It follows immediately that $F$ has $L^p$-boundary values with $|F| = h$ on $\partial\mathbb{D}$, therefore we actually have $\|F\|_{H^p} = \|h\|_{L^p}$. ∎

An analytic function $F$ on the unit disc is called *an outer* function, if there exists a non-negative function $h$ on $\partial\mathbb{D}$ such that $|\log h| \in L^1$ and

$$F(z) := e^{\left(\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \log h(\zeta)\, dm(\zeta)\right)}.$$

## 2. Entire functions

**2.1. Some basic facts.** We start with some definitions. A function $f : \mathbb{C} \to \mathbb{C}$ is called entire, if it is holomorphic in the whole complex plane. A typical example would be a polynomial $\sum_{n=0}^{N} a_n z^n$, or an exponential function $e^z$. Given an entire function $f$ define

$$M_f(r) := \max_{|z|=r} |f(z)|.$$

A polynomial of degree $N$ has exactly $N$ zeros and grows like $|z|^N$. For a general entire function the situation is much more complicated. The main goal here is to establish some basic facts about connection between zeros of an entire function and its growth.

**Proposition 2.1** *Assume that $f$ is entire and*

$$\liminf_{r\to\infty} \frac{M_f(r)}{r^p} = 0$$

*for some non-negative $p$. Then $f$ is a polynomial of degree no larger than $p$.*

**Proof.** The statement follows immediately from the Cauchy inequality

$$|c_n| \leq \frac{M_f(r)}{r^n}, \quad n \geq 0,$$

where $f(z) = \sum_{n=0}^{\infty} c_n z^n$. ∎ The *order* $\rho_f$ of the entire function $f$ is the infimum of such $\lambda > 0$ that

$$M_f(r) \lesssim e^{r^\lambda}.$$

In other words, for every positive $\varepsilon$ one has $M_f(r) \lesssim e^{r^{\rho_f+\varepsilon}}$ and there is a sequence $\{r_k\} \to \infty$ such that $M_f(r) \gtrsim e^{r^{\rho_f-\varepsilon}}$, or, finally,

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}.$$

We complement this characteristic by measuring the growth in a finer way. Namely, let $\sigma_f$ be the infimum of such $A > 0$ that

$$M_f(r) \lesssim e^{Ar^{\rho_f}},$$

we call it the *type* of $f$. As before,

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}}.$$

If $\sigma_f = \infty$, then the function $f$ is of *maximal type*, for $\sigma_f = 0$ it is of *minimal type*, otherwise it is of *normal type*. Entire functions of order $\rho_f = 1$ and normal type $\sigma_f$ are called *entire functions of exponential type $\sigma$*.

**Lemma 2** *Let $f(z) = \sum_{n=0}^{\infty} z^n$ be an entire function. If*

$$M_r(f) \lesssim e^{Ar^p}$$

*for some $A, p > 0$, then*

$$|c_n| \lesssim \left(\frac{eAp}{n}\right)^{\frac{n}{p}}.$$

*On the other hand, if the coefficients satisfy the asymptotic inequality above, then one has*

$$M_f(r) \lesssim e^{(A+\varepsilon)r^p}$$

*for every positive $\varepsilon$.*

**Proof.** *Part 1.* By Cauchy inequality and our assumption on the growth we have

$$|c_n| \lesssim e^{Ar^p - n\log r}.$$

The right-hand side is minimal for $r_n^p = \frac{n}{Ap}$. Plugging this into the estimate for the coefficients we obtain the desired inequality.

*Part 2.* We may assume that $c_0 = 0$, and the coefficient estimate holds for all $n$ (we can always substract a polynomial). In that case we have for $|z| = r$

$$|f(z)| \le \sum_{n=1}^{\infty} \left(\frac{eAp}{n}\right)^{\frac{n}{p}} r^n = \sum_{n=1}^{\infty} \left(\frac{eAr^p}{\frac{n}{p}}\right)^{\frac{n}{p}} \le \sum_{m=1}^{\infty} \left(\frac{eAr^p}{m}\right)^{m+1}$$

after a suitable change of variable. By Stirling's formula

$$|f(z)| \le C \sum_{m=1}^{\infty} \frac{\left(A + \frac{\varepsilon}{2}\right)^{m+1} r^{p(m+1)}}{m!} \le Ce^{(A+\varepsilon)r^p},$$

with $C = C(\varepsilon)$. ■

**Corollary 2.1** *The order and type of an entire function can be computed as follows*

$$\rho_f = \limsup_{n\to\infty} \frac{n\log n}{\log\left(\frac{1}{|c_n|}\right)},$$

$$\sigma_f = \frac{1}{\rho_f e} \limsup_{n\to\infty} \left(n|c_n|^{\frac{p}{n}}\right).$$

2.2. **Jensen formula.** Assume $f$ is an entire function. Fix some $R > 0$. By rescaling in the Poisson representation formula on the unit disc we obtain

$$\Re f(z) = \frac{1}{2\pi}\int_0^{2\pi} \Re f(Re^{i\theta}) \frac{R^2 - r^2}{R^2 - 2Rr\cos(t-\theta) + r^2}\, d\theta$$

for $z = re^{it}$ with $0 \le r < R$. Since

$$\frac{R^2 - r^2}{R^2 - 2Rr\cos(t-\theta) + r^2} = \Re \frac{\zeta + z}{\zeta - z}$$

with $\zeta = Re^{i\theta}$, one has

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \Re f(Re^{i\theta}) \frac{\zeta + z}{\zeta - z}\, d\theta + i\Im f(0).$$

Now let $\{z_n\}$ be the zeros of $f$, counted with respect to their multiplicity, and ordered according to increasing modulus. Fix some $R > 0$ such that $|z_n| \ne R, n \in \mathbb{N}$, and consider

$$\varphi(z) := f(z) \cdot \prod_{n=1}^{N} \frac{R^2 - \overline{z}_n z}{R(z - z_n)},$$

where $N = N(R)$ is the amount of zeros inside $\{|z| < R\}$. The second term in the product is just the inverse of rescaled (to the disc of radius $R$) Blaschke product for the unit disc, generated by first $N$ zeros of $f$. Clearly, $\varphi$ does not vanish in this disc, and

$$\log\varphi(z) = \frac{1}{2\pi}\int_0^{2\pi} \Re(\log\varphi(Re^{i\theta})) \frac{Re^{i\theta} + z}{Re^{i\theta} - z}\, d\theta + iC,$$

where $C = \Im \log\varphi(0)$. Hence for $z \notin \{z_n\}$ one has

$$\log f(z) = \frac{1}{2\pi}\int_0^{2\pi} \log|f|(Re^{i\theta}) \frac{Re^{i\theta} + z}{Re^{i\theta} - z}\, d\theta + \sum_{n=1}^{N} \log \frac{R(z - z_n)}{R^2 - \overline{z}_n z} + iC,$$

so we obtain

$$\log|f|(z) = \frac{1}{2\pi}\int_0^{2\pi} \log|f|(Re^{i\theta}) \frac{Re^{i\theta} + z}{Re^{i\theta} - z}\, d\theta + \sum_{n=1}^{N} \log\left|\frac{R(z - z_n)}{R^2 - \overline{z}_n z}\right|,$$

or the *Poisson-Jensen* formula.

Assume for a moment that $f(0) \neq 0$. Then, plugging $z = 0$ into the formula above we get

$$\log|f|(0) = \frac{1}{2\pi}\int_0^{2\pi} \log|f|(Re^{i\theta})\,d\theta + \sum_{n=1}^{N} \log\frac{|z_n|}{R}.$$

Define $N_r$ to be the counting function of zeros of $f$, i.e. $N_r = \sharp(\{z_n\}_1^\infty \cap \{|z| \le r\})$. Then

$$-\sum_{n=1}^{N} \log\frac{|z_n|}{R} = \int_0^R \log\frac{R}{r}\,dN_r = \int_0^R \frac{N_r}{r}\,dr$$

after integrating by parts. Therefore

$$\int_0^R \frac{N_r}{r}\,dr = \frac{1}{2\pi}\int_0^{2\pi} \log|f|(Re^{i\theta})\,d\theta - \log|f|(0), \tag{14}$$

which is *Jensen* formula. In particular, if $f(0) = 1$, then

$$\int_0^{eR} \frac{N_r}{r}\,dr \le \frac{1}{2\pi}\int_0^{2\pi} \log M_f(eR)\,d\theta = \log M_f(eR).$$

On the other hand, $\int_0^{eR} \frac{N_r}{r}\,dr \ge N_R$, therefore we have

$$N_r \le \log M_f(er).$$

2.3. **Factorization of entire functions.** Assume that a sequence $\{z_n\}$ of complex numbers not containing 0 satisfies

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{p+1}} < +\infty$$

for some $p \in \mathbb{Z}_+$. We define the *Weierstrass primary factor* $G(w,p)$ as follows

$$G(w,p) := 1 - w, \quad p = 0,\ w \in \mathbb{C};$$

$$G(w,p) := (1-w)e^{\sum_{k=1}^{p} \frac{w^k}{k}}, \quad p > 0,\ w \in \mathbb{C}.$$

**Proposition 2.2** *The infinite product*

$$\Pi(z) := \prod_{n=1}^{\infty} G\left(\frac{z}{z_n}, p\right), \quad z \in \mathbb{C},$$

*converges uniformly on compact sets to a non-zero entire function.*

**Proof.** If $|w| \le \frac{1}{2}$, then, clearly,

$$|\log G(w,p)| \le \sum_{k=p+1}^{\infty} \frac{|w|^k}{k} \le 2|w|^{p+1}.$$

It follows immediately that $\prod_{n=1}^{N} G\left(\frac{z}{z_n}, p\right)$ converges absolutely in closed disc of radius $R > 0$ for every $R$.
■

**Theorem 2.1** (Hadamard) *Assume that $f$ is an entire function of order $\rho = \rho_f$, and $\{z_n\}$ is the sequence of its non-zero roots. Then there exist integer numbers $p, q \le \rho$, and a polynomial $P_q$ of degree $q$ such that*

$$f(z) = z^m e^{P_q(z)} \prod_{n=1}^{\infty} G\left(\frac{z}{z_n}, p\right), \quad z \in \mathbb{C}, \tag{15}$$

*where $m$ is the multiplicity of the root of $f$ at the origin.*

**Proof.** By removing the zero at 0 if needed we may assume that $f(0) \neq 0$, i.e. $m = 0$. Recall that

$$\log f(z) = \frac{1}{2\pi} \int_0^{2\pi} \log |f|(Re^{i\theta}) \frac{Re^{i\theta} + z}{Re^{i\theta} - z} \, d\theta + \sum_{|z_n|<R} \log \frac{R(z - z_n)}{R^2 - \overline{z}_n z} + iC$$

for any $R$ such that the respective circle does not hit the zero set. After differentiating this formula $p + 1 := [\rho] + 1$ times we obtain

$$\log f(z)^{(p+1)} = \frac{(p+1)!}{2\pi} \int_0^{2\pi} \log |f|(Re^{i\theta}) \frac{2Re^{i\theta}}{(Re^{i\theta} - z)^{p+2}} \, d\theta + \sum_{|z_n|<R} \left( \frac{p! \overline{z}_n^{p+1}}{(R^2 - \overline{z}_n z)^{p+1}} - \frac{p!}{(z_n - z)^{p+1}} \right).$$

Taking $r := |z|$ we see that

$$\left| \log f(z)^{(p+1)} + \sum_{|z_n|<R} \frac{p!}{(z_n - z)^{p+1}} \right| \leq C_p \left( \log M_f(R) \frac{R}{(R-r)^{p+2}} + \frac{N_R}{(R-r)^{p+1}} \right).$$

By previous estimates it follows that the right-hand side tends to zero, as $R$ goes to infinity. In particular,

$$\log f(z)^{(p+1)} = - \sum_{n=1}^{\infty} \frac{p!}{(z_n - z)^{p+1}},$$

and this series converges absolutely for $z \notin \{z_n\}$. Now we integrate back $p$ times (along some curve from 0 to $z$, taking care not to intersect any cuts that were made to define the logarithms), arriving at

$$\log f(z) = \sum_{n=1}^{\infty} \left( \log \left( 1 - \frac{z}{z_n} \right) + \sum_{k=1}^{p} \frac{z^k}{k z_n^k} \right) + P_q(z)$$

for some $q \leq p$. ■ Now consider a sequence $\{z_n\} \subset \mathbb{C}$ of non-zero complex numbers, such that

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^\lambda} < \infty$$

for some $\lambda > 0$. The smallest such $\lambda$ is called the *convergence exponent* of the sequence $\{z_n\}$. Let $N_r$ be the counting function of $\{z_n\}$ and let

$$\rho_1 := \limsup_{r \to \infty} \frac{\log N_r}{\log r}.$$

**Lemma 3** *Let the series* $\sum_{n=1}^{\infty} \frac{1}{|z_n|^\lambda}$ *converge for some positive* $\lambda$. *Then the integral* $\int_0^\infty \frac{N_r}{r^{\lambda+1}} \, dr$ *converges, and*

$$\lim_{r \to \infty} \frac{N_r}{r^\lambda} = 0.$$

*Also the convergence of* $\{z_n\}$ *coincides with the order of its counting function.*

**Proof.** Clearly

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^\lambda} = \int_0^\infty \frac{dN_r}{r^\lambda}.$$

Integrating by parts we obtain

$$\int_0^R \frac{dN_r}{r^\lambda} = \frac{N_R}{R^\lambda} + \lambda \int_0^\infty \frac{N_r}{r^{\lambda+1}} \, dr.$$

The second statement follows from the estimate $\frac{N_r}{r^{\lambda+1}} \leq \lambda \int_r^\infty \frac{N_r}{r^{\lambda+1}}$.
Let $\lambda_0$ be the convergence exponent of the sequence. The inequality $\rho_1 \leq \lambda_0$ follows immediately from the previous statement. On the other hand, $N_r \lesssim r^{\rho_1 + \varepsilon}$ for any $\varepsilon > 0$. It follows that the integral $\int_0^\infty \frac{N_r}{r^{\lambda+1}} \, dr$ converges for $\lambda = \rho_1 + \varepsilon$, and also $\frac{N_r}{r^\lambda}$ tends to zero. We are done. ■

Since the representation in Hadamard's theorem is not, strictly speaking, unique, we assume that $p$ is the smallest integer s.t.

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{p+1}} < +\infty .$$

We call $g = \max(p,q)$ – the <u>genus</u> of $f$.

Clearly, Hadamard's representation holds for such a number $p$.

<u>Example</u>

$f(z) = \dfrac{\sin \pi \sqrt{z}}{\pi \sqrt{z}}$ is entire (Show it!), and of order $\rho = \frac{1}{2}$, with zeros at $\{n^2\}_{n=1}^{\infty}$.

By Hadamard

$$f(z) = C \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2}\right) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2}\right),$$

Since $f(0) = 1$.

It follows $(z \to z^2)$ that

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Let $\{z_n\}$ be a sequence of complex numbers, $N(r) = N_r$ its counting function, $p \geq 0$ s.t.

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{p+1}} < +\infty .$$

Denote

$$\Pi(z) = \prod_{n=1}^{\infty} G\left(\frac{z}{z_n}, p\right).$$

to be the canonical product generated by $\{z_n\}$. We want to estimate its growth in terms of the counting function $N_r$. We start with the following lemma.

Lemma (Borel).

Let $z \in \mathbb{C}$. Then one has

$$(i)\ \log|G(z,0)| \leq \log(1+|z|)$$

$$(ii)\ \log|G(z,p)| \leq C_p \frac{|z|^{p+1}}{1+|z|}, \quad p > 0$$

with $C_p \approx \log p$.

Proof.

(i) is obvious. Let $p > 0$. If $|z| < p/p+1$, then

$$\log|G(z,p)| \leq \sum_{n=p+1}^{\infty} \frac{|z|^n}{n} \leq |z|^{p+1}.$$

If $|z| \geq \frac{p}{p+1}$, then $\log(1+|u|) < |u|$,

and

$$\log|G(z,p)| \leq 2|z| + \frac{|z|^2}{2} + \dots + \frac{|z|^p}{p} =$$

$$= |z|^p \left( \frac{1}{p} + \frac{1}{(p-1)|z|} + \dots + 2\frac{1}{|z|^{p-1}} \right) \leq$$

$$\leq |z|^p \left( \frac{p+1}{p} \right)^{p-1} \left( 2 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{p} \right) \lesssim$$

$$\lesssim C_p \frac{|z|^{p+1}}{1+|z|} \qquad \boxed{3}$$

Theorem.

$$\Pi(z) = \prod_{n=1}^{\infty} G\left(\frac{z}{z_n}, p\right)$$

converges uniformly on compact sets, and

$$\log|\Pi(z)| \leq K_p r^p \left( \int_0^r \frac{N_t}{t^{p+1}} dt + r \int_r^\infty \frac{N_t}{t^{p+2}} dt \right),$$

where $K_p \approx p \log p$, $|z| = r$.

Proof.

For $p=0$ one has

$$\log|\Pi(z)| \leq \sum_{n=1}^{\infty} \log\left(1+\frac{r}{|z_n|}\right) =$$

$$= \int_0^\infty \log\left(1+\frac{r}{t}\right) dN_t \leq \int_0^r \frac{N_t}{t}\,dt + r\int_r^\infty \frac{N_t}{t^2}\,dt.$$

If $p \geq 1$, then by previous lemma we see that

$$\log|\Pi(z)| \lesssim C_p \sum_{n=1}^{\infty} \frac{r^{p+1}}{|z_n|^p(r+|z_n|)} = C_p r^{p+1}\int_0^\infty \frac{dN_t}{t^p(t+r)} =$$

$$= C_p r^{p+1} \frac{N_t}{t^p(t+r)}\Big|_0^\infty + C_p r^{p+1}\int_0^\infty \left(\frac{p}{t^{p+1}(t+r)} + \frac{1}{t^p(t+r)^2}\right) N_t\,dt \overset{\copyright}{}$$

$$\overset{\copyright}{\leq} p\,C_p\, r^p\left(\int_0^r \frac{N_t\,dt}{t^{p+1}} + r\int_r^\infty \frac{N_t}{t^{p+2}}\,dt\right).$$

$\boxtimes$

Theorem (Borel)

The order of a canonical product is the convergence exponent of its zeros.

Proof. Let $p$ be the smallest integer s.t.

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{p+1}} < +\infty .$$

Then

$$p \leq \rho_1 = \inf\{\lambda : \sum_{n=1}^{\infty} \frac{1}{|z_n|^{\lambda+1}} < +\infty\} \leq p+1 .$$

- Assume $\rho_1 < p+1$. Fix $\varepsilon > 0$ s.t. $\rho_1 + \varepsilon < p+1$. Then

$$N_r \overset{as}{<} r^{\rho_1+\varepsilon},$$

and by Theorem above we have

$$\log M_\Pi(r) \lesssim K_p r^p \left(1 + \frac{r^{\rho_1+\varepsilon-p}}{\rho_1+\varepsilon-p} + \frac{r^{\rho_1+\varepsilon-p}}{p+1-\rho_1-\varepsilon}\right) \lesssim$$

$$\lesssim r^{\rho_1+2\varepsilon}.$$

- Now let $\rho_1 = p+1$. Since

$$\frac{N_r}{r^{p+1}}, \int_R^{\infty} \frac{N_r}{r^{p+2}}\, dr \longrightarrow 0,$$ it follows that

$$\log M_\Pi(r) \lesssim \varepsilon r^{p+1} = \varepsilon r^{\rho_1} \quad \forall \varepsilon > 0.$$

We see that

$$\rho_n \leq \rho_1 .$$

The converse follows immediately from Jensen's formula. $\boxtimes$

---

# Phragmén-Lindelöf theorems.

**Theorem**

Assume $f$ is analytic inside the angle $A_\lambda = \{re^{i\theta} : r \geq 0, |\theta| < \pi/2\lambda\}$, $\lambda \geq 1$, $|f|$ is bounded by some constant $M$ on the sides of this angle, and

$$\log M_f(r) \lesssim r^p$$

for some $p < \lambda$. Then $|f(z)| < M$ in $A$.

**Proof.** Consider $\varepsilon > 0$ s.t. $p+\varepsilon < \lambda$, and some $\delta > 0$. Then

$$\varphi_\delta(z) := f(z) e^{-\delta z^{p+\varepsilon}}$$

is analytic in $A$, and

$$|\varphi_\delta(z)| \lesssim e^{|z|^p - \delta |z|^{p+\varepsilon} \cos(p+\varepsilon)\pi/2\lambda}, \quad z \in A.$$

It follows immediately that

$$|\varphi_\delta(Re^{i\theta})| \le M, \quad |\theta| < \alpha$$

for any large $R$, hence by Maximum Principle

$|\varphi_\delta| \le M$ in the whole angle $A$, which is

$$|f(z)| \le M e^{\delta |z|^{\rho+\varepsilon}}, \quad z \in A.$$

Since $\delta > 0$ is chosen arbitrarily, we have $|f| \le M$.

$\square$

## Theorem

Assume $f$ is analytic inside

$$A = \{re^{i\theta} : r \ge 0,\ |\theta| < \frac{\pi}{2\lambda}\}, \quad \lambda \ge 1$$

and

$$\log M_f(r) \le (\sigma + \varepsilon) r^\lambda$$

for any $\varepsilon > 0$, and also $|f|$ is bounded by $M$ on the sides of $A$. Then one has

$$|f(re^{i\theta})| \le M e^{\sigma r^\lambda \cos \lambda\theta}, \quad re^{i\theta} \in A.$$

Proof. Consider

$$\varphi_\varepsilon(z) := f(z)\, e^{-(\sigma+\varepsilon) z^\lambda}$$

It is analytic inside $A$, it is bounded on $\mathbb{R}_+$, and on the sides of $A$. By previous Theorem one has

$$|\varphi_\varepsilon| \leq \text{Const} \quad \text{in each half-angle } A^\pm,$$

where

$$A^\pm = \left\{ re^{i\theta} : r \geq 0, \begin{array}{l} 0 < \theta < \frac{\pi}{2\lambda} \\ -\frac{\pi}{2\lambda} < \theta < 0 \end{array} \right\}.$$

We apply that theorem once more to see that this constant is actually $M$. The statement follows immediately $\boxed{\ }$

Corollary.

If $f(z) = f(x+iy), y > 0$, is analytic in the upper half-plane $\mathbb{R}^2_+$, ~~...~~ s.t.

$$M_f(r) \lesssim e^{(\sigma+\varepsilon) r}$$

for any $\varepsilon > 0$, and $|f(x)| \leq M$ on $\mathbb{R}$, then

$$|f(x+iy)| \leq M e^{\sigma y}, \quad x \in \mathbb{R}, y > 0.$$

Proof.

Take $\lambda = 1$ in previous Theorem, and apply it to $f(-iz)$. $\boxed{\ }$