Introduction

Canonical systems $\leftrightsquigarrow$ de Branges spaces

Louis de Branges (b. 1932)

1959 – 1965

de Branges, Hilbert Spaces of Entire Functions, 1968

R. Romanov, CS and de Branges spaces

Ch. Remling, Spectral theory of CS. (dB spaces)

I. Spectral theory for CS — an overview.

What is a canonical system?

$$J Y'(x) = z H(x) Y(x), \quad x \in [0, L)$$

(L can be $+\infty$.)

- $Y = \begin{pmatrix} Y_+ \\ Y_- \end{pmatrix}$, $Y : [0, L) \to \mathbb{C}^2$ abs. continuous vector-valued function

- $H(x)$ — 2×2 matrix valued function (Hamiltonian) with real entries, $H \geq 0$, $H \neq 0$ on any set of positive measure, $H \in L^1[0, \ell]$ $\forall \ell < L$.

- $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ matrix imaginary unit, $J^2 = -I$

- $z \in \mathbb{C}$ — spectral parameter

The problem has unique solution for any initial data $Y(0) = Y_0 \in \mathbb{C}^2$.

$$Y = Y(x, z) = \begin{pmatrix} Y_+(x,z) \\ Y_-(x,z) \end{pmatrix}$$

Moreover, $\forall x \in (0, L)$ $Y_+(x,z)$, $Y_-(x,z)$ - entire functions of z.

We can also consider this equation for matrices: $J M' = z H M$

The solution with $M(0,z) = I$ is called the <u>fundamental solution</u>.

The matrix $M(L,z)$ (or $M(x,z)$) is called the <u>monodromy matrix</u>

$$M = \begin{pmatrix} \theta_+ & \varphi_+ \\ \theta_- & \varphi_- \end{pmatrix},$$ where $\theta = \begin{pmatrix} \theta_+ \\ \theta_- \end{pmatrix}$, $\varphi = \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix}$

solve the equation $JY' = zHY$ with $Y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Moreover, $\forall x \in (0, L)$ $Y_+(x, z)$, $Y_-(x, z)$ - entire functions of z. (*) Про матрицу монодромии

II. ST for CS - an overview

Two cases:

1. $H \in L^1[0, L)$ regular, Weyl's limit circle case

$H \notin L^1[0, L)$ - singular, Weyl's limit point case

Regular case: Let $Y(0) \in \mathbb{R}^2$.

$E_x(z) := Y_+(x, z) + i\, Y_-(x, z)$, $x \in (0, L]$ - entire function of the Hermite-Biehler class.

$H \longleftrightarrow E_L$ (or $\mathcal{H}(E_L)$ - dB space)

Inverse theorem: any E_L from some class corresp to some (unique) Hamiltonian.

Moreover, $\mathcal{H}(E_x) \subset \mathcal{H}(E_L)$ isometrically, $0 < x < L$

Singular case: $H \longleftrightarrow \mu$ - spectral measure

μ - meas. on $\mathbb{R}$, $\int \frac{d\mu(t)}{t^2+1} < \infty$

$M = \begin{pmatrix} \Theta_+ & \varphi_+ \\ \Theta_- & \varphi_- \end{pmatrix}$. Then $\forall z \in \mathbb{C} \setminus \mathbb{R}$ $\forall \tau \in \mathbb{R}$

$$\exists\ q(z) = \lim_{x \nearrow L} \frac{\tau \Theta_+(x, z) + \varphi_+(x, z)}{\tau \Theta_-(x, z) + \varphi_-(x, z)}$$

q - Weyl coeff, independ. on τ.

q is a Herglotz function: $\operatorname{Im} q(z) > 0$, $z \in \mathbb{C}^+$

$\overline{q(\bar{z})} = q(z)$

Then $g(z) = pz + c + \int \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\mu(t)$,

$p \geq 0$, $c \in \mathbb{R}$, $\int_{\mathbb{R}} \frac{d\mu(t)}{t^2+1} < \infty$

$H \rightsquigarrow g \rightsquigarrow \mu$. Invers thm says that any such μ corresponds to the unique Hamiltonian.

<u>Example:</u> $H(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$J Y' = z H Y$ $\quad \begin{cases} -Y_-' = z Y_+ \\ Y_+' = z Y_- \end{cases}$

Solution with $Y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \Theta(x, z) = \begin{pmatrix} \cos xz \\ -\sin xz \end{pmatrix}$

$Y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad \Phi(x, z) = \begin{pmatrix} \sin xz \\ \cos xz \end{pmatrix}$

$E_x(z) := \Theta_+(x,z) + i\Theta_-(x,z) =$

$= \cos xz - i \sin xz = e^{-ixz}$

Let $f \in L^2(\mathbb{C}^2, [0, L)) = L^2(H, [0, L))$

$\qquad \qquad \qquad \qquad \qquad \qquad \| \quad \{ f : [0, L) \to \mathbb{C}^2 : \int_0^L (Hf, f) < \infty \}$

$f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \xmapsto{\mathcal{U}} \frac{1}{\sqrt{\pi}} \int_0^L (f_+(t) \underbrace{\cos zt}_{\Theta_+(t,z)} - f_-(t) \underbrace{\sin zt}_{-\Theta_-(t,z)}) dt =$

$= \frac{1}{2\sqrt{\pi}} \int_{-L}^{L} g(t) e^{-izt} dt \qquad g \in L^2(-L, L)$

$f_+(t) = \frac{g(t) + g(-t)}{2}$

$f_-(t) = \frac{g(t) - g(-t)}{2}$

Thus, this mapping $\mathcal{U}$ is a unitary operator from the space associated to a Hamiltonian to some de Branges space

$\mathcal{F}\, L^2(-L,L)$ – Paley-Wiener space PW_L – is a special case of a de Branges. In general we also construct a <u>Fourier-type transform</u> from $L^2(H,[0,L))$ to $\mathcal{H}(E_L)$ (in the regular case)

II Examples of canonical systems.

1. Dirac equation

Q locally summable 2×2 matrix function on $[0,L)$, $Q\in L^1_{loc}[0,L)$

$$JX' + QX = zX$$

Let X^0 be the solution of $J(X^0)' + QX^0 = 0$ with $\det X^0 = 1$

$$X := X^0 Y \qquad J(X^0)'Y + JX^0Y' + QX^0Y = zX^0Y$$

$$JY' = (X^0)^T J X^0 Y' = z(X^0)^T X^0 Y$$

$H = X_0^T X_0$, rank $=2$ $\quad A^T J A = J$ (iff $\det A = 1$)

2. Schrödinger equation

$$-y'' + qy = zy, \quad q\in L^1_{loc}[0,L)$$

Choose y_1, y_2 – solutions of $-y''+qy=0$ with

$$y_1(0) = y_2'(0) = 1$$
$$y_1'(0) = y_2(0) = 0$$

Then $y_1' y_2 - y_1 y_2' \equiv -1$

$$Y := \begin{pmatrix} y_1, & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y_2 y' - y_2' y \\ y_1' y - y_1 y' \end{pmatrix}$$

Then Y is a solution of $-y''+qy=zy$ iff

$$JY' = zHY, \quad H=\begin{pmatrix} y_1^2 & y_1y_2 \\ y_1y_2 & y_2^2 \end{pmatrix} \quad \text{rank}=1$$

$$(JY')_1 = y''y_1 - y_1''y = (qy-zy)y_1 - qy_1y = -zyy_1$$

$$y_1^2(y_2y'-y_2'y)+y_1y_2(y_1'y-y_1y') =$$
$$= -y_1^2y_2'y + y_1y_2y_1'y = y_1y\underbrace{[y_1'y_2-y_1y_2']}_{-1}.$$

3. String equation.

$\rho(x)>0$, $\rho\in L^1_{loc}[0,L)$

$$-y''=z^2\rho y \qquad Y:=\begin{pmatrix} zy \\ y' \end{pmatrix}$$

$$JY'=z\begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix}Y$$

IV. Some technicalities

Proposition: For any CS there exist an equivalent CS with $\operatorname{Tr} H \equiv 1$.

$JY'=zHY$, $Y(0)=Y_0$

$JY_{(x,z)} - \tilde{J}Y_0 = z\int_0^x H(t)Y(t,z)\,dt$

Proof: $\xi(x) := \int_0^x \operatorname{tr} H(t)\,dt$

The set $\{\xi(x) : \xi(x)=\xi(x') \text{ for some } x'\neq x\}$

$\operatorname{tr} H(t)=0 \Rightarrow H(t)=0 \Rightarrow \xi$ increases strictly

$\tilde{H}(\xi(x)) := H(x)$ defines $\tilde{H}$ on $(0,L')$

$L'=\xi(L)$ Let Y be a solution to

$$JY'=zHY.$$

$\tilde{Y}(\xi(x)) := Y(x)$. Consider $\dfrac{\tilde{H}(t)}{\operatorname{tr}\tilde{H}(t)} =: H_0(t)$ – new Hamiltonian with trace 1.

$$\int_0^S H_0(t)\tilde{Y}(t)\,dt = \int_0^S \frac{\tilde{H}(t)}{\operatorname{tr}\tilde{H}(t)}\tilde{Y}(t)\,dt = \qquad t=\xi(x), \quad S=\xi(x_S)$$

$$= \int_0^{x_S} \frac{\tilde{H}(\xi(x))}{\cancel{\operatorname{tr}\tilde{H}(\xi(x))}}\cdot\tilde{Y}(\xi(x))\cdot \underbrace{\cancel{\operatorname{tr}H(x)}}_{\operatorname{tr}\tilde{H}(\xi(x))}\,dx =$$

$$= \int_0^{x_S} H(x)Y(x)\,dx = \frac{1}{z}\int_0^{x_S} JY'(x)\,dx =$$

$$= \frac{1}{z}\left(JY(x_S) - JY(0)\right) = \frac{1}{z}\left(J\tilde{Y}(S) - JY(0)\right)$$

$$J\tilde{Y}' = zH_0\tilde{Y}, \quad \operatorname{tr}H_0 \equiv 1.$$

$H \rightsquigarrow H_0$ Regular case $\leftrightarrow L' < \infty$

Singular case $\leftrightarrow L' = +\infty$.

Th. Let H be a Hamiltonian with $\operatorname{tr}H \equiv 1$.
Then the equation $JY' = zHY$, $Y(0) = Y_0$, has unique solution $Y = Y(x,z)$, for any fixed $x \in [0, L]$ $Y_+(x,z)$, $Y_-(x,z)$ are entire functions of ~~fo~~ exponential type $\leq x$ and $\|Y(x,z)\| \leq \|Y_0\| e^{x|z|}$.

<u>Proof</u>: $JY(x,z) = JY_0 + z\int_0^x H(t)Y(t,z)\,dt$

Usual iteration procedure

$JY_1(x,z) := JY_0 + z\int_0^x H(t)\,Y_0\,dt$

$JY_{n+1}(x,z) := JY_0' + z\int_0^x H(t)\,Y_n(t,z)\,dt$

Then $\|JY_1(x,z) - JY_0\| \le |z|\cdot x\,\|Y_0\|$

Note that $\|H(t)\| \le 1$

$$\|Y_{n+1}(x,z) - Y_n(x,z)\| \le \frac{|z|^{n+1}x^{n+1}\|Y_0\|}{(n+1)!}$$

$$Y_{n+1}(x,z) - Y_n(x,z) = z\int_0^x H(t)\big(Y_n(t,z) - Y_{n-1}(t,z)\big)\,dt$$

and induction.

Now, $Y_n(x,z) = Y_0 + \sum_{k=1}^{n}\big(Y_k(x,z) - Y_{k-1}(x,z)\big)$

Y_n converges uniformly on $0 \le x \le L$, $|z| < R$

to a solution. $Y_n(x,z)$ - pols in $z \Rightarrow$

$\Rightarrow$ solution is entire on z

$$\|Y(x,z)\| \le$$
$$\le \|Y_0\| + \sum_{k=1}^{\infty}\frac{|z|^k x^k\|Y_0\|}{k!} = \|Y_0\|\,e^{x|z|}$$

<u>Rem</u>. $Y_0 \in \mathbb{R}^2 \Rightarrow Y(x,z)$ is real for real z.