1. Hardy spaces on $\mathbb{D}$ and $\mathbb{R}^2_+$

1.1. **Basic definitions.** Given $1 \le p < \infty$ we define

$$H^p(\mathbb{D}) := \left\{ f \in Hol(\mathbb{D}) : \|f\|_{H^2(\mathbb{D})} := \frac{1}{2\pi} \sup_{0<r<1} \left(\int_0^{2\pi} |f(re^{i\theta})| \, d\theta \right)^{\frac{1}{p}} < \infty \right\},$$

and

$$H^p(\mathbb{R}^2_+) := \left\{ f \in Hol(\mathbb{R}^2_+) : \|f\|_{H^2(\mathbb{R}^2_+)} := \sup_{0<y<\infty} \left(\int_{\mathbb{R}} |f(x+iy)| \, dx \right)^{\frac{1}{p}} < \infty \right\}.$$

For $p = \infty$ we just consider bounded holomorphic functions in respective domains. Note that in $\mathbb{R}^2_+$ one has to consider all values of $y > 0$, case in point $f(z) = \frac{e^{-i\frac{z}{p}}}{(z+i)^{\frac{2}{p}}}$. These classes are almost images of each other w.r.t. a conformal map $\mathbb{D} \mapsto \mathbb{R}^2_+$, however, say, non-zero constants do not belong to $H^p(\mathbb{R}^2_+)$ for finite p. We consider both these spaces simultaneously, using one of them where it is more convenient.

1.2. **Poisson kernel representation.** Assume that $f = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$, continuous up to the boundary. Then, letting $u := \Re f$, we have

$$u(re^{i\theta}) = \sum_{n\in\mathbb{Z}} A_n r^{|n|} e^{in\theta}, \quad 0 \le r < 1, \ 0 \le \theta < 2\pi,$$

with

$$\begin{aligned} A_n &= \frac{1}{2} a_n, \quad n > 0 \\ A_0 &= \Re a_0 \\ A_n &= \frac{1}{2} \overline{a}_n, \quad n < 0. \end{aligned}$$

Hence for $r < 1$ one has

$$u(re^{i\theta_0}) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \sum_{n\in\mathbb{Z}} r^{|n|} e^{in(\theta_0-\theta)} \, d\theta,$$

and

$$\sum_{n\in\mathbb{N}} r^{|n|} e^{in\theta_0} = \frac{1-r^2}{1+r^2-2r\cos\theta_0}, \quad 0 \le r < 1, \ 0 \le \theta_0 < 2\pi,$$

which is the *Poisson kernel* for the unit disk $\mathbb{D}$. One can look at this representation in a different way. Assume u is harmonic on $\mathbb{D}$, continuous up to the boundary, then

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \, dt$$

by MVP. Now fix a point $z_0 = re^{i\theta_0}$, $0 \le r < 1$, $0 \le \theta_0 < 2\pi$, and consider the Möbius transformation $\tau = \tau_{z_0}$ that moves z_0 to 0, so that

$$\tau(z) = \frac{z - z_0}{1 - \overline{z_0} z}.$$

Remark. Möbius transformations are conformal self-maps of the unit disc, they have the following form

$$\tau_{w,\theta}(z) = e^{i\theta} \frac{z-w}{1-\overline{w}z}, \quad w \in \mathbb{D}, \ \theta \in [0, 2\pi).$$

Clearly $\tau(\partial\mathbb{D}) = \mathbb{D}$ and $u(\tau^{-1}(\cdot))$ is again a harmonic function in $\mathbb{D}$, continuous up to the boundary. Hence

$$u(z_0) = u(\tau^{-1}(0)) = \frac{1}{2\pi} \int_0^{2\pi} u(\tau^{-1}(e^{it}))dt = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \frac{1-r^2}{1-2r\cos(\theta-\theta_0)+r^2} \, d\theta$$

by change of variables. In other words,

$$u(re^{i\theta_0}) = (\varphi * P_r)(\theta_0)$$

for $\varphi(\theta) := u(e^{i\theta})$ and

$$P_r(\theta) := \frac{1-r^2}{1-2r\cos\theta + r^2}, \quad 0 \le r < 1,\ 0 \le \theta < 2\pi.$$

On the other hand,

$$P_r(\theta - \theta_0) = \Re \frac{e^{i\theta} + re^{i\theta_0}}{e^{i\theta} - re^{i\theta_0}},$$

so for any continuous (and 2π-periodic) function φ one has that $u(re^{i\theta_0}) := (\varphi * P_r)(\theta_0)$ defines a function, harmonic in the unit disc and continuous up to the boundary.
Now if we consider the map $w : \mathbb{D} \mapsto \mathbb{R}^2_+$ with

$$w(z) := i\frac{1-z}{1+z},$$

we get that (after another change of variables)

$$U(w) = \int_{\mathbb{R}} U(t) P_y(x-t)\, dt, \quad w = x + iy \in \mathbb{R}^2_+,$$

if U is harmonic in $\mathbb{R}^2_+$ and continuous in $\overline{\mathbb{R}}^2_+ \cup \{\infty\}$ with

$$P_y(t) = \frac{1}{\pi}\frac{y}{t^2+y^2}, \quad y > 0,\ t \in \mathbb{R}.$$

As before, since

$$P_y(x-t) = \frac{1}{\pi}\Im \frac{1}{t-(x+iy)},$$

we see that $U(x+iy) := (\varphi * P_y)(x)$ defines a function which is harmonic in the upper halfplane and continuous up to the boundary (plus $\{\infty\}$), if $\phi \in C_0(\mathbb{R})$.
We actually can say more than that. Assume that $\varphi \in L^p(\mathbb{R})$ for some $1 < p \le \infty$, or μ is a finite measure on $\mathbb{R}$ (or a positive measure on $\mathbb{R}$ with $\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty$), then, still, $\varphi * P_y$ defines a harmonic function on the upper halfplane. Indeed, for $y > 0, p$ fixed the Poisson kernel P_y (along with its derivatives) is in $L^q(\mathbb{R})$, where $\frac{1}{q} + \frac{1}{p} = 1$, and is smooth in each variable, so by general theory one can differentiate under the integral.

Theorem 1.1 *Assume that u is a harmonic function in the upper halfspace $\mathbb{R}^2_+$. Then*

- *There exists a function $\varphi \in L^p$, $1 < p \le \infty$ such that*

  $$u(x,y) = (\varphi * P_y)(x), \quad y > 0,\ x \in \mathbb{R},$$

  if and only if

(1) $$\sup_{y>0} \|u(\cdot,y)\|_{L^p(\mathbb{R})} < \infty.$$

  In that case one has

(2) $$\lim_{y\to 0} \|u(\cdot,y) - \varphi\|_{L^p(\mathbb{R})} = 0$$

  for $1 < p < \infty$, and

(3) $$\lim_{y\to 0} \left| \int_{\mathbb{R}} u(t,y) g(t)\, dt - \int_{\mathbb{R}} \varphi(t) g(t)\, dt \right| = 0, \quad g \in L^1(\mathbb{R}),$$

  for $p = \infty$.

- *There exists a finite measure μ on $\mathbb{R}$ such that*

  $$u(x,y) = (\mu * P_y)(x), \quad y > 0,\ x \in \mathbb{R},$$

  if and only if

(4) $$\sup_{y>0} \|u(\cdot,y)\|_{L^1(\mathbb{R})} < \infty.$$

In that case one has

$$\lim_{y\to 0}\left|\int_{\mathbb{R}} u(t,y)g(t)\,dt - \int_{\mathbb{R}} g(t)\,d\mu(t)\right| = 0, \quad g \in C_0(\mathbb{R}). \tag{5}$$

- *The function u is positive if and only if there exists a positive measure μ on $\mathbb{R}$ and a non-negative constant c such that*

$$u(x,y) = cy + (\mu * P_y)(x), \quad y > 0,\ x \in \mathbb{R}, \tag{6}$$

with

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < +\infty. \tag{7}$$

Proof. ***The "only if" part.***
We already know that the convolution $\varphi * P_y$ or $\mu * P_y$ defines a harmonic function on the upper half-plane. To estimate norms one only has to apply Minkowski's inequality to the convolution representation,

$$\left(\int_{\mathbb{R}} |\varphi * P_y|^p(x)\,dx\right)^{\frac{1}{p}} \le \|\varphi\|_{L^p(\mathbb{R})}, \quad 1 < p < \infty,$$

and

$$\int_{\mathbb{R}} |\mu * P_y|(x)\,dx \le \int_{\mathbb{R}} |d\mu|.$$

For $p = \infty$ it follows from the fact that $P_y > 0$ and $\int_{\mathbb{R}} P_y(t)\,dt \equiv 1,\ y > 0$.
It remains to show that harmonic extensions to $\mathbb{R}^2_+$ converge to φ (re. μ) in appropriate norms. Observe first that shifts are continuous in L^p, $1 \le p < \infty$ (but not at the ends of the scale — hence slightly different convergence there!). In other words,

$$\lim_{x\to 0} \|\varphi(x - \cdot) - \varphi(\cdot)\|_{L^p(\mathbb{R})} = 0, \quad \varphi \in L^p(\mathbb{R}).$$

It follows that

$$\|\varphi * P_y - \varphi\|_{L^p(\mathbb{R})} \le \int_{\mathbb{R}} P_y(t)\|\varphi(\cdot - t) - \varphi\|_{L^p(\mathbb{R})}\,dt = \left(\int_{|t|\le\sqrt{y}} + \int_{|t|>\sqrt{y}}\right) P_y(t)\|\varphi(\cdot - t) - \varphi\|_{L^p(\mathbb{R})}\,dt$$

by Minkowski's inequality. The first term tends to zero by continuity of shifts, and the second one by the estimate on the tail of Poisson kernel. In addition, if φ is uniformly continuous on $\mathbb{R}$, then its shifts are continuous in L^∞-norm, and arguing as above we arrive to the uniform convergence of $\varphi * P_y$ to φ. The properties (3) and (5) follow by duality. The "only if" part of the third point is obvious.
The 'if' part
We first need the following lemma.

Lemma 1 *Assume u is bounded harmonic function in the upper halfplane and is continuous up to the boundary (so no condition at infinity!). Then*

$$u(x+iy) = \int_{\mathbb{R}} P_y(x-t)u(t)\,dt = (u * P_y)(x), \quad x \in \mathbb{R},\ y > 0.$$

Proof. Put

$$U(x+iy) = u(x+iy) - (u * P_y)(x).$$

Since u is continuous in $\overline{\mathbb{R}}^2_+$, we clearly have that $u * P_y \to u$ pointwise. In particular, U is bounded and harmonic in $\mathbb{R}^2_+$, continuous up to the boundary, and $U \equiv 0$ on $\mathbb{R}$. Reflecting U on $\mathbb{R}^2_-$ we see that the resulting function is bounded and harmonic in $\mathbb{R}^2$, hence it is constant. ■ Assume now that, say, (1) holds, and fix some $y_n > 0$. By mean value property and Cauchy-Schwartz one has

$$|u(x+iy)| \le C(y) \sup_{y>0} \|u(\cdot, y)\|_{L^p(\mathbb{R})}, \quad x \in \mathbb{R},$$

for some $C(y)$. Lemma 1 implies that

$$u(x + i(y + y_n)) = (u(\cdot, y_n) * P_y)(x), \quad y > 0, q \in \mathbb{R}.$$

The sequence $u(\cdot, y_n)$ is bounded in $L^p(\mathbb{R})$, so by Banach-Alaoglu it has a weak limit. Since Poisson kernels are in respective L^q, we obtain the existence of $\varphi = \lim_{y_n \to 0} u(\cdot, y_n)$. The same works for $p = 1$ and weak convergence of measures. The third point is proven by going to the unit disc (where it follows from the second point) by a conformal map, and then going back. ■

Theorem 1.2 (F. and M. Riesz, 1917) *Assume that μ is a singular finite measure on $\partial\mathbb{D}$ such that*

$$\int_0^{2\pi} e^{-nt}\, d\mu(t) = 0, \quad n < 0.$$

Then $\mu \equiv 0$

1.3. **Blaschke products.**

Theorem 1.3 *Assume $\{z_n\}_{n=1}^{\infty}$ is a sequence of points in the unit disc. Then the Blaschke product*

$$B(z) := \prod_{n=1}^{\infty} \frac{-\overline{z}_n}{|z_n|} \frac{z - z_n}{1 - \overline{z}_n z} \tag{8}$$

converges uniformly on compact subsets of $\mathbb{D}$ to a nonzero holomorphic function in $\mathbb{D}$ if and only if

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty. \tag{9}$$

In this case the zeros of B are precisely the points z_n with respective multiplicity.

Theorem 1.4 (F. Riesz) *Assume $f \in H^p(\mathbb{D})$ with $1 \le p \le \infty$ and f is not identically zero. Let $\{z_n\}_{n=1}^{\infty}$ be the sequence of zeros of f (w.r.t. their multiplicity). Then*

$$\sum_{n \in \mathbb{N}} (1 - |z_n|) < \infty, \tag{10}$$

and if B is the Blaschke product with zeros $\{z_n\}$, then $\frac{f}{B}$ belongs to $H^p(\mathbb{D})$, has no zeros in $\mathbb{D}$, and

$$\left\| \frac{f}{B} \right\|_{H^p(\mathbb{D})} = \|f\|_{H^p(\mathbb{D})}. \tag{11}$$