Spectral Theory.

Example. Dirichlet problem in $\mathbb{C}_+$:

$$\begin{cases} (\partial_{xx} + \partial_{yy})\, v(x+yi) = 0 & \text{in } \mathbb{C}_+ = \{x+iy : y>0\} \\ v(x+i\cdot 0) = f(x) & \text{on } \mathbb{R} \\ v \text{ "not too large at } y = +\infty\text{"} \end{cases}$$

Formal solution.

Observe that $-\partial_{xx} = A \geq 0$ in the sense that for $f \in C_c^2(\mathbb{R}, \mathbb{C})$ with compact support we have

$$(Af, f)_{L^2} = \int_{-\infty}^{+\infty} -\partial_{xx} f \cdot \bar{f}\, dy =$$

$$= \int_{-\infty}^{+\infty} \partial_x f \cdot \partial_x \bar{f}\, dy \geq 0$$

If we could treat A as a real number (or as a nonnegative function):

$$v(x,y) = \left(e^{-y\sqrt{A}} f\right)(x)$$

Then $v(x,0) = f(x)$

$$\partial_y \partial_y v(x,y) = \partial_y\left(-\sqrt{A}\, e^{-y\sqrt{A}}\right) f = A e^{-y\sqrt{A}} f$$
$$= -\partial_{xx} v(x,y),$$

i.e. $\partial_{xx} v + \partial_{yy} v = 0$ for $y > 0$.

Obs. We can make sense of this using Fourier transforms:

$$(Af)^\wedge(t) = \int_{-\infty}^{+\infty} Af(x)\, e^{-2\pi i x t}\, dx$$

$$= -(-2\pi i t)^2 \hat{f}(t) = 4\pi^2 t^2 \hat{f}(t)$$

and this suggests to consider

$$\hat{v}(t,y) = e^{-2\pi|t|y} \hat{f}(t) \quad \text{so that}$$

$$\partial_{yy} \hat{v}(t,y) = 4\pi^2 t^2 \hat{f}(t)\, e^{-2\pi|t|y}$$

$$= 4\pi^2 t^2 \hat{v}(t,y)$$

$$= (Av)^\wedge(t,y)$$

F.T. w.r.t. the 1st variable

and, after posing

$$v(x,y) = \int_{-\infty}^{+\infty} \hat{v}(t,y)\, e^{2\pi i t x}\, dt \quad \text{(Fourier inversion)}$$

one has $\partial_{yy} v(x,y) = Av(x,y) = -\partial_{xx} v(x,y)$.

Also: $$\int_{-\infty}^{+\infty} |v(x,y)|^2 dx = \int_{-\infty}^{+\infty} |\hat{v}(t,y)|^2 dt =$$

$$= \int_{-\infty}^{+\infty} e^{-4\pi|t|y} |\hat{f}(t)|^2 dt \xrightarrow[y \to +\infty]{} 0$$

by M.C.T. if (e.g.) $\hat{f}$ (hence f) belongs to $L^2(\mathbb{R})$.

After we have formally solved the Dirichlet problem, we might want a more direct formula:

$$v(x,y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x-s)\,\frac{s}{s^2+y^2}\,ds,$$ a formula we have already met.

Check it (exercise)

Hint. (i) $(f * g)^\wedge = \hat{f}\cdot\hat{g}$

(ii) $$\int_{-\infty}^{+\infty} e^{-2\pi|t|y}\, e^{2\pi i t x}\,dt =$$

$$= \left[\frac{e^{2\pi t(ix-y)}}{2\pi(ix-y)}\right]_0^{+\infty} + \left[\frac{e^{2\pi t(ix+y)}}{2\pi(ix+y)}\right]_{-\infty}^{0}$$

$$= \frac{1}{2\pi}\left(\frac{1}{ix+y} - \frac{1}{ix-y}\right) = \frac{2y}{2\pi(x^2+y^2)}.$$

The advantage of the formal solution is that it works whenever $(Af,f)\geq 0$ $\forall f\in H$: a suitable Hilbert space.

e.g. $A=-\Delta_M$ when Δ_M is the Laplace-Beltrami operator on a Riemannian manifold, etc.

Exercise. Formally solve the Dirichlet problem for the wave equation

$$\begin{cases} \partial_{yy} v(x,y) = \partial_{xx} v(x,y) & \text{for } x\in\mathbb{R},\ y>0 \\ v(x,0) = \varphi(x) & \varphi\in L^2(\mathbb{R}) \\ \partial_y v(x,0) = \psi(x) & \psi\in L^2(\mathbb{R}) \end{cases}$$

Observe that the solution works for $x\in M$, a Riemannian manifold.

Above we "computed":

$\sqrt{-\Delta}$ $\quad (\Delta = \partial_{xx})$

e^{-yA} $\quad (A=\sqrt{-\Delta})$

$\cos(yA)$, $\sin(yA)$ (in the exercise).

How do we make sense of them?

General idea: 1^{st}: compute a formal solution to a problem;

2^{nd}: make it less formal;

3^{rd}: study its properties.

Spectral Theory of Normal and self-adjoint matrices.

$A \in M_n(\mathbb{C})$ is self-adjoint if $A = A^*$

it is unitary if $A^*A = I$

it is normal if $A^*A = A^*A$

unitary $\Longrightarrow$ normal

self-adjoint $\Longrightarrow$

Here $A^* \in M_n(\mathbb{C})$: $(A^*u, v) := (u, Av)$, $\forall u, v \in \mathbb{C}^n$.

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix} = D(\lambda_1, \dots \lambda_n) \text{ is}$$

$\lambda_1 \dots \lambda_n \in \mathbb{C}$

diagonal: $D(\lambda_1, \dots, \lambda_n)$ is normal

and it is self-adj $\Leftrightarrow \lambda_1, \dots \lambda_n \in \mathbb{R}$

it is unitary $\Leftrightarrow |\lambda_1| = \dots = |\lambda_n| = 1$

Prop. $A \in M_n(\mathbb{C})$ is unitary

$\Leftrightarrow (Au, Av) = (u, v) \ \forall u, v \in \mathbb{C}^n$

$\Leftrightarrow |Au|^2 = |u|^2 \quad \forall u \in \mathbb{C}^n$

Pf. Unitary $\Leftrightarrow A^*A = I \Leftrightarrow$

$\forall u, v \in \mathbb{C}^n$: $(u, v) = (A^*Au, v) = (Au, Av)$

$\Rightarrow |u|^2 = |Au|^2 \quad \forall u \in \mathbb{C}^n$

$\Rightarrow \forall u, v \in \mathbb{C}^n$: $8(u, v) \overset{*}{=}$ Polarization Identity Exercise

$= |u+v|^2 - |u-v|^2 + i|u+iv|^2 - i|u-iv|^2$

$= |Au+Av|^2 - |Au-Av|^2 + i|Au+iAv|^2 - i|Au-iAv|^2$

$= 8(Au, Av)$, by the same polarization.

The characteristic polynomial of $A \in M_n(\mathbb{C})$ is $p(z) = \det(zI - A)$:

$\deg(p) = n$, hence, p has n solutions $z = \lambda_1, \dots, \lambda_n$ (counting multiplicities).

Obs. $z = \lambda_j$ $(\exists j) \Leftrightarrow zI - A$ is not invertible $\Leftrightarrow \exists v \in \mathbb{C}^n, v \neq 0$:

$Av = zv$ (z is an eigenvalue of A and v is one of the corresponding eigenvectors).

Theorem (I) $A \in M_n(\mathbb{C})$ is normal

$\Leftrightarrow \exists U$ unitary s.t. $A = U^* \cancel{\text{Diag}}\, D U$ for a diagonal matrix.

Moreover, if $D = D(\lambda_1 \dots \lambda_n)$

then $\lambda_1 \dots \lambda_n$ are the eigenvalues of A; and $\dim\{v \in \mathbb{C}^n : Av = \lambda_j v\} = H(\lambda_j)$ the dimension of the eigenspace relative to λ_j, is the multiplicity of λ_j as root of $p(z) = 0$.

Moreover, $H(\lambda_j) \perp H(\lambda_i)$ if $\lambda_j \neq \lambda_i$.

(II) If A is self-adjoint, the above holds and, moreover, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Viceversa, if A is normal and $\lambda_1 \dots \lambda_n \in \mathbb{R}$, then A is self-adjoint.

Sketch of the proof. (I) $A = U^* D U \Rightarrow A^* A = U^* \bar{D} U U^* D U = U^* |D|^2 U = AA^*$

Lemma. $A \in M_n(\mathbb{C}) \Rightarrow \exists U : U^* U = I$ and $\exists T$ upper triangular:

$$T = \begin{bmatrix} t_1 & & \dots \\ & t_2 & \\ 0 & & \ddots \end{bmatrix} \quad \text{s.t.} \quad A = U^* T U$$

Given the Lemma: A is normal $\Rightarrow$

$$\left.\begin{array}{l} A^*A = U^* T^* T U \\ \shortparallel \\ AA^* = U^* T T^* U \end{array}\right\} \Rightarrow T^* T = T T^* \Rightarrow T \text{ is diagonal}$$

In fact $T^* T = \begin{bmatrix} \bar{t}_1^t \,\Big|\, \begin{matrix} 0 \\ \bar{t}_2^t \end{matrix} \,\Big|\, \begin{matrix} 0 \\ 0 \\ \bar{t}_3^t \end{matrix} \,\Big|\, \dots \end{bmatrix} \begin{bmatrix} t_1 \\ \hline 0 \,|\, t_2 \\ \hline 0 \;\, 0 \,|\, t_3 \\ \hline \dots \end{bmatrix}$

$$= \begin{bmatrix} |t_{11}|^2 & & * \\ & |t_{12}|^2 + |t_{22}|^2 & \\ & & |t_{13}|^2 + |t_{23}|^2 + |t_{33}|^2 & \\ * & & & \ddots \end{bmatrix}$$

while $TT^* = \begin{bmatrix} |t_1|^2 & & * \\ & |t_2|^2 & \\ * & & |t_3|^2 \\ & & & \ddots \end{bmatrix}$

Comparing elements on the diagonal:

$t_{12} = \dots = t_{1n} = 0$ i.e. T is diagonal.

$t_{23} = \dots = t_{2n} = 0$

$\vdots$

Pf. of the Lemma. Let $v \in \mathbb{C}^n$, $|v| = 1$, s.t. $Av = \lambda v$ $(\exists \lambda)$ and find $U = (v | v_2 | \dots | v_n)$ unitary, so that $(v, v_j) = 0$.

Then $U^* A U = \begin{pmatrix} \bar{v}^t \\ \bar{v}_2^t \\ \vdots \end{pmatrix} A (v | v_2 | \dots) =$

$= \begin{pmatrix} \bar{v}^t \\ \bar{v}_2^t \\ \vdots \end{pmatrix} (Av | Av_2 | \dots) = \begin{pmatrix} \bar{v}^t \\ \bar{v}_2^t \\ \vdots \end{pmatrix} (\lambda v | Av_2 | \dots) =$

$= \begin{pmatrix} \lambda & * & * & \dots \\ 0 & & * & \\ 0 & & & \\ \vdots & & & \end{pmatrix} = \left(\begin{array}{c|c} \lambda & a \\ \hline 0 & B \end{array}\right) =$ by induction $= \left(\begin{array}{c|c} \lambda & a \\ \hline 0 & V^* T V \end{array}\right)$ with T upper tr.

and V unitary

$$= \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & V^* \end{array}\right)\left(\begin{array}{c|c} \lambda & aV^* \\ \hline 0 & T \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & V \end{array}\right)$$

and the Lemma is proved ~~with~~

Since $A = \left[\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & V \end{array}\right) U^*\right]^* \left(\begin{array}{c|c} \lambda & aV^* \\ \hline 0 & T' \end{array}\right)\left[\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & V \end{array}\right) U^*\right]$ ∎

Since $A = U^* D(\lambda_1 \cdots \lambda_n) U$, the properties of eigenvectors / eigenspaces can be proved directly for $D(\lambda_1 \cdots \lambda_n)$. That's easy (exercise). ∎

Let $\lambda_1 = \cdots = \lambda_{m_1} = \mu_1$; $\lambda_{m_1+1} = \cdots = \lambda_{m_2} = \mu_2 ; \ldots ; \mu_\ell$ be the distinct eigenvalues and ; $H_1, \ldots, H_m$ the eigenspaces ;

$\mathbb{C}^n \xrightarrow{\Pi_j} H_j$ the orthogonal projection:

$$\Pi_j = U^* \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & I_{m_j} & 0 \\ \hline 0 & 0 & 0 \end{array}\right) U \quad \text{where}$$

$$I = \left(\begin{array}{c|c|c|c} I_{m_1} & & 0 & \\ \hline 0 & I_{m_2} & & \\ \hline \vdots & & \ddots & \\ \hline 0 & & & I_{m_\ell} \end{array}\right)$$

We can write $A = \sum_{j=1}^{\ell} \lambda_j \Pi_j$.

$\sigma(A) = \{\mu_1, \ldots, \mu_\ell\}$ is the <u>spectrum</u> of A while $A = \sum_{j=1}^{\ell} \lambda_j \Pi_j$ is the <u>spectral decomposition</u> of A

We have $\Pi_i \Pi_j = \begin{cases} \Pi_i & i = j \\ 0 & i \neq j \end{cases}$

If q is a polynomial,
$$q(z) = \sum_{j=0}^{N} c_j z^j$$

Then
$$q(A) = \sum_{j=0}^{N} c_j \left(\sum_{i=1}^{\ell} \lambda_i \Pi_i\right)^j =$$
$$= \sum_{j=0}^{N} c_j \sum_{i=1}^{\ell} \lambda_i^j \Pi_i =$$
$$= \sum_{i=1}^{\ell} \left(\sum_{j=0}^{N} c_j \lambda_i^j\right) \Pi_i = \sum_{i=1}^{\ell} q(\lambda_i) \Pi_i$$

i.e. $q(A)$ has the same spectral decomposition as A, with $q(\lambda_i)$ instead of λ_i :

$$\sigma(q(A)) = q(\sigma(A)) \qquad \text{<u>Spectral Mapping Theorem</u>}$$

Also observe that
$$q_1 = q_2 \text{ on } \sigma(A) \Rightarrow q_1(A) = q_2(A).$$

i.e. operators of the form $q(A)$ might be identified with $\sigma(A) \xrightarrow{q} \mathbb{C}$.

Since $\forall\, \sigma(A) \xrightarrow{\phi} \mathbb{C}$ there is a polynomial q s.t. $q|_{\sigma(A)} = \phi$, the spectral theory of normal metrices is simpler than the general one. (There are non-trivial problems).

e.g. $A = \begin{pmatrix} 1 & 0 \\ 0 & e^{is} \end{pmatrix} = \Pi_1 + e^{is}\Pi_2$

$$e^A = \sum_{n=0}^{\infty} \frac{(\Pi_1 + e^{is}\Pi_2)^n}{n!} = e\,\Pi_1 + e^{e^{is}}\Pi_2 =$$

$= q(A)$ where $q(1) = e$, $q(e^{is}) = e^{e^{is}}$

clearly we can realize it with $q(z) = az + b$ with $a, b \in \mathbb{C}$.

In the infinite dimensional case things are not so straightforward, but most of the theory can be recovered, with useful applications.

§1- Bounded Operators and adjoints.

H Hilbert, $H \xrightarrow{T} H$ (linear) bounded.

Then $H \xrightarrow{T^*} H$, the adjoint of T, is defined as: $(T^*x, y) := (x, Ty)$.

$\|T^*\| = \|T\|$; $(ST)^* = T^*S^*$; $(\alpha T)^* = \bar{\alpha}T^*$;

$(T^{-1})^* = (T^*)^{-1}$; $T^{**} = T$

$\|T^*T\| = \|T\|^2$: $\|T^*T\| \geq \sup_{|x|=1} (x, T^*Tx) = \|T\|^2$

Self-Adj: $T = T^*$

Orthogonal projection: $P^2 = P$, $P = P^*$

Normal $T^*T = TT^*$.

<u>Resolvent</u> $\lambda \in \rho(T) \iff \lambda I - T$ is (a) a bijection, (b) with bounded inverse.

bijection $\Rightarrow$ bndd inv.

$\lambda \in \sigma(T)$ (<u>spectrum</u>) iff $\lambda \notin \rho(T)$

For $\lambda \in \sigma(T)$ there are several possibilities:

(i) $\exists x \in H, x \neq 0$: $Tx = \lambda x$: $\lambda I - T$ is not injective. We say $\lambda \in \sigma_p(T)$: <u>point spectrum</u> and λ is an eigenvalue; x " " " vector.

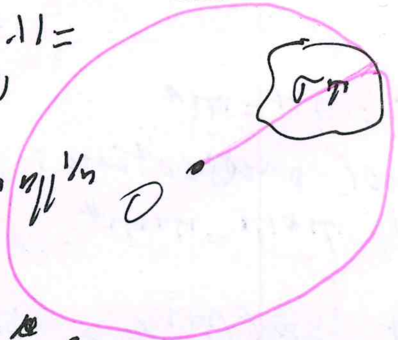
(ii) λ is not an eigenvalue and Range $(\lambda I - T)$ is not dense: residual spectrum

(iii) λ is an eigenvalue and Range $(\lambda I - T) \underset{\neq}{\subseteq} H$ is dense.

Spectral radius.

$r(T) := \sup_{\lambda \in \sigma(T)} |\lambda| =$

Thm. $= \lim_{n \to \infty} \|T^n\|^{1/n}$



If $T = T^*$: $r(T) = \|T\|$

Thm. (Phillips) $\sigma(T^*) = \overline{\sigma(T)}$

$R_{\bar{\lambda}}(T^*) = [R_\lambda(T)]^*$

Theorem: $T = T^* \Rightarrow T$ has empty residual spectrum.

Theorem $\sigma(T) \neq \emptyset$.

Pf. $\frac{1}{\lambda - T} = \frac{1}{\lambda} \frac{1}{1 - T/\lambda} = \frac{1}{\lambda} \sum_0^\infty \left(\frac{T}{\lambda}\right)^n$

If we could do it $\forall \lambda$, then $\lambda \mapsto \frac{1}{\lambda - T}$ would be "analytic" and bounded, hence constant. ■

~~Relationship with invariant subspaces.~~

§2 - Spectral Thm.

$\{\Pi_\Omega\}_{\Omega \text{ Borel in } \mathbb{R}}$: projections on H

is a **projection valued measure** (p.v.m.) if $\Pi_\emptyset = 0$; $\Pi_{(-a,a)} = I \; \exists a \geq 0$

$\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ (disj) $\Rightarrow$

$$\Pi_\Omega = s\text{-}\lim_{n\to\infty} \sum_{j=1}^{n} \Pi_{\Omega_j}$$

$$\Pi_{\Omega_1} \Pi_{\Omega_2} = \Pi_{\Omega_1 \cap \Omega_2}$$

Obs. $x, y \in H$: $\Omega \mapsto (\Pi_\Omega x, y)$ is a (σ-val.) meas.

Supp. $f : \mathbb{R} \to \mathbb{C}$ is bdd

Then $\exists! B$ s.t.

$$(Bx, y) = \int f(\lambda) \, d(\Pi_\lambda x, y)$$

Spectral Thm.

$A = A^*$ bdd $\longleftrightarrow$ $\{\Pi_\Omega\}$ p.v.m. s.t.

$\Pi_\Omega = \chi_\Omega(A)$ $\qquad A = \int \lambda \, d\Pi_\lambda$

Moreover: $f(A) = \int f(\lambda) \, d\Pi_\lambda = \vec{\Phi}(f)$

defines a map $\vec{\Phi}$

$$f \text{ Borel on } \mathbb{R} \xmapsto{\vec{\Phi}} \vec{\Phi}(f) \text{ Bdd on } H$$

s.t.

(a) $\vec{\Phi}$ is a $*$-hom.

(b) $\|\vec{\Phi}(f)\|_{\mathcal{B}(H)} \leq \|f\|_{L^\infty}$

(c) $\vec{\Phi}(x \mapsto x) = A$

(d) $f_n \xrightarrow[n\to\infty]{} f$ pointwise and

$\|f_n\|_{L^\infty} \leq C \Rightarrow \vec{\Phi}(f_n) \xrightarrow[n\to\infty]{} \vec{\Phi}(f)$ in the strong topology

i.e. $\vec{\Phi}(f_n) h \xrightarrow[n\to\infty]{} \vec{\Phi}(f) h$ in H.

More:

(e) $A\psi = \lambda \psi \Rightarrow \vec{\Phi}(f)\psi = f(\lambda)\psi$

(f) $f \geq 0 \Rightarrow \vec{\Phi}(f) \geq 0$; i.e. $\langle \vec{\Phi}(f) h, h \rangle \geq 0 \; \forall h \in H$

(g) $BA = AB \Rightarrow \vec{\Phi}(f) B = B \vec{\Phi}(f)$

Remarks on spectra.

(1) When $\dim(H) = +\infty$, it can happen that $\lambda \in \sigma(A)$, $A = A^*$, but there is no $h \in H\setminus\{0\}$ s.t.

(*) $Ah = \lambda h$

More: it is possible that (*) has no solution $\lambda \in \mathbb{C}$, $h \in H\setminus\{0\}$. This is in sharp contrast with the case $\dim(H) < +\infty$.

Example. $H = L^2[0,1]$ and

$A = M_x : f \longmapsto f \cdot x$

A is selfadjoint:

$$\int_0^1 x\varphi(x)\overline{\psi(x)}\,dx = \int_0^1 \varphi(x)\overline{\psi(x)x}\,dx$$

hence $\underset{\mathbb{R}}{\sigma(A)} \neq \emptyset$. But $A\varphi = \lambda\varphi$

means $x\varphi(x) = \lambda\varphi(x)$ a.e. $x \in [0,1]$ and this implies $\varphi = 0$ a.e. $x \in [0,1]$.

Exercise: show that $\sigma(M_x) = [0,1]$

(2) If $Ah = \lambda h$ has solutions ($A \in \mathcal{B}(H)$, $\lambda \in \mathbb{C}$, $h \in H$, $h \neq 0$) then, if $\mathcal{M} = \operatorname{span}\{h\}$, we have that $\mathcal{M}$ is an invariant subspace of A:

(i) $A(\mathcal{M}) \subseteq \mathcal{M}$

(ii) $0 \neq \mathcal{M} \subsetneq H$

(iii) $\mathcal{M}$ is closed

An old open problem in Operator Theory asks if all $T \in \mathcal{B}(H)$ ($\dim(H) = +\infty$) admit invariant subspaces.

- If $\dim(H) < +\infty$ they do since $\det(\lambda I - T) = 0$ has solutions.
- If $A = A^*$ they do as a consequence of the spectral theorem (Exercise).