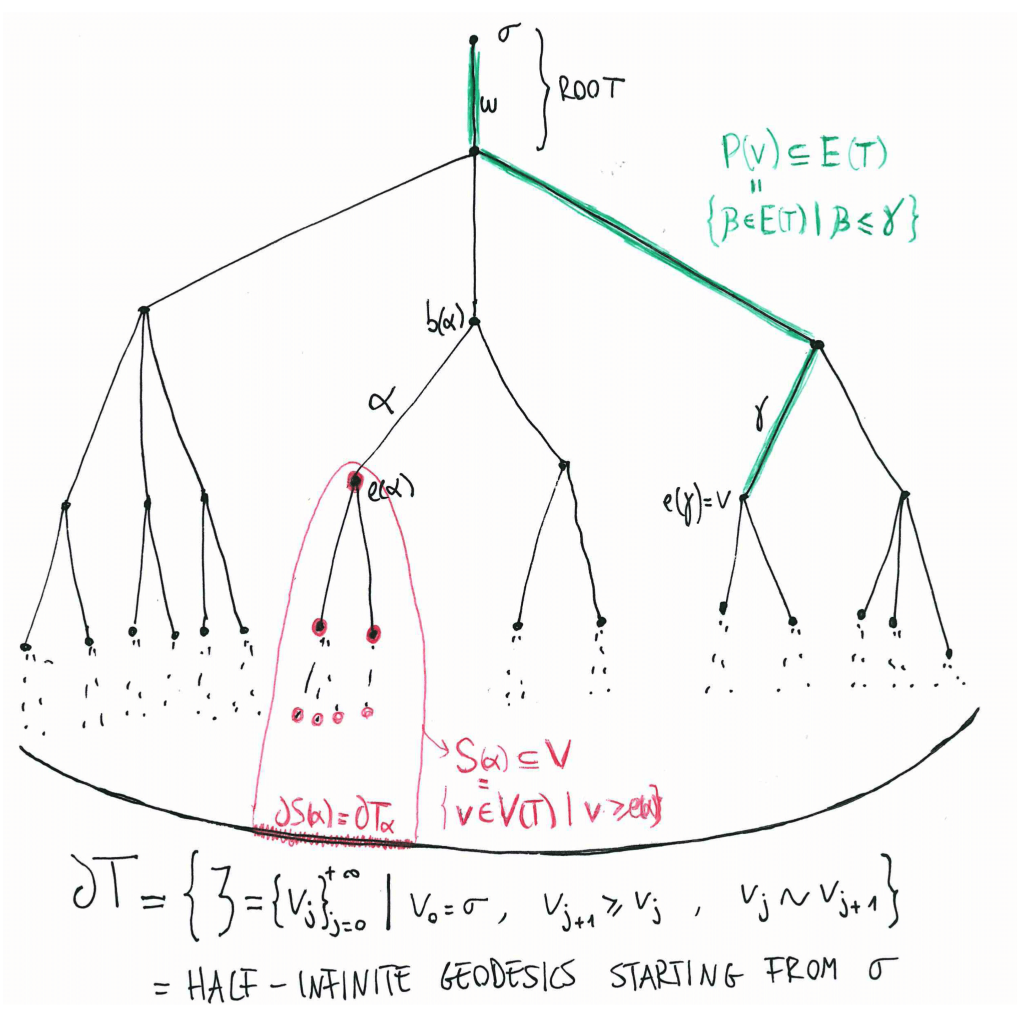
EQUILIBRIUM MEASURES ON TREES

MATTEO LEVI (Università di Bologna)
Joint work with NICOLA ARCOZZI

INTRODUCTION

According to physics laws, given a positive charge distribution (Radon measure) μ on a compact conductor $K \subseteq \mathbb{R}^3$, it will get a new equilibrium configuration μ^K which is the one minimizing the energy associated to the electrostatic potential and making the latter essentially constant on K (say $\equiv V_K$). The quantity $\mu^K(K)/V_K$ is the **capacity** of K. Generalizing from Physics to Mathematics, in any locally compact metric space (X, d) one can set up an L^p potential theory and, roughly speaking, try to solve (P): given $E \subseteq X$, minimize the p-energy arising from some potential over a set of admissible functions on E. This is equivalent to solving a p-Laplace equation, and the minimum energy is called the capacity of E. In the Euclidean space one has existence and uniqueness of a p-harmonic function being the minimizer. By a minimax duality argument, (P) is equivalent to maximize the total mass of E over a set of admissible measures. The unique measure solving the problem is called the **equilibrium measure** for E. At present, we lack a complete description of such measures in the Euclidean space. In this work, we give a characterization of them when the metric space X, is an infinite, locally finite rooted tree T. In particular, we show that equilibrium measures are exactly those solving a nonlinear discrete integro-differential equation.

POTENTIAL THEORY ON THE TREE



Definition. Let $f : E(T) \to \mathbb{R}$ and $\mu \in \mathcal{M}^+(\partial T)$. We define:

The **potential** of f, $If : V(T) \cup \partial T \to \mathbb{R}$, $If(\zeta) = \sum_{\alpha \in P(\zeta)} f(\alpha)$.

The **co-potential** of μ, $I^*\mu : E(T) \to \mathbb{R}$, $I^*\mu(\alpha) = \mu(\partial T_\alpha)$.

The **energy** of μ, $\mathcal{E}_p(\mu) = \sum_{\beta \in E(T)} M_p(\beta)^p$,

where $M_p(\beta) := I^*\mu(\beta)^{p'-1}$ is the edge function associated to μ.

Definition (p-capacity of $E \subseteq \partial T$). Set $\Omega_E = \{f \in \ell^p : If \geq 1 \text{ on } E\}$.

$$c_p(E) \overset{def}{=} \inf_{f \in \Omega_E} \|f\|^p_{\ell^p} = \min_{f \in \overline{\Omega_E}^{\ell^p}} \|f\|^p_{\ell^p} = \|f^E\|^p_{\ell^p},$$

where $\overline{\Omega_E}^{\ell^p} \overset{[1]}{=} \{f \in \ell^p : If \geq 1\ c_p - a.e. \text{ on } E\}$. The function f^E is *unique* ([1]) and it is called the **equilibrium function** for E. It holds $If^E = 1$ on E but for a set of null-capacity.

EQUILIBRIUM MEASURES

Theorem (Dual definition of p-capacity, [1]). *Let $E \subseteq \partial T$ be a capacitable set. Then,*

$$c_p(E) = \sup_{\mu \in \mathcal{M}_E} \mu(E)^p,$$

*where $\mathcal{M}_E := \{\mu \in \mathcal{M}^+(\partial T) : \operatorname{supp}(\mu) \subseteq E,\ \mathcal{E}_p(\mu) \leq 1\}$. Moreover, there exists a unique measure $\mu^E \in \mathcal{M}^+(\partial T)$, $\operatorname{supp}(\mu^E) \subseteq \overline{E}$, called p-**equilibrium measure** for E, such that M_p^E is the p-equilibrium function for E, i.e.*

$$c_p(E) = \mu^E(\overline{E}) = \|M_p^E\|^p_{\ell^p}.$$

THE MAIN RESULT

Theorem 1 (Characterization of equilibrium measures, [2]).

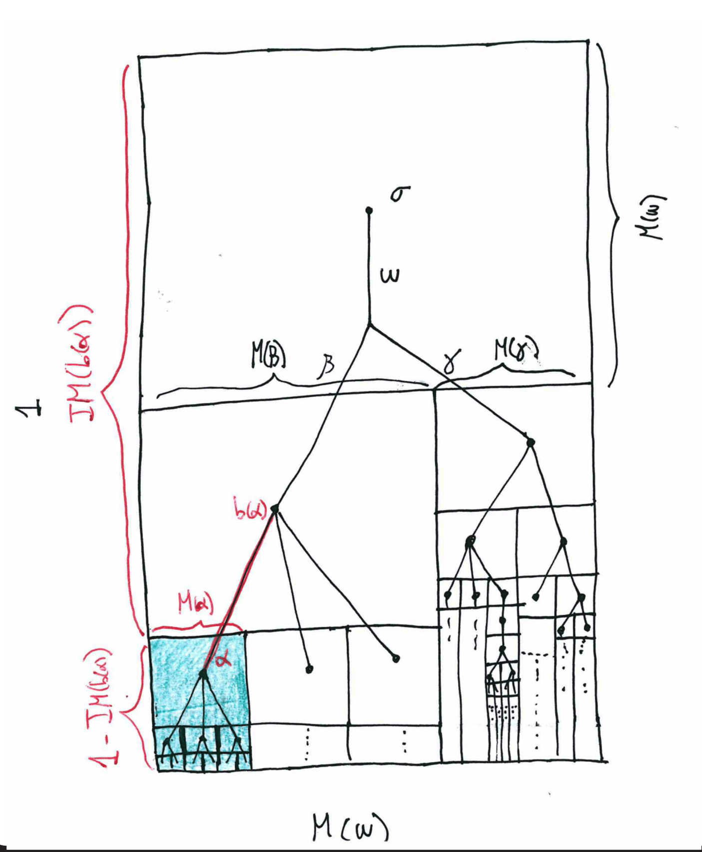
(i) Let $E \subseteq \partial T$ and $\mu = \mu^E$ be its p-equilibrium measure. Then, for every $\alpha \in E(T)$, μ solves the following equation:

$$M_p(\alpha)^{p/p'}\Big(1 - IM_p(b(\alpha))\Big) = \mathcal{E}_{p,\alpha}(\mu). \tag{1}$$

(ii) Let $\mu \in \mathcal{M}^+(\partial T)$ be a solution of (1). Then, there exists an $\mathcal{F}_\sigma$ set E such that μ is its p-equilibrium measure.

The necessary condition (i) is a quite straight forward consequence of some rescaling properties of capacity on subtrees. The main result is the opposite implication, where one has to cerefully deal with the so called **irregular points**, namely that exceptional set on which the potential is not equal to one. According to the **beautiful probabilistic** interpretation of capacity given in [3], (ii) can be reformulated saying that it exists an F_σ set such that μ is given by the distribution $\mathbb{P}(\lim_{n\to\infty} X_n = \zeta \in E;\ X_n \neq o\ \forall n)$, where X_n is the symmetric random walk on T, starting at $e(\omega)$.

THE CASE $p = 2$: SQUARE TILINGS



In the linear case $p = 2$ our result can be re-interpreted in terms of square tilings of a rectangle. The relationship between graphs and square tilings has already been studied by many authors. In particular, an important result [4] consists in associating a square tiling to the boundary of a planar graph. However, the opposite implication seems to be new, even in the linear case.

Theorem 2 ([2]). *(i) Let μ^E be the equilibrium measure for $E \subseteq \partial T$. Then, there exists a square tiling of a rectangle R having sides 1 and $c_2(E)$, where the combinatorics of the tilings are prescribed by T and the square $Q(\alpha)$ associated with the edge α has side $\mu^E(\partial T_\alpha)$. Moreover, if we replace T by its subtree T_E having edges α s.t. $\mu^E(\partial T_\alpha) > 0$, the tiling does not have degenerate squares.*

(ii) Viceversa, suppose a rectangle R is square-tiled with combinatorics given by a rooted tree T, and assume the tiles are not degenerate. Then there exists an F_σ subset E of ∂T such that $c_2(E \cap \partial T_\alpha) > 0$ for all edges α, and that the measure $\mu(\partial T_\alpha) = |Q(\alpha)|^{1/2}$, where $|Q|$ is the area of Q, is the equilibrioum measure of E.

REFERENCES

[1] David R Adams and Lars I Hedberg. *Function spaces and potential theory*, volume 314. Springer Science & Business Media, 2012.

[2] Nicola Arcozzi and Matteo Levi. Equilibrium measures on trees. *In preparation.*

[3] Kai Lai Chung. Probabilistic approach in potential theory to the equilibrium problem. In *Annales de l'institut Fourier*, volume 23, pages 313–322, 1973.

[4] Itai Benjamini, Oded Schramm, et al. Random walks and harmonic functions on infinite planar graphs using square tilings. *The Annals of Probability*, 24(3):1219–1238, 1996.