Chapter 5. Direct and inverse spectral theory in the singular case.

I. Generalized Fourier transform in the singular case.

Let H be a Hamiltonian on $[0;+\infty)$ ($\operatorname{tr} H \equiv 1$),

$L^2([0;+\infty), H) =: L^2(H)$

$\mathcal{H} := \{ f \in L^2(H) : f = \text{const a.e. on any } H\text{-indiv. } I \}$

↖ as equivalence class in $L^2(H)$.

$$\mathcal{D} := \left\{ f \in \mathcal{H} : \begin{array}{l} f \text{ abs continuous on } [0;+\infty) \\ \exists g \in \mathcal{H} : Jf' = Hg \\ f_-(0) = 0 \end{array} \right\}$$

Th. Let $H \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ on any $(0, \varepsilon)$ and assume that $(b;+\infty)$ is not H-indiv. for any b. Then

$$D : \mathcal{D} \to \mathcal{H}, \quad f \mapsto g$$

is a correctly defined selfadjoint operator on $\mathcal{H}$

For a compactly supported $f \in \mathcal{H}$ we define GFT as before:

$$(\mathcal{U}f)(z) = \frac{1}{\sqrt{\pi}} \langle f, \Theta_{\bar{z}} \rangle_H =$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{+\infty} (H(t) f(t), \Theta(\bar{z}, t))\, dt.$$

Our aim is to prove

Theorem. $\exists\ \mu$ - measure on $\mathbb{R}$, $\int \frac{d\mu(t)}{t^2+1} < \infty$, such that $\mathcal{U}$ can be extended to a unitary operator from $\mathcal{H}$ onto $L^2(\mu)$. Moreover, if we denote by S_x the operator of multiplication by the independent variable on $L^2(\mu)$, then $\mathcal{D} = \mathcal{U}^{-1} S_x \mathcal{U}$.

II. Construction of the Weyl-Titchmarsh function.

Lem. $\forall z \notin \mathbb{R}$ the system $JY' = zHY$ has ~~a~~ nontrivial solution which belong to $\mathcal{H}$ (i.e $\int_0^\infty (HY, Y) < \infty$)

Proof. $\mathcal{D}$ is selfadj, hence $\forall\ h \in \mathcal{H}$
$\exists\ f \in \mathcal{D}$: $(\mathcal{D} - zI) f = h$.

Let h be compactly supported in $[0, x_0)$.

$Jf' = Hg$ $g \in \mathcal{H}$, $\mathcal{D}f = g$

$h = g - zf$, $Hh = Hg - zHf = Jf' - zHf$.

Since $h \equiv 0$ on $[x_0; +\infty)$, we see that $Jf' = zHf$, so f is the solution on $[x_0; +\infty)$.

Now, we can find a solution of $J\tilde{f}' = zH\tilde{f}$ on $[0, x_0]$ with the data $\tilde{f}(x_0) = f(x_0)$.

Combining these two solutions, we get a solution f on $[0;+\infty)$ with $f\in\mathcal{H}$.

Note that $\Theta(x,z)\notin\mathcal{H}$ for $z\in\mathbb{R}$. Otherwise $\Theta(x,z)$ would be an eigenfunction with a nonreal eigenvalue. Note that for any fixed z, any solution $f(x,z)=a(z)\Theta(x,z)+b(z)\Phi(x,z)$

$$\left(f(0,z)=a(z)\begin{pmatrix}1\\0\end{pmatrix}+b(z)\begin{pmatrix}0\\1\end{pmatrix}\right).$$

If $f(x,z)\in\mathcal{H}$ then $\exists\, m(z)\in\mathbb{C}$ such that

$$f(x,z)=\Phi(x,z)-m(z)\Theta(x,z)$$

For $X>0$ define $m_X(z)=\dfrac{\Phi_-(X,z)}{\Theta_-(X,z)}$

$f(x,z)$ is small, $x\to\infty$, $\Theta(x,z)$ is not, so one can expect that $\dfrac{\Phi_-(X,z)}{\Theta_-(X,z)}\xrightarrow[X\to\infty]{} m(z)$.

Th. $m_X \underset{X\to+\infty}{\rightrightarrows} m$ on compact sets in $\mathbb{C}\setminus\mathbb{R}$ uniformly.

Lemma. Let $S=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, $\det S=1$, $\beta(z)=\beta_S(z):=\dfrac{az+b}{cz+d}$. TFAE:

1. $\frac{1}{i}(S^*JS-J)\geq 0$ (S-J-contractive)
2. $\beta(\mathbb{C}^+)\subset\mathbb{C}^+$.

Proof: $1\Rightarrow 2$

$$\operatorname{Im}\beta(z)=\operatorname{Im}\frac{az+b}{cz+d}=\operatorname{Im}\frac{(az+b)(\overline{cz+d})}{|cz+d|^2}$$

$$\frac{1}{i}\left(S^*JS\begin{pmatrix}z\\1\end{pmatrix}, \begin{pmatrix}z\\1\end{pmatrix}\right) = \frac{1}{i}\left(JS\begin{pmatrix}z\\1\end{pmatrix}, S\begin{pmatrix}z\\1\end{pmatrix}\right) =$$

$$= \frac{1}{i}\left(\begin{pmatrix}-cz-d\\az+b\end{pmatrix}, \begin{pmatrix}az+b\\cz+d\end{pmatrix}\right) = \frac{2i}{i}\,\mathrm{Im}\,(az+b)(\overline{cz}+\overline{d})$$

By J-contractivity

$$\frac{1}{i}\left(S^*JS\begin{pmatrix}z\\1\end{pmatrix}\begin{pmatrix}z\\1\end{pmatrix}\right) \geq \frac{1}{i}\left(J\begin{pmatrix}z\\1\end{pmatrix}, \begin{pmatrix}z\\1\end{pmatrix}\right) = 2\,\mathrm{Im}\, z.$$

Thus, $\mathrm{Im}\,\frac{az+b}{cz+d} > 0$ if $\mathrm{Im}\, z > 0$.

$2 \Rightarrow 1$ – Exercise.

II. Proof of the theorem that $m_X \rightrightarrows m$.

$$\beta_{X,z}(w) := \frac{\overline{\varphi_-(X,z)}\, w - \overline{\varphi_+(X,z)}}{-\overline{\theta_-(X,z)}\cdot w + \overline{\theta_+(X,z)}}$$

The matrix

$$G_X(z) = \begin{pmatrix}\overline{\varphi_-(X,z)} & -\overline{\varphi_+(X,z)}\\ -\overline{\theta_-(X,z)} & \overline{\theta_+(X,z)}\end{pmatrix} = \overline{M^{-T}(X,z)}$$

Since $M(X,z)$ was J-contractive, $\overline{M^{-1}(X,z)}$ also is.

So $\beta_{X,z}(\mathbb{C}^+) \subset \mathbb{C}^+$. Note also, that

$$\beta : \frac{\overline{\theta_+(X,z)}}{\overline{\theta_-(X,z)}} \longmapsto \infty$$

$\in \mathbb{C}^-$

So $\beta(\mathbb{C}^+)$ is a disc on $\mathbb{C}^+$, denote it $D_{X,z}$

Nesting property: $X > X' \Rightarrow D_{X,z} \subset D_{X',z}$

$M(X,z) = N(X,z)M(X',z)$, $N(x,z)$ is the fundamental solution for the interval (X',X)

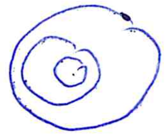
Then $G_X(z) = G_{X'}(z)\,\overline{N^{-1}(X,z)}$.

$\Omega = \overline{N^{-1}(x,z)}$ is J contractive

$$\beta_{x,z}(w) = \beta_{x',z} \circ \beta_{\Omega}(w)$$

Exerc: $\beta_{AB} = \beta_A \circ \beta_B$

$$\beta_{x,z}(\mathbb{C}^+) = \beta_{x',z}\Big(\underbrace{\beta_{\Omega}(\mathbb{C}^+)}_{\subset \mathbb{C}^+}\Big) \Rightarrow \beta_{x,z}(\mathbb{C}^+) \subset \beta_{x',z}(\mathbb{C}^+).$$

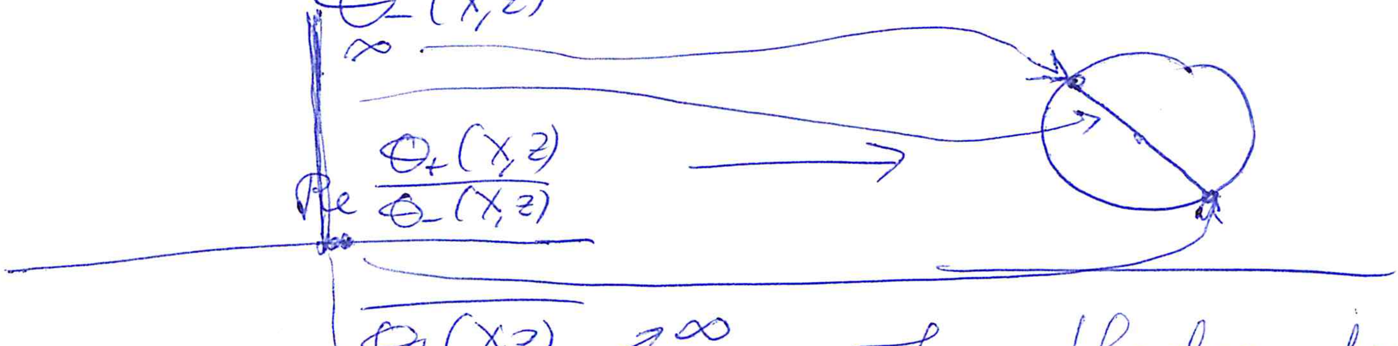


Clearly, such a family converges either to a limit circle, or to a point (limit circle case and limit point case).

We show that in our case the radii $R(x,z)$ converge to 0 uniformly in z on compacts of $D_{x,z}$ on $\mathbb{C}^+$ and that $\bigcap_{x>0} D_{x,z} = \{-\overline{m(z)}\}$.

$$\overline{\frac{\Theta_+(x,z)}{\Theta_-(x,z)}} \longmapsto \infty, \qquad \overline{-\frac{\Phi_-(x,z)}{\Theta_-(x,z)}} = \beta_{x,z}(\infty)$$

So $\overline{-\dfrac{\Phi_-(x,z)}{\Theta_-(x,z)}}$ lies on the boundary of the disc



$\overline{\dfrac{\Theta_+(x,z)}{\Theta_-(x,z)}} \nearrow \infty$

Thus, the diameter is the distance between $\beta(\infty)$ and $\beta\left(\mathrm{Re}\,\dfrac{\Theta_+(x,z)}{\Theta_-(x,z)}\right)$.

$$\operatorname{diam} D_{x,z} = \left| \overline{-\frac{\Phi_-(x,z)}{\Theta_-(x,z)}} - \overline{\frac{\Phi_-(x,z)\,\mathrm{Re}(\;) - \Phi_+(x,z)}{-\Theta_-(x,z)\,\mathrm{Re}(\cdot\cdot) + \Theta_+(x,z)}} \right| =$$

$$= \left| \frac{1}{\Theta_-^2(X,z)\, \mathrm{Im}\, \frac{\Theta_+(X,z)}{\Theta_-(X,z)}} \right| = \frac{1}{\mathrm{Im}\left(\Theta_+(X,z)\overline{\Theta_-(X,z)}\right)} =$$

$$= \frac{1}{\mathrm{Im}\, z \int_0^X \Theta^*(t,z)\, H(t)\Theta(t,z)\, dt}$$

But $\Theta(t,z) \notin \mathcal{H}$, $\int_0^X (H(t)\Theta(t,z), \Theta(t,z))\,dt \to \infty$, $X \to +\infty$.

From continuity of the dependance of $\Theta(t,z)$ on z. it is easy to see that the convergence is uniform on compacts on $\mathbb{C} \setminus \mathbb{R}$.

Thus, diam $D_{X,z} \to 0$, and the centers $\to m_*(z)$

$$-\overline{\frac{P_-(X,z)}{\Theta_-(X,z)}} \to m_*(z)$$

But also $-\overline{\frac{P_+(X,z)}{\Theta_+(X,z)}} \in \partial D_{X,z}$, as image of 0.

So, $-\overline{\frac{P_+(X,z)}{\Theta_+(X,z)}} \to m_*(z)$

Thus, $P(X,z) = \Theta(X,z)\left(-\overline{m_*(z)} + o(1)\right)$ as $X \to +\infty$

$f(X,z) = P(X,z) - m(z)\Theta(X,z) =$

— solution on $\mathcal{H}$

$= \Theta(X,z)\left(-\overline{m_*(z)} - m(z) + o(1)\right)$

Since $f(X,z) \in \mathcal{H}$, $\Theta(X,z) \notin \mathcal{H}$, we have

$$m(z) = -\overline{m_*(z)}.$$

All m_X are Herglotz functions, so is m

Def. $m(z)$ – Weyl-Titchmarsh function for (H,∞).

Let μ_X and μ be the measures on the represent. of m_X and m.

$$\frac{\varphi_-(X,z)}{\theta_-(X,z)} = \frac{1}{\pi}\int\left(\frac{1}{t-z}-\frac{t}{t^2+1}\right)d\mu_X(t) + \cancel{c_X z}+d_X$$ by theorem

$$m(z) = \frac{1}{\pi}\int\left(\frac{1}{t-z}-\frac{t}{t^2+1}\right)d\mu(t)+\cancel{cz}+d$$

IV. Proof of the theorem about GFT.

Let $f\in\mathcal{H}$, compact support, in $[0,X)$.

$$\|\mathcal{U}f\|^2_{L^2(\mu_X)} = -\pi\sum_j \frac{\varphi_-(X,t_j)}{\dot\theta_-(X,t_j)}|\mathcal{U}f(t_j)|^2 =$$

What is μ_X? $\mu_X(t_j) = -\pi\dfrac{\varphi_-(X,t_j)}{\dot\theta_-(X,t_j)}$

where $\{t_j\} = Z_{\theta_-(X,z)}$

$$= -\sum_j \frac{1}{\theta_+(X,t_j)\dot\theta_-(X,t_j)}|\langle f,\theta_t\rangle_H|^2$$

$\theta_+(X,z)\varphi_-(X,z)-\theta_-(X,z)\varphi_+(X,z)\equiv 1$, take $z=t_j$

$$\theta_+(X,t_j)\varphi_-(X,t_j)=1$$

Note also, that if $E_X(z)=\theta_+(X,z)+i\theta_-(X,z)$,

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then $\|K_{t_j}\|^2_{\mathcal{H}(E_X)} = K_{t_j}(t_j) =$

$$= -\frac{1}{\pi}\dot{\Theta}_-(X,t_j)\Theta_+(X,t_j).$$

Thus, $\|\mathcal{U}f\|^2_{L^2(\mathbb{R},d\mu_X)} = \sum_j \frac{|(\mathcal{U}f)(t_j)|^2}{\|K_{t_j}\|^2} =$

$$= \|\mathcal{U}f\|^2_{\mathcal{H}(E_X)} = \|f\|_{L^2([0,X],H|_{[0,X]})} =$$

$$= \|f\|_{L^2([0,+\infty),H)},$$ since f is compactly supported in $[0,X)$.

Take $z=i$: $\sup_X \int \frac{d\mu_X(t)}{t^2+1} < \infty$.

The measures μ_X converge to μ $(*)$ weakly in the sense that $\int \frac{g(t)}{t^2+1} d\mu_X(t) \to \int \frac{g(t)}{t^2+1} d\mu(t)$ for any $g \in C_0(\mathbb{R})$.

Thus, $\mathcal{U}f \in L^2(\mu)$ $\forall f \in \mathcal{H}$, compactly supported, and $\|\mathcal{U}f\|_{L^2(\mu)} \le \|f\|_H$.

We need to show the equality.

Assume that $f \in \mathcal{D}$ and $g = (D+i)f$.
comp. supported.

g is also compactly supported and $(\mathcal{U}g)(z) = (z+i)(\mathcal{U}f)(z)$. $(*)$

$(\mathcal{U}g)(z) = \frac{1}{\sqrt{\pi}} \int_0^L (H(t)g(t), \Theta_{\bar{z}}(t))\,dt =$

$HDf = Jf'$

$$= \frac{1}{\sqrt{\pi}} \int_0^L \left(J f'(t) + i H f(t), \theta_{\bar z}(t) \right) dt =$$

$$= \frac{1}{\sqrt{\pi}} \left(J f(t), \theta_{\bar z}(t) \right) \Big|_0^L + \underbrace{\frac{1}{\sqrt{\pi}} \int_0^L \left(f(t), J \theta'_{\bar z}(t) \right) dt}_{\substack{= z(Uf)(z)}} +$$

$$J\theta'_{\bar z}(t) = \bar z H \theta_{\bar z}(t)$$

$$+ i (Uf)(z)$$

$$\left(\underbrace{Jf_-(0)}_{= \binom{0}{?}}, \underbrace{\theta_{\bar z}(0)}_{= \binom{1}{0}} \right) = 0$$

The identity $(*)$ is proved.

Then $\sup\limits_X \int\limits_{|t|>A} |(Ug)(t)|^2 d\mu_X(t) =$

$$= \sup_X \int\limits_{|t|>A} \frac{|(Ug)(t)|^2}{t^2+1} d\mu_X(t) \leq \frac{\|Ug\|^2_{L^2(\mu_X)}}{A^2}.$$

It follows that $\|Uf\|_{L^2(\mu_X)} \to \|Uf\|_{L^2(\mu)}$.

for any $f \in D$, compactly supported. Such functions are dense in $\mathcal{H}$ (we will not prove it), so U extends to an isometry.

$UDf = S_X Uf \quad \forall f$ as above (verified above).

This holds for continuation.

It remains to verify that Ran U is dense on $L^2(\mu)$ (that is, we are onto). See details on [Romanov].

V. Inverse spectral problem – solution in finite-dimensional case.

Th Let $\mu = \sum_{j=0}^{n} \mu_j \delta_{t_j}$, $t_0 = 0$. Then there exists a canonical system (H, L) such that

$$\frac{\varphi_-(L,z)}{\Theta_-(L,z)} = \sum_j \frac{\mu_j}{t_j - z} + cz + d, \quad c \geq 0,\ d \in \mathbb{R}.$$

Proof: Put $\Theta_-(z) = -\frac{1}{\mu_0} z \prod_{j=1}^{n}(1 - z/t_j)$

$$\Theta_+(z) := \Theta_-(z)\left(-\sum_{j=0}^{n} \frac{1}{\mu_j \cdot \dot{\Theta}_-^2(t_j)(z - t_j)} + d_1\right)$$

Clearly, $\frac{\Theta_+}{\Theta_-}$ is Herglotz and so $\Theta_+ + i\Theta_- = E \in HB$

By construction $E(0) = 1$, E-polynomial.

So $\exists\ (H, L)$, $L < \infty$ such that $\Theta(L,z) = \begin{pmatrix} \Theta_+(z) \\ \Theta_-(z) \end{pmatrix}$.

Let $\varphi(x,z)$ be the solution with $\varphi(0,z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $\frac{\varphi_-(L,z)}{\Theta_-(L,z)}$ is Herglotz

by J-contractivity of $M(x,z)$

$$\frac{\varphi_-(L,z)}{\Theta_-(L,z)} = -\sum_{j=0}^{n} \frac{\varphi_-(L,t_j)}{\dot{\Theta}_-(L,t_j)(t_j - z)} + cz + d$$

Since $\det M(x,z) \equiv 1$, we have

$$\Theta_+(L,t_j)\varphi_-(L,t_j) - \underbrace{\Theta_-(L,t_j)}_{=0}\varphi_+(L,t_j) = 1$$

Hence $\frac{\varphi_-(L,t_j)}{\dot{\Theta}_-(L,t_j)} = \frac{1}{\Theta_+(L,t_j)\dot{\Theta}_-(L,t_j)}$

On the other hand,

$$\Theta_+(t_j,\lambda) = \frac{1}{\mu_j \dot{\Theta}_-(\lambda, t_j)} \quad \text{and so} \quad \frac{\varphi_-(\lambda, z)}{\dot{\Theta}_-(\lambda, z)}$$

has the required representation.

VI. Inverse problem – solution in the general case by approximation argument.

Let μ be a measure on $\mathbb{R}$: $\int \frac{d\mu(t)}{t^2+1} < \infty$, $\mu(\{0\}) = 0$.

Construct a sequence of measures with the properties:

a) Each μ_N is supported on finitely many points

b) $\mu_N \to \mu$ (*) weakly in the sense that $\forall g \in C_0(\mathbb{R})$ $\int_{\mathbb{R}} \frac{g(t)\, d\mu_N(t)}{t^2+1} \to \int_{\mathbb{R}} \frac{g(t)\, d\mu(t)}{t^2+1}$

c) $\sup_N \int_{|t|>s} \frac{d\mu_N(t)}{t^2} \xrightarrow[s\to\infty]{} 0$

d) $\mu_N(\{0\}) > 0$.

It is easy to see that such sequence exists.

Let (H_N, L_N) be the canonical system for μ_N.

Notice that $L_N \to \infty$. Indeed, $\mu_N(0) = -\frac{1}{\dot{\Theta}_N^-(0)} \to$ $\to \mu(\{0\}) = 0$. Thus, $-\dot{\Theta}_N^-(0) \to +\infty$.

But $L_N = \underbrace{\dot{\varphi}_N^+(\lambda, 0)}_{\geq 0} - \dot{\Theta}_N^-(\lambda, 0)$, so $L_N \to \infty$.

$G_N(x) := \int_0^x H_N(t)\,dt$, $0 \le x \le L_N$, for $x > L_N$ define G_N so that it is continuous on $\mathbb{R}$.

Then $\{G_N\}$ is compact in $C[0,A]$ for any $A > 0$.

Applying diagonal process we get a subsequence (still denoted G_N) such that $G_N \rightrightarrows G$ on any $[0,A]$ for some G. G is increasing and absolutely continuous (as in regular case), put $H(x) = G'(x)$, $x \ge 0$.

Then H is defined a.e., $\operatorname{tr} H = 1$ a.e., $H \ge 0$.

Let $M_N(x,z)$, $M(x,z)$ be the monodromy matrices for (H_N, L_N), $(H, +\infty)$ respectively.

As in the regular case, passing to a subsequence we get that $M_N(x,z) \rightrightarrows \tilde{M}(x,z)$ on $[0,A]$ for any $A > 0$, and, integrating by parts we get that $\tilde{M}(\cdot, z)$ satisfies the same equation as $M(\cdot, z)$, so $\tilde{M} = M$. Thus, the limit does not depend on a subsequence,

$$M_N(x,z) \to M(x,z) \quad \forall z \in \mathbb{C}.$$

Let $m_N(x,z) = \dfrac{\varphi_N^-(x,z)}{\theta_N^-(x,z)}$, $x \le L_N$;

and let $m(x,z)$ be the Weyl-Titchmarsh function of the system (H, ∞).

We will show that $\underset{\substack{\shortparallel \\ m_N(L_N,z)}}{m_N(z)} \xrightarrow[N\to\infty]{} m(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$.

Let $x < L_N$, $z \in \mathbb{C}^+$.

$$|m_N(z) - m(z)| \le |m_N(z) - m_N(x, z)| + \left| m_N(x, z) - \frac{\varphi_-(x, z)}{\theta_-(x, z)} \right| + \left| \frac{\varphi_-(x, z)}{\theta_-(x, z)} - m(z) \right| = $$

$$= I + \overline{II} + \overline{III}$$

$$I = |m_N(L_N, z) - m_N(x, z)| = \left| \overline{\frac{\varphi_N^-(L_N, z)}{\theta_N^-(L_N, z)} - \frac{\varphi_N^-(x, z)}{\theta_N^-(x, z)}} \right|$$

Let $D_{N,x,z}$ be the disc constructed as in the direct spectral problem.

Then $-\dfrac{\varphi_N^-(L_N, z)}{\theta_N^-(L_N, z)} \in \partial D_{N, L_N, z}$

$-\dfrac{\varphi_N^-(x, z)}{\theta_N^-(x, z)} \in \partial D_{N, x, z}$ and $D_{N, L_N, z} \subset D_{N, x, z}$ (nesting property).

Thus, $I \le \operatorname{Diam} D_{N, x, z} =$ see proof in section $\overline{III}$

$$= \frac{1}{\operatorname{Im}\left(\theta_N^+(x, z) \overline{\theta_N^-(x, z)}\right)}.$$

$$\operatorname{Im}\left(\theta_N^+(x, z) \overline{\theta_N^-(x, z)}\right) \xrightarrow[N \to \infty]{} \operatorname{Im}\left(\theta_+(x, z) \overline{\theta_-(x, z)}\right) =$$

$$= \operatorname{Im} z \int_0^x \theta^*(t, z) H(t) \theta(t, z) \, dt$$

Analogous estimate holds for $\overline{III}$.

$$\overline{III} = \Bigg| \underbrace{-m(z)}_{\in D_{x,z}} - \Bigg(\underbrace{-\frac{\varphi_-(x, z)}{\theta_-(x, z)}}_{\in D_{x,z}} \Bigg) \Bigg| = \frac{1}{\operatorname{Im}\left(\theta_+(x, z) \overline{\theta_-(x, z)}\right)}$$

Since $M_N(x,z) \to M(x,z)$, we have $\mathrm{II} \to 0$.

Thus, $\overline{\lim_{N\to\infty}} |m_N(z) - m(z)| \le \frac{2}{\operatorname{Im} z} \int_0^x \Theta^* H \Theta.$

for any $x > 0$ (since $L_N \to \infty$). Hence,

$$m_N(z) \to m(z) \quad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

By the choice of μ_N,

$$\frac{y}{\pi} \int_{\mathbb{R}} \frac{d\mu_N(t)}{(t-x)^2 + y^2} \xrightarrow[N\to\infty]{} \frac{y}{\pi} \int \frac{d\mu(t)}{(t-x)^2+y^2}, \quad z = x+iy \in \mathbb{C}^+$$

We use that $\sup\limits_N \int\limits_{|t|>A} \frac{d\mu_N(t)}{t^2+1} \xrightarrow[A\to\infty]{} 0$,

while for any fixed A $\int_{-A}^{A} g(t)\, d\mu_N(t) \to \int_{-A}^{A} g(t)\, d\mu(t)$

for any $g \in C[-A,A]$.

We have $\operatorname{Im} m_N(z) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{d\mu_N(t)}{|t-z|^2} + c_N \operatorname{Im} z$.
$c_N \ge 0$

Since $m_N(z) \to m(z)$ and the integrals converge we conclude that $c_N \to c \ge 0$.

Passing to the limit we get

$$\operatorname{Im} m(z) = \frac{y}{\pi} \int \frac{d\mu(t)}{|t-z|^2} + c \operatorname{Im} z, \text{ thus}$$

μ is the measure in the spectral representation of $m(z)$.

<u>Rem.</u> We solved the problem assuming that the limit Hamiltonian satisfies conditions $H \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ on $(0,\varepsilon)$, and H has no singular ray. These conditions are not essential, the Weyl-Titchmarsh function can be defined whenever $H \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. See details in [Romanov].