Carleson Measures for the Dirichlet space on the Bidisc

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Dirichlet space on $\mathcal{D}(\mathbb{D}^2)$

- $\mathcal{D}(\mathbb{D}^2) = \mathcal{D}(\mathbb{D}) \otimes \mathcal{D}(\mathbb{D})$
- $\|\Sigma_{n_1,n_2\geq 0} a(n_1,n_2) z_1^{n_1} z_2^{n_2}\|_{\mathcal{D}}^2 := \Sigma_{n_1,n_2\geq 0} |a(n_1,n_2)|^2 (n_1+1)(n_2+1)$
- Reproducing kernel is

$$K_{w_1,w_2}(z_1,z_2) = (C + \log\frac{1}{1-z_1\bar{w}_1})(C+\log\frac{1}{1-z_2\bar{w}_2})$$

where C depends on the choice of the norm.

Multipliers

A measure μ on $\bar{\mathbb{D}}^2$ is Carleson for $\mathcal{D}(\mathbb{D}^2)$, if the embedding $Id : \mathcal{D}(\mathbb{D}^2) \to L^2(\bar{\mathbb{D}}^2, d\mu)$ is bounded,

$$\int_{\bar{\mathbb{D}}^2} |f|^2\,d\mu \leq C_\mu \|f\|_{\mathcal{D}}^2.$$

The operator $M_m : f \mapsto mf$ is bounded on $\mathcal{D}(\mathbb{D}^2)$ if and only if

- $m \in H^\infty(\mathbb{D}^2)$, and
- the measures $|\partial_{z_1,z_2} m(z_1,z_2)|^2 dA(z_1)dA(z_2)$, $|\partial_{z_1} m(z_1,0)|^2 dA(z_1)dA(z_2)$, $|\partial_{z_2} m(0,z_2)|^2 dA(z_1)dA(z_2)$ are Carleson for $\mathcal{D}(\mathbb{D}^2)$

Bitree

- A bitree T^2 is a Cartesian product of two dyadic trees T. It is not a tree (or a planar graph).
- We identify T with $V(T)$ and T^2 with $V(T^2)$ (we don't consider the edges at all), in other words $\alpha = (\alpha_x, \alpha_y) \in T^2$, if $\alpha_x \in T = V(T), \alpha_y \in T = V(T)$.
- The natural order on T is defined as follows: $\alpha_x \leq \beta_x$, if β_x lies on the unique geodesic between α_x and the root.
- One can define a natural order relation on T^2 inherited from T, given $\alpha, \beta \in T^2$ we say that $\alpha \leq \beta$, if $\alpha_x \leq \beta_x$ and $\alpha_y \leq \beta_y$.
- A standard model of a dyadic tree is the collection of dyadic intervals on the unit interval. We put $[0,1)$ as a root, and every interval $\Delta_{j,k} = [j2^{-k}, (j+1)2^{-k})$, $k \geq 0,\ 0 \leq j < 2^k$ has exactly two sons $\Delta_{2j,k+1}$, $\Delta_{2j+1,k+1}$.
- Similarly, the elements (vertices) of T^2 can be represented as dyadic rectangles in $[0,1)^2$, namely for $\alpha \in T^2$ we put $\Delta_\alpha = \Delta_{\alpha_x} \times \Delta_{\alpha_y}$, where Δ_{α_x} (cf. Δ_{α_y}) is the dyadic interval corresponding to the vertex α_x (α_y).
- Given $\alpha_x, \beta_x \in T$ we define their confluent $\alpha_x \wedge \beta_x$ to be their least common ancestor (the point where the root geodesics from α_x and β_x meet). Again, the same can be done for T^2: for $\alpha, \beta \in T^2$ we define $\alpha \wedge \beta = (\alpha_x \wedge \beta_x, \alpha_y, \wedge \beta_y)$.

The problem

Describe Carleson measures for the Dirichlet space on the bidisc.

Theorem (Stegenga-type condition)

A measure μ on $\bar{\mathbb{D}}^2$ is Carleson for $\mathcal{D}(\mathbb{D}^2)$ if and only if for any finite collection $\{I_k\}_1^N, \{J_k\}_1^N$ of the arcs on $\mathbb{T}$ one has

$$\mu\left(\cup_{k=1}^N Q(I_k) \times Q(J_k)\right) \leq C_\mu \operatorname{Cap}_{\mathbb{D}^2}\left(\cup_{k=1}^N Q(I_k) \times Q(J_k)\right),$$

where $Q(I)$ is a Carleson square corresponding to the arc I, and $\operatorname{Cap}_{\mathbb{D}^2}$ is the capacity generated by the kernel $\Re K$.

Scheme of the proof

1. Pass to the dual problem using the adjoint operator $\Theta = Id^*$, arriving at

$$\int_{\bar{\mathbb{D}}^2} g^2\,d\mu \gtrsim \int_{\bar{\mathbb{D}}^2}\int_{\bar{\mathbb{D}}^2} g(w)g(z)\Re K_z(w)\,d\mu(z)d\mu(w) \qquad (1)$$

for any non-negative $g \in L^2(\bar{\mathbb{D}}^2, d\mu)$.

2. Dicretize the inequality by moving from the bidisc to the bitree (cartesian product of two uniform dyadic trees).
3. Write (1) in potential-theoretic terms and use the strong capacitory inequality to obtain the Stegenga-type characterization.
4. Go back to the bidisc.

Obstructions

- Neither the kernel $\Re K$ nor its discrete version satisfies the Maximum Principle

$$\sup_{\omega \in \operatorname{supp}\mu} V_K^\mu(\omega) \sim \sup V_K^\mu,$$

where V_K is the potential generated by $\Re K$ (or discrete counterpart). Known SCI proofs rely on the Maximum Principle in one way or another.
- Not enough is known about the weighted maximal inequalities (weak $1-1$ or L^p) in the bilinear setting.
- Sawyer's bilinear scheme also does not seem to work.

Potential Theory on the Bitree

- First we define the Hardy operator and its adjoint: given $f : T^2 \to \mathbb{R}$ we let

$$(\mathbb{I}f)(\alpha) = \Sigma_{\beta\geq\alpha} f(\beta), \quad (\mathbb{I}^* f)(\beta) = \Sigma_{\alpha\leq\beta} f(\alpha).$$

Given $\alpha_x, \beta_x \in T$ we define $d_T(\alpha_x \wedge \beta_x)$ to be the distance (in T) from their confluent to the root, and we let

$$d_{T^2}(\alpha\wedge\beta) := d_T(\alpha_x\wedge\beta_x)d_T(\alpha_y\wedge\beta_y) = \sharp\{\gamma \in T^2 :\ \gamma\geq\alpha,\ \gamma\geq\beta\}.$$

The (discrete) logarithmic potential is

$$\mathbb{V}^f(\alpha) = (\mathbb{I}\mathbb{I}^*)f(\alpha) = \Sigma_{\beta\in T^2} f(\beta) d_{T^2}(\alpha\wedge\beta).$$

- The potential $\mathbb{V}$ gives rise to the (discrete) logarithmic capacity Cap: for a set $E \subset T^2$ we let

$$\operatorname{Cap} E = \inf\{\mathcal{E}[f] : \mathbb{V}^f \geq 1 \text{ on } E\},$$

here $\mathcal{E}[f] = \Sigma_{\alpha\in T^2} \mathbb{V}^f(\alpha) f(\alpha)$ is the energy of f.
- By general theory there exists a unique equilibrium measure μ_E that realizes the $\inf$ above (measures and functions are essentially the same on T^2). Moreover $\mathbb{V}^{\mu_E} \equiv 1$ q.a.e. on $\operatorname{supp}\mu_E$.
- However (unlike the one-dimensional case) it could happen that $\mathbb{V}^{\mu_E}$ is arbitrarily large on the sets of positive capacity outside the support of μ_E (lack of Maximum Principle).

Dicretization Scheme

- Let I_{jk} be a j-th dyadic arc on $\mathbb{T}$ of generation $k \geq 0$, $I_{jk} = \{e^{2\pi i\theta} : j2^{-k} \leq \theta < (j+1)2^{-k}\}$. The system of these arcs can be represented by a dyadic tree T: for every pair $(j,k),\ k\geq 0,\ 0\leq j < 2^k$ there exists a unique vertex $\tau = \tau_{jk} \in T$ that corresponds to $I_{jk} = I_\tau$. Given $\tau \in T$ define by $S_\tau = S(I_\tau)$ the upper half of the respective Carleson square

$$S_\tau = \{re^{2\pi i\theta} \in \mathbb{D} : 1 - |I_\tau| \leq r \leq 1 - \frac{|I_\tau|}{2},\ e^{2\pi i\theta} \in I_\tau\}.$$

- Now consider the bitree $T^2 = T\times T$. For $\alpha = (\alpha_1,\alpha_2) \in T^2$ we let $\mathbb{S}_\alpha := S_{\alpha_1}\times S_{\alpha_2} \subset \mathbb{D}^2$. Observe that $\Re K_w(z) = \Re K_{w_1,w_2}(z_1,z_2)$ is more or less constant on the Carleson boxes $\mathbb{S}_\alpha \ni z$ and $\mathbb{S}_\beta \ni w$. Hence (1) can be rewritten as

$$\Sigma_{\alpha\in T^2}\tilde{g}^2(\alpha)\tilde{\mu}(\alpha) \gtrsim \Sigma_{\alpha\in T^2}\Sigma_{\beta\in T^2}\tilde{g}(\alpha)\tilde{g}(\beta)K_\beta(\alpha)\tilde{\mu}(\alpha)\tilde{\mu}(\beta), \qquad (2)$$

where $\tilde{\mu}(\alpha) := \mu(\mathbb{S}_\alpha)$, $\tilde{g}(\alpha) = \frac{1}{\tilde{\mu}(\alpha)}\int_{\mathbb{S}(\alpha)} g\,d\mu$, and $K_\beta(\alpha) = \sup_{z\in\mathbb{S}(\alpha), w\in\mathbb{S}_\beta} \Re K_w(z)$.

Capacitory Condition via Potential Theory

- For $\alpha,\beta \in T^2$ one has at best $d_{T^2}(\alpha\wedge\beta) \leq K_\beta(\alpha)$ (and not the other way around!). However, if we pass from pointwise estimates to average, we get the reverse inequality as well, in particular (2) is equivalent to

$$\|\tilde{g}\|^2_{L^2(T^2,\tilde{\mu})} \gtrsim \Sigma_{\alpha\in T^2}\Sigma_{\beta\in T^2}\tilde{g}(\alpha)\tilde{g}(\beta)d_{T^2}(\alpha\wedge\beta)\tilde{\mu}(\alpha)\tilde{\mu}(\beta)$$

- A dual version of the inequality above is

$$\Sigma_{\alpha\in T^2}(\mathbb{I}h)^2(\alpha)\tilde{\mu}(\alpha) \lesssim \|h\|^2_{l^2(T^2)} \qquad (3)$$

Strong Capacitory Inequality

Theorem

For any $f : T^2 \to \mathbb{R}_+$ one has

$$\Sigma_{k\in\mathbb{Z}} 2^{2k} \operatorname{Cap}\{\mathbb{I}f \geq 2^k\} \lesssim \|f\|_{l^2(T^2)}. \qquad (4)$$

- Given a set $E \subset T^2$ define its successor set to be $E_s = \{\beta \in T^2 : \beta \leq E\}$. Assume that $\tilde{\mu}$ satisfies

$$\tilde{\mu}E_s \lesssim \operatorname{Cap} E_s \qquad (5)$$

for any finite $E \subset T^2$.
- Then (3) follows from the strong capacitory inequality (4).
- On the other hand, if we have (3), then (5) follows from testing on equilibrium functions.
- Finally, (5) turns into a Stegenga-type condition when lifted back to the bidisc.

SCI: sketch of the proof

- The usual arguments have to be heavily modified, since Maximum Principle is unavailable.
- First we estimate the capacity of the excess sets: given $E \subset T^2$ and $\lambda \geq 1$ we put $E_\lambda = \{\mathbb{V}^{\mu_E} \geq \lambda\}$, where μ_E is equilibrium for E. We have

$$\operatorname{Cap} E_\lambda \lesssim \lambda^{-3} \operatorname{Cap} E$$

(which is an improvement over the trivial estimate with λ^{-2} on the right-hand side).
- This, in turn, implies the mixed energy estimate

$$\int \mathbb{V}^{\mu_E}\,d\mu_F \lesssim (\operatorname{Cap} F)^{\frac{2}{3}}(\operatorname{Cap} E)^{\frac{1}{3}},$$

where $F \subset E \subset T^2$, and μ_F, μ_E are the respective equilibrium measures.
- From now on we follow the standard arguments (level sets energy estimates).