsilon). \end{cases}$$

Let $c_{2k-1} = \min \mathcal{P}_{2k-1}$.

Example

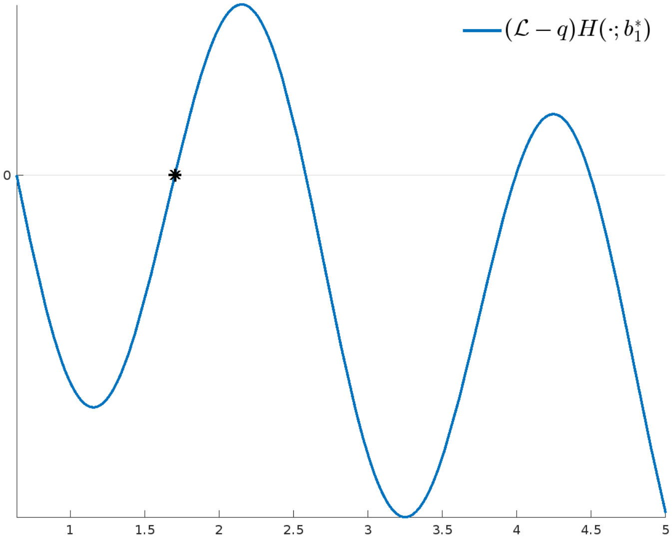


Figure: The black star shows c_1.

Optimal multi-barrier strategy

Algorithm

(4) Let

$$\mathcal{D}_{2k-1} := \{v \in [\mathscr{b}^*_{2k-1}, \mathscr{c}_{2k-1}] : v < z_{2k-1}(v)\},$$

where

$$z_{2k-1}(v) := \sup\{y > v : F(v, y; \mathbb{b}^*_{2k-1}) \geq F(v, z; \mathbb{b}^*_{2k-1}), \text{ for all } z > v\},$$

for $v \geq \mathscr{b}^*_{2k-1}$. Let

$$\mathscr{b}^*_{2k} := \inf \mathcal{D}_{2k-1} \quad \text{and} \quad \mathscr{b}^*_{2k+1} := z_{2k-1}(\mathscr{b}^*_{2k}).$$

Example

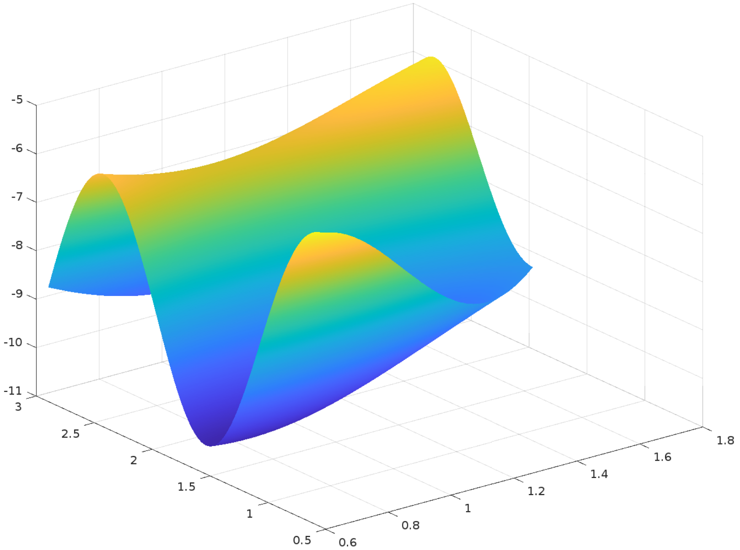


Figure: Surface of the function $F(v, z; b_1^*)$ for $b_1^* \leq v < z$

Example

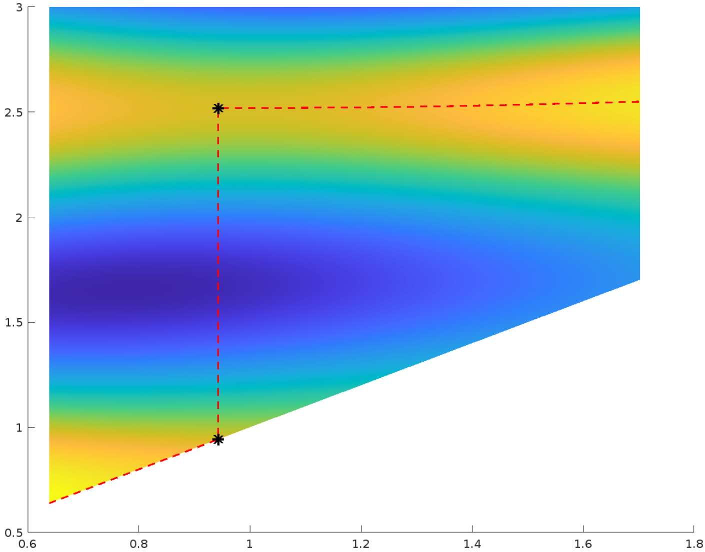


Figure: Plot (red dotted line) of $z_1(v)$ for $b_1^* \leq v$. Black stars show b_2^* and b_3^*.

Optimal multi-barrier strategy

Algorithm

(5) If
$$(\mathcal{L}-q)H(x;\mathbb{b}^*_{2k+1}) \leq 0 \quad \text{for } x \geq b^*_{2k+1},$$
choose the barriers $\mathbb{b}^*_{2k+1} = \{b^*_i\}_{i=1}^{2k+1}$, $n = k$ and stop. Otherwise, let $k = k+1$ and return to (3).

- A sufficient condition is $\partial_z F(b^*_{2k}, z; \mathbb{b}^*_{2k-1}) \leq 0$ for $z > b^*_{2k+1}$.

Example

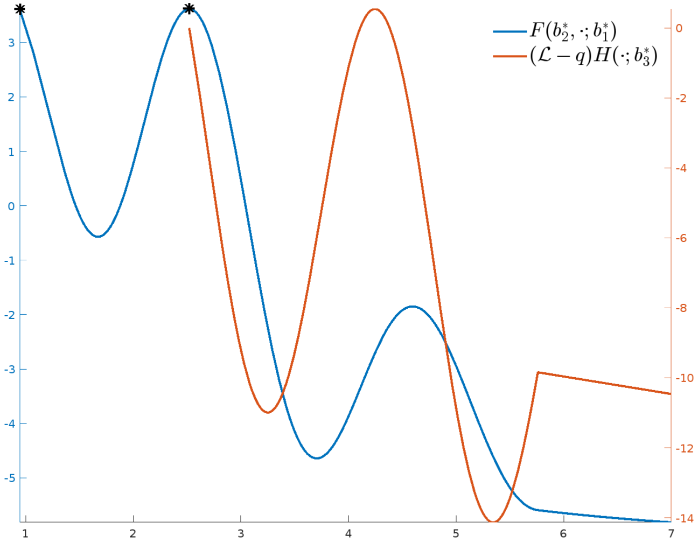


Figure: Black stars show b_2^* and b_3^*. $(\mathcal{L} - q)H(\bar{x}; b_3^*) > 0$ for some $\bar{x} > b_3^*$.

Example

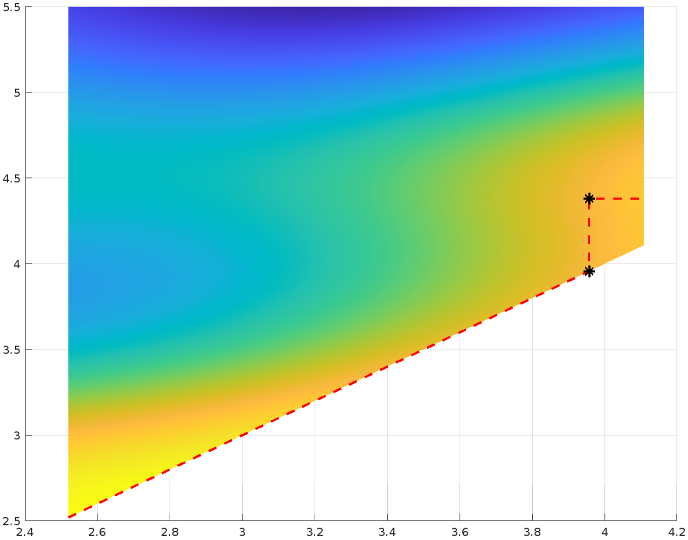


Figure: Plot (red dotted line) of $z_3(v)$ for $b_3^* \leq v$. Black stars show b_4^* and b_5^*.

Example

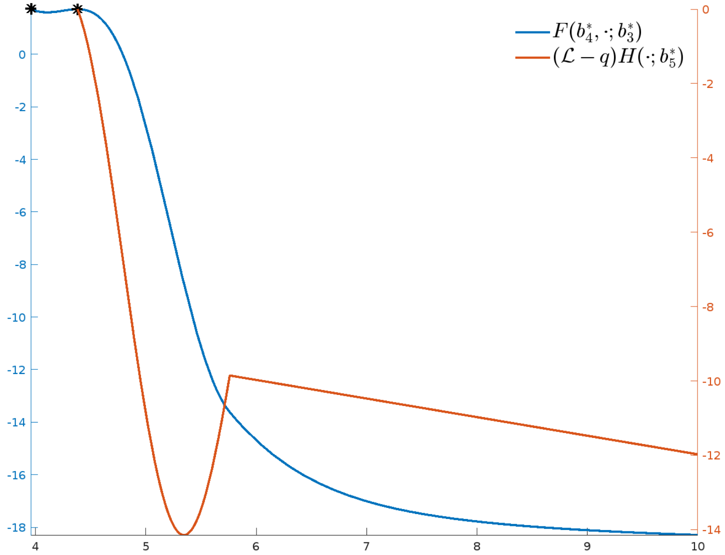


Figure: Black stars show b_4^* and b_5^*. $(\mathcal{L} - q)H(x; b_5^*) \leq 0$ for $x > b_3^*$.

Optimality verification

Proposition

Under some technical conditions, the points $\mathscr{b}^*_{2k}$ and $\mathscr{b}^*_{2k+1}$, for $1 \le k \le n$, given by the algorithm are well defined and satisfy

$$\mathscr{b}^*_{2k-1} < \mathscr{b}^*_{2k} < \mathscr{b}^*_{2k+1} \quad \text{and} \quad \mathscr{b}^*_{2k} < \mathscr{c}_{2k-1}.$$

Moreover,

$$F(\mathscr{b}^*_{2k}, \mathscr{b}^*_{2k}; \mathbb{b}^*_{2k-1}) = F(\mathscr{b}^*_{2k}, \mathscr{b}^*_{2k+1}; \mathbb{b}^*_{2k-1}).$$

This proposition allows to show that $V_{\mathbb{b}^*} \in C^2(\mathbb{R} \setminus \{\mathscr{b}_{2k}\}_{k=1}^n)$. The function also satisfies:

- As before $V_{\mathbb{b}^*}$ satisfies (HJB) on $(0, \mathscr{b}^*_1)$.
- $V'_{\mathbb{b}^*}(x) = g(x)$ for $x \in \bigcup_{k=2}^n [\mathscr{b}^*_{2k-1}, \mathscr{b}^*_{2k}] \cup [\mathscr{b}^*_{2n+1}, \infty)$.
- By the harmonic property of the scale functions $(\mathcal{L} - q)V_{\mathbb{b}^*}(x) = 0$ for $x \in \bigcup_{k=1}^n (\mathscr{b}^*_{2k}, \mathscr{b}^*_{2k+1})$.

Optimality verification

- Since $F(b^*_{2k}, x; \mathbb{b}^*_{2k-1}) \leq F(b^*_{2k}, b^*_{2k+1}; \mathbb{b}^*_{2k-1})$ for $x \in (b^*_{2k}, b^*_{2k+1})$ and $k \in \{1, \ldots, n\}$, then

$$\begin{aligned} V'_{\mathbb{b}^*}(x) &= qH(b^*_{2k}; \mathbb{b}_{2k-1}) W^{(q)}(x - b^*_{2k}) + F(b^*_{2k}, b^*_{2k+1}; \mathbb{b}_{2k-1}) W^{(q)\prime}(x - b^*_{2k}) \\ &\geq qH(b^*_{2k}; \mathbb{b}_{2k-1}) W^{(q)}(x - b^*_{2k}) + F(b^*_{2k}, x; \mathbb{b}_{2k-1}) W^{(q)\prime}(x - b^*_{2k}) \\ &= g(x). \end{aligned}$$

- By construction of the Algorithm, for both $x \in [b^*_{2k+1}, b^*_{2k+2}]$, with $k \in \{0, \ldots, n-1\}$, and $x \in [b^*_{2n+1}, \infty)$, we have that

$$(\mathcal{L} - q) V_{\mathbb{b}^*}(x) = (\mathcal{L} - q) H(x; \mathbb{b}^*_{2k-1}) \leq 0.$$

Theorem

Let $\mathbb{b}^* = \{b^*_k\}_{k=1}^{2n+1}$ be defined by the algorithm. Then $V_{\mathbb{b}^*}(x) = V(x)$ for all $x \geq 0$ and the multi-barrier $\mathbb{b}^*$-strategy is optimal.

Future work

- Consider singular control problems with terminal cost
- Consider impuse control problems
- Consider Itô difussions

Thank you for your attention!