Problems to the course

All dB spaces $\mathcal{H}(E)$ are assumed to satisfy $E \neq 0$ on $\mathbb{R}$.

1. Let $E, E_1 \in HB$ such that $\Theta = \Theta_1$, $\Theta = E^\# / E$, $\Theta_1 = E_1^\# / E_1$. Show that there exist entire S which has no zeros, real on $\mathbb{R}$ such that $E_1 = SE$ and that the mapping $f \mapsto Sf$ is a unitary operator from $\mathcal{H}(E)$ onto $\mathcal{H}(E_1)$.

2. Let $\Theta(z) = e^{iaz} B(z)$ be meromorphic inner function in $\mathbb{C}$ (i.e. zeros z_n of B tend to infinity). Show that $\Theta = E^\# / E$ for some entire E.

3. Prove the theorem: if $E = A - iB$, $E_1 = A_1 - iB_1$ then $\mathcal{H}(E) = \mathcal{H}(E_1)$ (as Hilbert spases with the equality of norms) iff $(A_1\ B_1) = (A\ B)M$, where M is constant matrix with real enties and $\det M = 1$.
<u>Hint:</u> if — show that the repr. kernels coincide
only if — write equality of kernels at two different points and express A_1, B_1 via A, B

4. Prove that the mapping $f \mapsto f/E$ is a unitary mapping from $\mathcal{H}(E)$ onto K_Θ, where $\Theta = E^\# / E$, $K_\Theta = H^2 \ominus \Theta H^2$.

5. Let $E = A - iB$ and $E_1 = A_1 - iB_1 \in HB$. Assume that $Z_A = Z_{A_1}$, $Z_B = Z_{B_1}$. Show that $\exists$ S-entire real on $\mathbb{R}$, without zeros such that $f \mapsto Sf$ is unitary from $\mathcal{H}(E)$ onto $\mathcal{H}(E_1)$. Does it imply that $E_1 = SE$?

6. Let $S_\alpha = e^{i\alpha} E - e^{-i\alpha} E^*$. Show that for $\alpha, \beta \in [0, \pi)$, $\alpha \neq \beta$, the zeros of S_α and S_β interlace.

7. Let $\frac{1}{i}(A^* J A - J) \geq 0$ and $\det A = 1$.

a) Show that $\overline{A}^{-1}$ also is J-contractive

b) Deduce from it that $\Im\left(A_{22}/A_{21}\right) \geq 0$, $\Im\left(A_{12}/A_{11}\right) \geq 0$

8. Let A, B be J-contractive, show that AB is J-contr.

9. Show that $f \in \mathcal{H}(E)$ is orthogonal to the domain of multiplication by z (i.e. to the set $\{g \in \mathcal{H}(E) : zg \in \mathcal{H}(E)\}$ iff $f = c\, S_\alpha$ for some α.

0. Let (x_1, x_2) be an indivisible interval for some Hamiltonian. Show that $\mathcal{H}(E_{x_1})$ is a subspace of $\mathcal{H}(E_{x_2})$ of codimension 1.

11. Let $\Theta = E^*/E$, $A \in H^\infty$, $\|A\| \leq 1$. If the parameter p in the representation $\operatorname{Re} \frac{1+\Theta A}{1-\Theta A}(z) = p \Im z + \frac{\Im z}{\pi} \int \frac{d\mu(t)}{|t-z|^2}$ is positive, then $\exists\, \alpha \in [0, \pi)$ such that $S_\alpha \in \mathcal{H}(E)$ (and $\frac{e^{2i\alpha} + \Theta}{e^{2i\alpha} - \Theta}$ also has linear term).

12. If $\beta(z) = \frac{az+b}{cz+d}$ maps $\mathbb{C}^+$ into $\mathbb{C}^+$, then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a J-contractive matrix if $\det A = 1$ (?)