Winter School on

Integrable Systems and Representation Theory

within the project "Current Trends in Mathematics"

# Abstracts of

# COURSES and INVITED TALKS

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Schedule

| Time | Monday | Tuesday | Wednesday | Thursday | Friday |
|---|---|---|---|---|---|
| **9.00-9.55** | *Registration* | **Feigin 3** | **Krichever 1** | **Reshetikhin 3** | **Calogero** |
| **10.00-10.55** | **Feigin 1** | **Lando 3** | **Reshetikhin 1** | **Krichever 2** | **Marshakov 3** |
| *11:00-11:30* | *coffee break* | *coffee break* | *coffee break* | *coffee break* | *coffee break* |
| **11:30-12:25** | **Feigin 2** | **Kac** | **Reshetikhin 2** | **Krichever 3** | **Gavrylenko** |
| *12:30-14:30* | *lunch break* | *lunch break* | *lunch break* | *lunch break* | *lunch break* |
| **14:30-15:25** | **Migliorini** | **Bershtein** | **Semenov-T-S 3** | **Tanzini** | **Contucci** |
| **15:30-16:25** | **Lando 1** | **Semenov-T-S 1** | **Zograf** | **Marshakov 1** | **Fioravanti** |
| *16:25-16:50* | *break* | *break* | *break* | *break* | *break* |
| **16:50-17:45** | **Lando 2** | **Semenov-T-S 2** | **Pogrebkov** | **Marshakov 2** | **Ravanini** |
| **17:50-18:45** | | | **Grinevich** | | |

Lectures in room Aula Tonelli (6th floor)

# Abstracts

# Courses

Boris Feigin
*HSE University*

**Affine and toroidal algebras and integrable systems.**

I plan to discuss the constructions of different quantum groups. Our main tool - shuffle algebras. Definition of such algebras is rather elementary and simple. In shuffle terms it is possible to get the simple construction of affine and toroidal quantum groups and also in some cases deformed $W$-algebras. Representations of quantum groups and $R$-matrices also naturally appear. Shuffle algebras are good for the constructions of coordinate rings of the quantizations of the different moduli spaces - for example instanton manifolds. I plan to explain how to do it. Actually such coordinate rings can be realised as quotients or subalgebras of the shuffle algebras. In some special case subalgebras in the suffle algebras become commutative. By this way we get "big" commutative subalgebras which in representations become the quantum hamiltonians of interesting integrable systems.

Igor Krichever
*Columbia University - HSE University - Skoltech*

**Algebraic-geometrical integration theory of soliton equations.**

A self-contained introduction to the theory of soliton equations with an emphasis on its applications to algebraic-geometry. Topics include:

1. General features of the soliton systems. Basic hierarchies of commuting flows: KP, 2D Toda, Bilinear Discrete Hirota hierarchy. Discrete and finite-dimensional integrable systems.

2. Algebraic-geometrical integration theory. Spectral curves. Baker-Akhiezer functions.

3. Hamiltonian theory of soliton equations.

Sergei Lando
*HSE University - Skoltech*

**Combinatorial integrability.**

The minicourse will be devoted to a description of combinatorial solutions to integrable hierarchies of Kadomtsev–Petviashvili type that arise naturally in enumeration of various topological and algebro-geometric objects. A preliminary layout includes

- *Permutations and their decompositions into products of transpositions* (simple Hurwitz numbers, Okounkov's theorem, Hurwitz formula, Bousquet-Mélou–Schaeffer formula, cut-and-join equation)
- *Symmetric group representations* (diagonilizability of the cut-and-join operator, the group algebra of the symmetric group, Schur polynomial, Jucys–Murphy elements)
- *The semi-infinite Grassmannian and the Kadomtsev–Petviashvili hierarchy* (Plücker embeddings, semi-infinite planes in the space of Laurent series, Orlov–Shcherbin family of solutions)
- *Ramified coverings of the 2-sphere* (coverings and ramified coverings, Hurwitz numbers and ramified coverings, Caley formula, genus expansion)

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Andrey Marshakov
*Skoltech - HSE University*

**Cluster varieties, integrable systems and supersymmetric gauge theories.**

The minicourse is devoted to integrable systems on cluster varieties, their deautonomization and connection with supersymmetric gauge theories. We start with the cluster Poisson varieties and describe their main properties, keeping as a basic example the Fock-Goncharov construction of cluster co-ordinates on the (affine, co-extended) Lie groups. Then we discuss how this construction leads to appearence of a completely integrable system on their Poisson subvarieties, with the most well-known example given by relativistic Toda chains, while generally these integrable systems can be alternatively defined a la Goncharov and Kenyon. The whole picture allows natural deautonomization, still keeping traces of integrability in the (discrete, non-autonomous) equations of the Painleve type, whose solutions can be constructed in terms of supersymmetric gauge theories. To do that we remind the connection between Seiberg-Witten prepotentials and algebraic integrable systems, introduce Nekrasov functions and show that their duals (just by Fourier transform) appear in this context as isomonodromic tau-functions, solving the Hirota equations for deautonomized cluster integrable systems.

Nicolai Reshetikhin
*University of California Berkeley - Saint Petersburg State University*

**Integrable and superintegrable systems on moduli spaces of flat connections.**

Let $G$ be a simple Lie group. For a compact topological surface the moduli space of $G$-flat connections has a natural symplectic structure when the surface is closed and a Poisson structure when the surface has boundary (Atyiah and Bott). Symplectic leaves of this Poisson variety are parametrized by conjugacy classes of monodromies along connected components of the boundary. The goal of these lectures is to explain that central functions on holonomies along any simple collection of curves on a surface define a superintegrable system and to describe these systems in some special cases. We will also see that these systems are natural generalizations of spin Calogero-Moser systems. The first lecture will focus on the definition and properties of superintegrable systems. In the second lecture spin Calogero-Moser systems will be introduced and it will be proven that they are superintegrable systems. After this superintegrable systems corresponding to simple curves on a surface will be introduced and the connection to Calogero-Moser type systems will be explained. The lectures are based on joint work with S. Artamonov and J. Stokman.

Michael Semenov-Tian-Shansky
*Steklov St. Petersburg - Université de Bourgogne*

**Quantum Toda Lattice and Representation Theory.**

Quantum Toda Lattice is a famous model which displays deep connections both with the Representation Theory of semisimple Lie groups and with the Quantum Inverse Scattering Method. Kostant's famous discovery linked Toda Lattice to the theory of generalized Whittaker functions. New formulae, due mainly to Lebedev and his collaborators, are based on Sklyanin's ideology of quantum separation of variables. Interaction between the two approaches sheds a new light on classical constructions of Representation Theory.

# Invited Talks

Mikhail Bershtein
*LITP-RAS, NRU-HSE, Skoltech*

**Shuffle algebras: main examples.**

My talk is supposed to be an addition to B. Feigin's lecture course. I plan to discuss in detail the connection between shuffle algebras and simplest quantum affine and toroidal algebras, (mainly following A. Negut.)

Francesco Calogero
*Università di Roma "La Sapienza" and INFN-Roma*

**Dynamical systems solvable by algebraic operations.**

In this talk I shall focus on systems of nonlinear Ordinary Differential Equations, and introduce the notion of their *solvability by algebraic operations*: implying that their *general* solution, considered as a function of *complex time*, feature at most a *finite* number of *rational branch points*, or equivalently define a Riemann surface with a *finite* number of sheets. Some properties of these systems shall be reviewed, including the subclasses of them featuring such remarkable properties as *isochrony* or *asymptotic isochrony* (as functions of *real* time). Techniques to identify such systems shall be reviewed, and several examples reported, including *new* classes of such systems.

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F. Calogero, *Zeros of Polynomials and Solvable Nonlinear Evolution Equations*, Cambridge University Press, 2018 (168 pages).

F. Calogero and F. Payandeh, "Polynomials with multiple zeros and solvable dynamical systems including models in the plane with polynomial interactions", J. Math. Phys. **60**, 082701 (2019).

F. Calogero, R. Conte and F. Leyvraz, "New *solvable* systems of two autonomous first-order ordinary differential equations with purely quadratic right-hand sides" (in preparation).

Pierluigi Contucci
*Università di Bologna*

**Exact solutions in mean-field disordered statistical mechanics.**

The talk will review some properties of the exact solutions of models like the Sherrington-Kirpatrick mean-field spin-glass. Generalisations to the higher dimensional cases will be presented together with open problems related to the Boltzmann Machines in artificial intelligence.

Davide Fioravanti
*INFN-Bologna*

**The strange case of 4D Wilson loops or a third way to TBA.**

I will show how the relativistic Thermodynamic Bethe Ansatz (TBA) by Al. Zamolodchikov, naturally attached to Dynkin diagrams, can be derived from an operator product expansion of null polygonal Wilson loops. This is a sort of third way by which TBA arises naturally, by counting as other means 1) the usual thermodynamics on the scattering theory and 2) its derivation from the functional equations of the so-called Ordinary Differential Equation/Integrable Models correspondence. I will illustrate both these ways.

Pavlo Gavrylenko
*Skoltech and NRU-HSE*

**From Nekrasov functions to theta functions.**

I'm going to demonstrate how to take an autonomous limit of the general solution of some ($q-$)isomonodromic system. Such solutions are usually given as Fourier transformations of Nekrasov functions. One can show using Seiberg-Witten equations that in the autonomous limit these Fourier transformations turn into Riemann theta functions, and thus satisfy Fay bilinear relations.

Petr Grinevich
*Steklov Math. Inst. Moscow, LITP-RAS and Moscow State University*

**Total positivity, M-curves and real regular Kadomtsev-Petviashvili II solutions.**

This lecture is based on joint results with Simonetta Abenda, Bologna University.

We establish a bridge between two approaches to construct real regular solutions of the Kadomtsev-Petviashvili equation. Multiline soliton solutions are constructed in terms of totally non-negative Grassmannians, and real regular finite-gap solutions correspond to spectral $M$-curves with divisors satisfying an extra condition. It is easy to construct soliton solutions by degenerating the spectral curves, but if we would like to stay in the real regular class, the problem becomes non-trivial.

We present a construction associating a degenerate $M$-curve and a divisor on it with reality and regularity condition to a point of a totally non-negative Grassmannian. This construction essentially uses the parametrization of the totally non-negative Grassmannians in terms of the Le-networks from the Postnikov's paper.

Victor Kac
*MIT*

**Cohomology of algebraic structures and integrability.**

I will explain an operadic approach to cohomology theory, which allows one to develop cohomology for Poisson vertex algebras and vertex algebras. This is applied to the proof of integrability for classical and quantum Hamiltonian PDE.

Luca Migliorini
*Università di Bologna*

**The Hitchin fibration as an integrable system.**

After recalling some well-known facts about period matrices of Riemann surfaces, I will show how abelianization, in the form of the Beauville-Narasimhan-Ramanan correspondence, produces Higgs bundles associated with spectral curves. The moduli space of Higgs bundles on a given Riemann surface is a partial compactification of the cotangent bundle of the moduli space of vector bundles, and is therefore endowed with a symplectic structure. The relative Picard scheme of the spectral family acts on this moduli space, thus giving the completely integrable structure. I will quickly discuss the structure of the singular fibres of the Hitchin map. For simplicity only the case of $GL_n$-Higgs bundles will be treated.

Andrei Pogrebkov
*Steklov Math. Inst. Moscow and NRU-HSE*

**Hirota Difference Equation and Darboux System: Mutual Symmetry.**

We considered the relation between two famous integrable equations: The Hirota difference equation (HDE) and the Darboux system that describes conjugate curvilinear systems of coordinates in R3. We demonstrated that specific properties of solutions of the HDE with respect to independent variables enabled introduction of an infinite set of discrete symmetries. We showed that degeneracy of the HDE with respect to parameters of these discrete symmetries led to the introduction of continuous symmetries by means of a specific limiting procedure. This enabled consideration of these symmetries on equal terms with the original HDE independent variables.

Francesco Ravanini
*Università di Bologna*

**Integrablity and Thermodynamic Bethe Ansatz in Generalised Quantum Hydrodynamics.**

I briefly review the topic of Generalised Hydrodynamics in the framework of Non Equilibrium Steady States (NESS) and how it is strongly connected to Integrability. As a result, the profiles of the steady currents of energy between two reservoirs at different temperatures can be expressed in terms of a Thermodynamic Bethe Ansatz (TBA) approach known as NESS-TBA. Various issues of this approach are presented.

Alessando Tanzini
*SISSA, Trieste*

**On Painlevé/gauge theory correspondence.**

I will give an introduction to the correspondence between Painlevé equations, the associated isomonodromy deformation problems and supersymmetric gauge theories. The relation to Hitchin's integrable system, (quantum) Toda chain and elliptic Calogero system will be highlighted.

Peter Zograf
*PDMI – RAS and Saint Petersburg State University*

**Mathematical Physics of Map Enumeration.**

Generating functions whose coefficients count maps (ribbon graphs) and hypermaps (Grothendieck's dessins d'enfant) satisfy several remarkable integrability properties. In particular, they obey Virasoro constraints, evolution equations, Kadomtsev-Petviashvili (KP) hierarchy and a topological recursion. (After a joint work with M. Kazarian)