Chapter 4. Uniqueness in the inverse spectral problem.

Th. Let (H, L), (H_1, L_1) be two canonical systems, $L_1, L < \infty$ (regular case), $H, H_1 \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ on any interval $(0, \varepsilon)$. Let Θ, Θ_1 be the solutions of the respective canonical systems with initial data $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at zero. If $\Theta(L, z) = \Theta_1(L_1, z)$ for any $z \in \mathbb{C}$, then $L = L_1$, $H = H_1$

This result heavily depends on the lattice structure:

Th. $\mathcal{H}$ - dB space, $\mathcal{H}_1, \mathcal{H}_2$ are dB subspaces of $\mathcal{H}$ (function E, E_1, E_2 have no real zeros). Then either $\mathcal{H}_1 \subset \mathcal{H}_2$ or $\mathcal{H}_2 \subset \mathcal{H}_1$.

A. $L = L_1$

Recall that $\quad J\dot{M}(x, 0) = \int_0^x H(t)\,dt$

and so $L = \dot{\varphi}_+(L, 0) - \dot{\theta}_-(L, 0)$.

Lemma. $\dot{\varphi}_+(L, 0)$ depends only on Θ_+, Θ_-.

Proof. Here we reproduce a part of the proof of the theorem about reconstruction of the monodromy matrix from its first column. Assume for the moment that $\Theta = \begin{pmatrix} \Theta_+ \\ \Theta_- \end{pmatrix}$ comes from some canonical system with the

Hamiltonian H and let $\mathcal{U}$ be the correspon-
ding generalized Fourier transform. If $\varphi = \begin{pmatrix}\varphi_+ \\ \varphi_-\end{pmatrix}$
is the second column of the monodromy matrix,
then we computed:

$$\left(\mathcal{U}\,\varphi_z\right)(\omega) = \frac{1}{\sqrt{\pi}} \cdot \frac{\varphi^T(x,z)\,\mathcal{J}\Theta(x,\omega) - 1}{\omega - z} =$$

$\varphi_z(x) = \varphi(x,z)$

$$= \frac{\varphi_-(x,z)\Theta_+(x,\omega) - \varphi_+(x,z)\Theta_-(x,\omega) - 1}{\sqrt{\pi}\,(\omega - z)}.$$

Let us compute $\langle \varphi_z, \varphi_0 \rangle_H$ in two different ways.
One one hand, by
$M^*(x,\lambda)\,\mathcal{J}\,M(x,z) - \mathcal{J} = (z - \bar{\lambda}) \int_0^x M^*(t,\lambda) H(t) M(t,z)\,dt$
we get for the entry 22:

$$-\varphi_+(x,\bar{\lambda})\,\varphi_-(x,z) + \varphi_-(x,\bar{\lambda})\,\varphi_+(x,z) =$$
$$= (z-\bar{\lambda}) \int_0^x \left(H(t)\varphi_z(t), \varphi_\lambda(t)\right) dt = (z-\bar{\lambda}) \langle \varphi_z, \varphi_\lambda \rangle_H; \quad \text{if } x = L$$

Taking $\lambda = 0$ $\left(\varphi_+(x,0) = 0,\ \varphi_-(x,0) = 1\right)$
we get $\dfrac{\varphi_+(L,z)}{z} = \langle \varphi_z, \varphi_0 \rangle_H$.

On the other hand,

$$\langle \varphi_z, \varphi_0 \rangle_H = \left(\mathcal{U}\varphi_z, \mathcal{U}\varphi_0\right)_{\mathcal{H}(E)} =$$
$$= \frac{1}{\pi} \left(\frac{\varphi^T(L,z)\,\mathcal{J}\Theta(L,t) - 1}{t - z}, \frac{\Theta_+(L,t) - 1}{t} \right)_{\mathcal{H}(E)}$$

$$= \frac{1}{\pi}\left(\frac{\varphi^T(L,z)\, J\, (\Theta_L(t) - \Theta_L(z))}{t-z}, \frac{\Theta_+(L,t)-1}{t} \right)_{\mathcal{H}(E)}$$

We used that $\varphi^T(L,z) J \Theta(L,z) = \det M(L,z) = 1$

Equating these two expressions we get

$$\frac{\varphi_+(L,z)}{z} = \varphi^T(L,z) J G(z),$$

Where $$G_\pm(z) = \frac{1}{\pi}\left(\frac{\Theta_\pm(L,t) - \Theta_\pm(L,z)}{t-z}, \frac{\Theta_+(L,t)-1}{t} \right)_{\mathcal{H}(E)}.$$

This is a system on φ_+, φ_-, which we can solve. Now, if we only have Θ and no canonical system, we still can use the same formulas for φ_+, φ_- and then show that the corresponding matrix $[\Theta, \varphi]$ will be J-contractive with required properties. See details in [Romanov].

Thus, L depends only on Θ_+, Θ_-, and so $L = L_1$.

Before we proceed, we need one important remark.

<u>Rem.</u> Let (H, L) be a canonical system, $E_t(z) = \Theta_+(t,z) \pm i\Theta_-(t,z)$ be the coresp. HB functio if <u>t is not an interior point</u> of some indivisible interval, then $\mathcal{H}(E_t) \subset \mathcal{H}(E_L)$ isometrically.

Indeed, we may consider $H|_{[0,t]}$. In the space $\mathcal{H}$, associated with H we may consider the subspace $\mathcal{H}_t$ of functions in $\mathcal{H}$ which are zero on $(t, L]$. Then, if $\mathcal{U}$ is the GFT associated with H, then $\mathcal{U}|_{\mathcal{H}_t}$ is the GFT associated with $H|_{[0,t]}$.

$\mathcal{U}$ is unitary in $\mathcal{H} \Rightarrow \mathcal{U}(\mathcal{H}_t)$ is a subspace in $\mathcal{H}(E_L)$, embedded isometrically. ($\mathcal{U}(\mathcal{H}_t) = \mathcal{H}(E_t)$)

Why our condition (t - not an interior point) is essential? Because otherwise

<u>$\mathcal{H}_t$ is not a subspace in $\mathcal{H}$.</u>

B: $H = H_1$

<u>Proof of the theorem:</u>

$E = E_L$, $E^1 = E^1_L$

Step 1

$E_t(z) = \Theta_+(t,z) + i\Theta_-(t,z)$

$E^1_t(z) := \Theta^1_+(t,z) + i\Theta^1_-(t,z)$

If t is not interior point for an indivis. interval for H, then $\mathcal{H}(E_t) \subset \mathcal{H}(E) = \mathcal{H}(E^1)$.

Thus, $\mathcal{H}(E_t)$ is a subspace in $\mathcal{H}(E^1)$.

Let $\mathcal{M} = [0, L] \setminus \cup I_j$, $\mathcal{M}_1$ - analogously.

I_j - H-indiv.

Let $t \in \mathcal{M}_1$, then $\mathcal{H}(E^1_t) \subset \mathcal{H}(E^1)$ isometrically. Put (we write $\mathcal{H}^1_t = \mathcal{H}(E^1_t)$, $\mathcal{H}_x = \mathcal{H}(E_x)$)

$$x(t) = \sup\{x \in \mathcal{M} : \mathcal{H}_x \subset \mathcal{H}^1_t\}$$

$\mathcal{M}$ is closed, so $x(t) \in \mathcal{M}$. We show:

$$\boxed{\mathcal{H}_{x(t)} = \mathcal{H}^1_t.}$$

<u>Exerc.</u> Let $\{\mathcal{H}_\alpha\}_{\alpha\in A}$ be a family of dB subspaces in a dB space $\mathcal{H}$. Then $\bigcap_\alpha \mathcal{H}_\alpha$ is a dB subspace (unless it is zero) and $\text{Clos}(\bigcup_\alpha \mathcal{H}_\alpha)$ is a dB subspace in $\mathcal{H}$. Use axiomatic description.

$\mathcal{H}_{x(t)}^{I} \subset \mathcal{H}_t^1$ is easy. Choose a sequence of $x_n \in M$ such that $x_n < x(t)$, $x_n \to x(t)$.

If $x(t)$ is the element of $\{x\in M: \mathcal{H}_x \subset \mathcal{H}_t\}$ then trivial.

It follows from the fact that

$$\mathcal{H}_{x_n} = \mathcal{U}\, \mathcal{H}_{H|[0,x_n]} \quad \text{and}$$

$$\text{Clos}\Big(\bigcup_n \mathcal{H}_{H|[0,x_n]}\Big) = \mathcal{H}_{H|[0,x(t)]}.$$

$\mathcal{H}_t^1 \subset \mathcal{H}_{x(t)}$. $\quad \mathcal{N} := \bigcap_{x > x(t)} \mathcal{H}_x \supset \mathcal{H}_1^t$, since

$\mathcal{H}_t^1 \subset \mathcal{H}_x$ as $x > x(t)$ $\left(\begin{array}{l}\mathcal{H}_t^1 \subset \mathcal{H}_x, \\ \text{or } \mathcal{H}_x \subset \mathcal{H}_t^1\end{array}\right)$ for $x > x(t)$ second is impossible.

If $\mathcal{H}_{x(t)} = \mathcal{N}$, we are done.

Otherwise $\mathcal{H}_{x(t)}$ is a proper subspace of $\mathcal{N}$.

$$x_* := \inf\{M \cap (x(t), L]\}, \quad x_* \neq x(t)$$

(otherwise use the argument on the side of $\mathcal{H}_H$). Then $(x(t), x^*)$ is an indivisible interval,

$\mathcal{N} = \mathcal{H}_{x_*}$ and so $\dim \mathcal{N} \ominus \mathcal{H}(x,t) = 1$.

Thus, $\mathcal{H}_t^1 = \mathcal{H}_{x(t)}$ or $\mathcal{H}_t^1 = \mathcal{N}$.

In the latter case $x(t)$ is not the supremum of $\{x \in M: \mathcal{H}_x \subset \mathcal{H}_t^1\}$.

Step 2. Since $\mathcal{H}_{x(t)}$ and $\mathcal{H}^1_t$ coincide as dB spaces, they have the same reproducing kernels. The repr. kernel at $\lambda=0$ for $\mathcal{H}_x$ is

$$\frac{i}{2\pi}\,\frac{\Theta_+(x,z)\overline{\Theta_-(x,0)}-\Theta_-(x,z)\overline{\Theta_+(x,0)}}{z}=\frac{1}{2\pi i}\,\frac{\Theta_-(x,z)}{z}.$$

(with $\overline{\Theta_-(x,0)}=0$, $\overline{\Theta_+(x,0)}=1$)

Thus, $\Theta_-(x(t),z)=\Theta_-^1(t,z)$. Fix some $\lambda\neq 0$.

$$\Theta_+(x(t),z)\overline{\Theta_-(x(t),\lambda)}-\Theta_-(x(t),z)\overline{\Theta_+(x(t),\lambda)}=$$
$$=\Theta_+^1(t,z)\overline{\Theta_-^1(t,\lambda)}-\Theta_-^1(t,z)\overline{\Theta_+^1(t,\lambda)}.$$

Thus, $\Theta(x(t),z)=\begin{pmatrix}1 & a_t\\ 0 & 1\end{pmatrix}\Theta^1(t,z)$

a_t depends only on t (and λ) but not on z.
We show that $a_t\equiv 0$.
By construction $x(t)$ is monotonically increasing on $\mathcal{M}_1$. Extend x by linearity to each interval (indivisible) I_j such that $\bigcup_j I_j=[0,L]\setminus\mathcal{M}_1$.

<u>Note:</u> If I is indivisible interval, then $\Theta(x,z)$ grows linearly on $x\in I$ (for fixed z).

So $\Theta_-(x(t),z)=\Theta_-^1(t,z)$ <u>$\forall t\in[0,L]$</u>.

Step 3. Let us differentiate the last equality over z twice and take $z=0$.

$$\ddot{\Theta}_-(x(t),0)=\ddot{\Theta}_-^1(t,0)$$

Now we claim that $x(t)$ is absolutely continuous on $[0,L]$ (for details see [Romanov]).
Then we can differentiate w.r. to t!

$$\ddot{\Theta}_-(x(t),0) = \ddot{\Theta}_-(t,0)$$

$$x'(t)\,\ddot{\Theta}_-'(x(t),0) = \left(\ddot{\Theta}_-^1\right)'(t,0) \quad \text{a.e. } t\in[0,L]$$

<u>Lemma.</u> If $\Psi(x,\lambda)$ is the solution to $J\Psi' = zH\Psi$ with $\Psi(0) = \begin{pmatrix}1\\0\end{pmatrix}$ then

$$\ddot{\Psi}_{1-}'(x,0) = 2\left(\dot{\Psi}_{1-}'(x,0)\dot{\Psi}_{1+}(x,0) - \dot{\Psi}_{1+}'(x,0)\dot{\Psi}_{1-}(x,0)\right).$$

<u>Proof</u>: Let $M(x,\lambda)$ be the fund. solution.

$$JM' = \lambda HM \Longrightarrow J\dot{M}' = (HM + \lambda H\dot{M})^{\cdot} =$$
$$= \lambda H\ddot{M} + 2H\dot{M}$$

$$J\ddot{M}'\big|_{\lambda=0} = 2H\dot{M}\big|_{\lambda=0} =$$

/Recall that $J\dot{M}(x,0) = \int_0^x H(s)\,ds$, so

$$J\dot{M}'(x,0) = H(x)$$ /

$$= 2J\dot{M}'(x,0)\dot{M}(x,0)$$

$$\ddot{M}'(x,0) = 2\dot{M}'(x,0)\dot{M}(x,0) \quad \text{Take 21 entry}$$

But recall: $J\dot{M}(0) \geq 0 \Rightarrow T_2\dot{M}(0) = 0.$

$$\begin{pmatrix} \dot{\Psi}_{1+}'(x,0) & ? \\ \dot{\Psi}_{1-}'(x,0) & -\dot{\Psi}_{1+}'(x,0) \end{pmatrix}\begin{pmatrix} \dot{\Psi}_{1+}(x,0) & ? \\ \dot{\Psi}_{1-}(x,0) & ? \end{pmatrix}.$$

<u>Step 4.</u> $-\dot{\Theta}_-(x,0) = \int_0^x H_{11}(s)\,ds \Rightarrow \dot{\Theta}_-(x,0)$ is nonzero (unless $H = \begin{pmatrix}0&0\\0&1\end{pmatrix}$ on some $(0,\varepsilon)$, but we excluded this possibility).

Then $$\ddot{\Theta}_-(x(t),0) = -2\left(\dot{\Theta}_-(x(t),0)\right)^2\left(\frac{\dot{\Theta}_+(x(t),0)}{\dot{\Theta}_-(x(t),0)}\right)'.$$

Since $\mathring{\Theta}_-(x(t), z) = \mathring{\Theta}_-^1(t, z)$, we have:

$$x'(t)\left(\frac{\mathring{\Theta}_+(x(t), z)}{\mathring{\Theta}_-(x(t), z)}\right)' = \left(\frac{\mathring{\Theta}_+^1(t, z)}{\mathring{\Theta}_-^1(t, z)}\right)'.$$

$$\frac{d}{dt}\left(\frac{\mathring{\Theta}_+(x(t), z)}{\mathring{\Theta}_-(x(t), z)} - \frac{\mathring{\Theta}_+^1(t, z)}{\mathring{\Theta}_-^1(t, z)}\right) = 0.$$

Thus, this is a constant on $[0, L]$.

Note that for $t \in \mathcal{M}_1$ the value of this function is exactly a_t:

$$\begin{cases} \Theta_+(x(t), z) = \Theta_+^1(t, z) + a_t \Theta_-^1(t, z) \\ \Theta_-(x(t), z) = \Theta_-^1(t, z) \end{cases}$$

Take derivative in z and divide by $\mathring{\Theta}_-^1(t, z)$

This is true for all $[0, L]$ since on an indivisible interval Θ, Θ^1 grow linearly in t.

Taking $t = L$ $(x(L) = L)$ we see that $a_t \equiv 0$.

Thus, $\Theta(x(t), z) = \Theta^1(t, z)$. By the explicit formulas for the second column we see that $M(x(t), z) = M^1(t, z)$, $t \in [0, L]$

$$x'(t) \cdot M'(x(t), z) = (M^1)'(t, z)$$

whence

$$x'(t) H(x(t)) M(x(t), z) = H_1(t) M^1(t, z)$$

$$x'(t) H(x(t)) = H_1(t)$$

Taking the trace we get $x'(t) = 1$ a.e. in t

so $x(t) = t$, and $H(t) = H_1(t)$.

Remark about constructivity:

In our solution of the inverse problem we had to choose subsequences. It was not clear whether different $\{E_{N_k}\}$, $\{H_{N_k}\}$ $\{F_{N_k}\}$ could lead to different solutions. But now we know that the solution is unique. Thus, the limiting functions F $(H = F')$ may differ only by the constant. But $F_N(0) = 0$, so $F = 0$, and, thus, F is unique. We conclude that the initial system $\{F_N\}$ itself converges in $C[0,L]$. Thus, the solution is actually constructive

Borg theorem. $q \in L^1(0,1)$, real, $h, h_1 \in \mathbb{R}$

$$u'(0) = h u(0), \quad u'(1) = h_1 u(1)$$

$$-u'' + qu = \lambda u \qquad \sigma(q, h, h_1) = \{\lambda \in \mathbb{R} : \text{there is a nontrivial solution}\}$$

Th. If $\sigma(q, h, h_1) = \sigma(\tilde{q}, \tilde{h}, \tilde{h}_1)$
$\sigma(q, h, h_2) = \sigma(\tilde{q}, \tilde{h}, \tilde{h}_2)$, $h_1 \neq h_2$,

then $\tilde{q} = q$.

Idea: pass to canonical systems:

$$H = \begin{pmatrix} y_1^2 & y_1 y_2 \\ y_1 y_2 & y_2^2 \end{pmatrix} \qquad \begin{matrix} y_1(0) = y_2'(0) = 1 \\ y_1'(0) = y_2(0) = 0 \end{matrix}$$

We get E and $\tilde{E}_1$

Equality of spectra $\Leftrightarrow$ $Z_{\Theta_+} = Z_{\tilde{\Theta}^1_+}$

$Z_{\Theta_-} = Z_{\tilde{\Theta}^1_-}$

From this (and regularity) $E = \tilde{E}_1$, $H = \tilde{H}_1$.
and finally $q = \tilde{q}$