In the equality

$$M^*(x,\lambda)J\,M(x,z) - J = (z-\bar\lambda)\int_0^x M^*(t,\lambda)H(t)M(t,z)\,dt$$

put $\lambda=0$ (note that $M(x,0)\equiv I$):

$$J M(x,z) - J = z\int_0^x H(t)M(t,z)\,dt$$

Differentiate w.r.to z at $z=0$:

$$J\dot M(x,0) = \int_0^x H(t)\underbrace{}_{M(t,0)=I}dt \geq 0$$

and has trace x

$$\begin{pmatrix} -\dot\Theta_-(x,0) & -\dot\Phi_-(x,0) \\ \dot\Theta_+(x,0) & \dot\Phi_+(x,0)\end{pmatrix}$$

<u>Cor.</u> $\dot\Theta_-(x,0)<0$, $\dot\Phi_+(x,0)>0$, $\operatorname{tr} J\dot M(x,0) = \dot\Phi_+(x,0) - \dot\Theta_-(x,0) = x$.

<u>Chapter 3.</u> Inverse spectral theory in the regular case.

<u>Plan</u>: 1. Solve the problem on one indivisible interval
2. Solve the problem for polynomial HB-functions
3. Solve the problem in the general case using polynomial approximations

<u>I. Chain rule.</u> $M(x,z)$ - fundamental solution

Let $0<a<b$ and let $N(x,z)$ be the solution

of $\quad JN' = zHN$ on (a,b) such that $N(a,z) = I$. Then

$$\boxed{M(b,z) = N(b,z)M(a,z)}$$

Proof: Consider $\widetilde{M}(x,z) = N(x,z)M(a,z)$

Then $J\widetilde{M}'(x,z) = JN(x,z)M(a,z) =$

$= zHN(x,z)M(a,z) = zH\widetilde{M}$

$\widetilde{M}(a,z) = M(a,z) \Longrightarrow \widetilde{M}(x,z) = M(x,z)$ everywhere on (a,b), $\widetilde{M}(b,z) = M(b,z)$

∏ One interval: $\quad I = (0,a)$, $e \in \mathbb{R}^2$, $\|e\| = 1$, $e = \begin{pmatrix} e_+ \\ e_- \end{pmatrix}$

$H(x) = (\cdot, e)e$, $x \in I$.

Then $M(x,\lambda) = I - \lambda x(\cdot, e)Je$.

Indeed $JM'(x,\lambda) = -J\lambda(\cdot, e)Je = \lambda(\cdot,e)e =$

$= \lambda H M(x,\lambda)$

Since $HJe = 0$.

Thus, $M(\lambda) = I - \lambda a(\cdot, e)Je =$

$$= I - \lambda a \begin{pmatrix} -e_+e_- & -e_-^2 \\ e_+^2 & e_+e_- \end{pmatrix} = I + \lambda \underbrace{\begin{pmatrix} ae_+e_- & ae_-^2 \\ -ae_+^2 & -ae_+e_- \end{pmatrix}}_{R}$$

The matrix R has the properties:

$R^2 = 0$, $R_{12} \geq 0$ $R_{21} \leq 0$. Conversely for any nonzero matrix R with these 3 properties $\exists$ a and e such that $I + \lambda R$ is the monodromy for H on $(0,a)$. Take $a = R_{12} - R_{21}$, $e_- = \sqrt{\frac{R_{12}}{a}}$

$e_+ = \operatorname{sign} R_{11} \sqrt{-R_{21}/a}$

III. Solution in the polynomial case

Let E be HB polynomial of degree n, $E(0)=1$, $E(t)\neq 0$ on $\mathbb{R}$. Then $\mathcal{H}(E)=\mathcal{P}_{n-1}$.

Consider $X=\mathcal{P}_{n-2}\subset\mathcal{H}(E)$. X is a de Branges subsp.

Hence $X=\mathcal{H}(E_1)$ with $E_1(z)=c(z-\bar{a})K_a^X(z)$, where $a\in\mathbb{C}^+$. We will show that E_1 can be expressed via E.

Important property: If $\exists\alpha: F=e^{i\alpha}E-e^{-i\alpha}E^*\in\mathcal{H}(E)$, then $\forall f\in\mathcal{H}(E)$ such that $zf\in\mathcal{H}(E)$ we have $f\perp F$.

Proof: $t_n\in Z_F \Rightarrow \dfrac{F(z)}{z-t_n}=c_n K_{t_n}(z)$

Then $(f,F)_E=\left(f(z-t_n),\dfrac{F}{z-t_n}\right)=0.$

Exerc. If $F\perp\underbrace{\{f\in\mathcal{H}(E): zf\in\mathcal{H}(E)\}}_{\mathcal{D}(M_z)}$, then $\exists\alpha: F=e^{i\alpha}E-e^{-i\alpha}E^*$.

$zX\subset\mathcal{H}(E)\Longrightarrow X\perp F=e^{i\alpha}E-e^{-i\alpha}E^*$ (for some α)

Clearly X has codim 1 in $\mathcal{H}(E)$.

Hence $P_X f=f-\dfrac{1}{\|F\|^2}(f,F)F$.

$K_w^X=P_X K_w=K_w-\dfrac{\overline{F(w)}}{\|F\|^2}F$

$$\underbrace{(z-\bar{a})K_a^X}_{\frac{1}{c}E_1(z)}=\frac{i}{2\pi}\Big(E(z)\overline{E(a)}-E^*(z)\overline{E^*(a)}-(z-\bar{a})\frac{\overline{F(w)}}{\|F\|^2}\big(e^{i\alpha}E-e^{-i\alpha}E^*\big)\Big).$$

We conclude $E_1(z) = (cz+d)E(z) + (c_1 z + d_1)E^*(z)$

$E_1(0) \neq 0$ (otherwise all functions on X, and vanish at 0)

so we may divide by $E(0)$.

So let, E, E_1 as before, Θ and Θ_1

$E = \Theta_+ + i\Theta_-$, $E_1 = \Theta_{1+} + i\Theta_{1-}$

Then $\Theta_1(\lambda) = (\Lambda_0 + \lambda\Lambda_1)\Theta(\lambda)$, where Λ_0, Λ_1 are constant matrices with real entries.

<u>Prop.</u> Let $n \geq 2$. Then a) Λ_0 is invertible

b) $S := \Lambda_0^{-1}\Lambda_1 = \begin{pmatrix} s & s_1 \\ s_2 & -s \end{pmatrix}$

Where $s^2 + s_1 s_2 = 0$, $s_2 \geq 0$, $s_1 \leq 0$

c) Let $\Psi = \Lambda_0^{-1}\Theta_1$. Then

$$\Theta(\lambda) = (I - \lambda S)\Psi(\lambda)$$

and $\Psi_+ + i\Psi_-$ is a Hermite-Biehler funct.

<u>Proof:</u> a) Θ_1 Θ-pols of degree $n-1$ and n respectively

Let η_n, η_{n-1} be coeff. of the vector polynomial Θ at $\lambda^n, \lambda^{n-1} \in \mathbb{R}^2$. Then $\eta_n \neq 0$

$$\begin{cases} \Lambda_1\eta_n = 0 \\ \Lambda_1\eta_{n-1} + \Lambda_0\eta_n = 0 \end{cases}$$

Then $-\Lambda_1\eta_{n-1} = \Lambda_0\eta_n$. If this is nonzero vector, then we see that $\operatorname{Ran}\Lambda_1 \subset \operatorname{Ran}\Lambda_0$

η_{n-1}, η_n - ЛНЗ

If $\operatorname{Ran} \Lambda_1 = \operatorname{Ran} \Lambda_0$, then $\Theta_1(\lambda) = p(\lambda) e$,
where p is a scalar pol-l, $e \in \mathbb{R}^2$ constant vector.

Wrong since $\Theta_{1+} + i\Theta_{1-} \in HB$.

Thus, $\operatorname{Ran} \Lambda_0 = 2$, i.e. Λ_0 invertible.

<u>What if</u> $-\Lambda_1 \eta_{n-1} = \Lambda_0 \eta_n = 0$?

Then $\eta_n \in \operatorname{Ker} \Lambda_0, \operatorname{Ker} \Lambda_1$, let $g \in \mathbb{R}^2, g \perp \eta_n, g \neq 0$.

Then $(\Lambda_0 + \lambda \Lambda_1) u = (u, g) \times$(linear vector function of λ)

Thus, $\Theta_1(\lambda) = (\Lambda_0 + \lambda \Lambda_1)\Theta(\lambda) = \underset{\overset{\|}{(\Theta(\lambda)\,g)}}{p(\lambda)} \cdot$(linea funt of λ)

$\Theta_1(\lambda) \neq 0$, Hence $p(\lambda)$ is a nonzero constant

Then all coefficients of $\Theta(\lambda)$ are orthogonal to g with numbers >0.

$\Theta(\lambda) = q(\lambda)\eta_n + \Theta(0)$ for a scalar pol. $q(\lambda)$

$\Theta(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Hence $\dfrac{\Theta_+}{\Theta_-} = \dfrac{q(\lambda)\eta_{n+} + 1}{q(\lambda)\eta_{n-}} =$

$= \underbrace{\dfrac{\eta_{n+}}{\eta_{n-}}}_{\in \mathbb{R}, \text{ const}} + \dfrac{1}{\eta_{n-} q(\lambda)}$ is a Herglotz function.

But $\dfrac{c}{q}$ is Hergl. for a pol q only if $\deg q \leq 1$.

Then $\deg \Theta = 1$, !!!

B) $S\eta_n = 0$, $S\eta_{n-1} = -\eta_n$. Hence $\underbrace{\det S = \operatorname{tr} S = 0}_{S^2 = 0}$

(S is similar to $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$) $(I + \lambda S)^{-1} = (I - \lambda S)$

$\Theta_1 = (\Lambda_0 + \lambda \Lambda_1)\Theta$ multiply by Λ_0^{-1} to get.

$= \Lambda_0^{-1}\Theta_1 = (I + \lambda S)\Theta$

We need to show $S_2 \geq 0$.

$E = \Theta_+ + i\Theta_- \in HB \iff \Im \frac{\Theta_+}{\Theta_-} > 0 \iff \Im(\Theta_+ \overline{\Theta_-})$

$$(J\Theta, \Theta) = \begin{pmatrix} -\Theta_- & \Theta_+ \\ \Theta_+ & \Theta_- \end{pmatrix} = -\Theta_- \overline{\Theta_+} + \Theta_+ \overline{\Theta_-} = 2i \Im(\Theta_+ \overline{\Theta_-}) =$$
$$= 2i \Im\left(|\Theta_-|^2 \frac{\Theta_+}{\Theta_-}\right)$$

$\cancel{\Theta_+}$ $$0 < \frac{1}{i}(J\Theta, \Theta) = \frac{1}{i}\left((J - \lambda JS)\Psi, (\;\; - \lambda S)\Psi\right) =$$
$$= \frac{1}{i}(J\Psi, \Psi) + \frac{\bar\lambda - \lambda}{i}(JS\Psi, \Psi)$$

$$-(\lambda JS\Psi, \Psi) - \bar\lambda(J\;\;, S\Psi)$$

$S = \begin{pmatrix} +s & s_1 \\ s_2 & -s \end{pmatrix} \Rightarrow JS$ - ~~self~~adj and $S^* J S = 0$ since S is of rank 1.

$(J\Psi, \Psi) = O(|\lambda|^{2n-2})$ Let Ψ_{max} - coeff. at λ^{n-1} in Ψ.

So, $\ldots = -2 \Im\lambda \, |\lambda|^{2n-2} (JS\Psi_{max}, \Psi_{max}) + o(|\lambda|^{2n-1})$

Thus, $(JS\Psi_{max}, \Psi_{max}) \leq 0$ $\quad \lambda = i\tau \to \infty$

JS - selfadj $\det JS = 0$ Hence $JS \leq 0$, or $JS \geq 0$.

If $(JS\Psi_{max}, \Psi_{max}) = 0$, then $JS\Psi_{max} = 0$

But $S\Psi_{max} = -\eta_n \neq 0$.

Thus, $(JS\Psi_{max}, \Psi_{max}) < 0$ and so $JS \leq 0$.

$JS = \begin{pmatrix} -s_2 & s \\ s & s_1 \end{pmatrix}$ so $s_2 \geq 0$, $s_1 \leq 0$.

---

c) It remains to show that $\Psi_+ + i\Psi_- \in HB$.

$\Psi = \Lambda_0^{-1}\Theta_1$ $\quad \Psi_+ = \alpha\Theta_{1+} + \beta\Theta_{1-}$

$\Psi_- = \gamma\Theta_{1+} + \delta\Theta_{1-}$

$$\frac{\Psi_+}{\Psi_-} = \frac{\alpha\Theta_{1+} + \beta\Theta_{1-}}{\gamma\Theta_{1+} + \delta\Theta_{1-}} = \frac{\alpha\frac{\Theta_{1+}}{\Theta_{1-}} + \beta}{\gamma\frac{\Theta_{1+}}{\Theta_{1-}} + \delta} =$$

$$= \frac{\alpha}{\gamma} + \frac{\beta - \frac{\alpha\delta}{\gamma}}{\gamma\left(\frac{\Theta_{1+}}{\Theta_{1-}} + \delta\right)} = \frac{\alpha}{\gamma} + \frac{\overbrace{(\gamma\beta - \alpha\delta)}^{= -\det \Delta_0}}{\gamma^2\left(\frac{\Theta_{1+}}{\Theta_{1-}} + \delta\right)}$$

So $\frac{\Psi_+}{\Psi_-}$ or $-\frac{\Psi_+}{\Psi_-}$ is a Herglotz depending on the sign of $\det \Delta_0$

We want to show that $\frac{\Psi_+}{\Psi_-}$ is Hergl.

Let $S = W\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} W^{-1}$ W const real matrix

$$W^{-1}\Psi(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} W^{-1}\Theta(\lambda)$$

$$\frac{(W^{-1}\Psi)_+}{(W^{-1}\Psi)_-} = \frac{(W^{-1}\Theta)_+}{(W^{-1}\Theta)_-} + \lambda$$

If $\det W > 0$, then RHS is Herglotz and so is Ψ_+/Ψ_-. If $\det W < 0$, then RHS $= \lambda -$ Herglotz. Assume that $-\Psi_+/\Psi_-$ is Hergl. then LHS = Hergl. Hence Hergl + Hergl $= \lambda$, then both functions are linear, contradiction with $n \geq 2$.

By const. $E_\Psi = \Psi_+ + i\Psi_-$ is HB pol of degree $n-1$, $E_\Psi(0) = 1$, since $\Psi(0) = \Theta(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Repeat with E_Ψ and we obtain

$$\Theta(\lambda) = (I + \lambda R_1) \dots (I + \lambda R_{n-1})\Theta_{n-1},$$ where

each of R_j has the form $\begin{pmatrix} \rho & \nu \\ \tau & -\rho \end{pmatrix}$, $\rho^2 + \nu\tau = 0$, $\tau \leq 0$, $\nu \geq 0$

as on the solution on one interval.

If Θ_{n-1} is of deg 1, $\Theta_{n-1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} a \\ b \end{pmatrix}$

$\Theta_{n-1}^{+} + i\Theta_{n-1-} \in HB$ so $b < 0$, $\Theta_{n-1} = I + \lambda\begin{pmatrix} a & -a^2/b \\ b & -a \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Th. Let $E = \Theta_+ + i\Theta_- \in HB$, E-pol, $\deg E = n$, $E(0) = 1$. Then there exist matrices $M_j = I + \lambda R_j$ such that $\det R_j = \operatorname{tr} R_j = 0$, $(R_j)_{12} \geq 0$, $(R_j)_{21} \leq 0$

and

$$\boxed{\Theta(\lambda) = M_1(\lambda) \dots M_n(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

Now invoke chain property.

For each M_j $\exists$ L_j, H_j - Hamiltonian on an interval (a_j, b_j) of length L_j (constant of rank 1), such that $M_j(\lambda)$ is the monodromy matrix: i.e. if $M_j(x,\lambda)$ is the solution of

$$\mathcal{J} M_j'(x,\lambda) = \lambda H M(x,\lambda) : M_j(a_j,\lambda) = I$$

and $M_j(\lambda) = M_j(b_j, \lambda)$

Then defining $H = H_j$ on (a_j, b_j) (assuming $b_j = a_{j+1}$) we get that $\Theta(\lambda) = M(b_n, \lambda)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Thus, our E is a dB function for some Hamiltonian consisting of n indivisible intervals.

IV. Solution in the general case – approximation argument.

Let $E = \Theta_+ + i\Theta_- \in HB$, $E(0)=1$, $\underline{E\text{-regular}}$, $E \neq 0$ on $\mathbb{R}$.

We want to find the Hamiltonian H, on $(0,L)$, such that $\Theta_L = \Theta$.

We will use existence of solutions in the polynomial case and approximate our function by polynomials.

Let $\{t_j\}$ be zeros of Θ_- (note that $t_0 = 0 \in Z_{\Theta_-}$)

$$\Theta_-(z) = \underbrace{\dot{\Theta}_-(0)}_{=:C} z \left(\prod_j \left(1 - z/t_j\right) e^{z/t_j}\right) \cdot e^{az} \qquad a \in \mathbb{R}$$

Put
$$\Theta_N^-(z) = C z \prod_{|t_j| \le N} \left(1 - z/t_j\right)$$

Then
$$\Theta_-(z) = \Theta_N^-(z) \, S_N(z) R_N(z)$$
$$S_N = e^{az + \sum_{|t_j| \le N} z/t_j}, \qquad R_N = \prod_{|t_j| > N} \left(1 - z/t_j\right) e^{z/t_j}$$

$R_N \rightrightarrows 1$ on compact sets in $\mathbb{C}$, S_N has no zeros, real

How we define Θ_N^+?

Let
$$\frac{\Theta_+(z)}{\Theta_-(z)} = \sum_j p_j \left(\frac{1}{z - t_j} + \frac{t_j}{1 + t_j^2} \right) + a + bz$$

$$\Theta_N^+(z) := \left(\sum_{|t_j| \le N} \ldots\ldots + a + bz \right) \Theta_N^-(z).$$

Then $E_N = \Theta_N^+ + i\Theta_N^-$ is a HB pol without real zeros and $E_N(0) = 1$ (since $p_0 = 1/\dot{\Theta}_-(0)$)

For E_N we have a canonical system (H_N, L_N).

Let M_N be the monodromy matrix for H_N,

$M_N(z) = M_N(L_N, z)$.

Fix $h \in (0, L_N)$ and consider the monodromy matrix M_N^h: $J(M_N^h)' = zH_N M_N^h$ on $(L_N - h, L_N)$,

and $M_N^h(L_N - h, z) = I$. $M_N^h(z) = M_N^h(L_N, z)$

Then $M_N(z) = M_N^h(z) M_N(L_N - h, z)$

Chain rule.

In what follows we will choose subsequences always keeping the same index N.

Fix $\ell > 0$: $L_N \geq \ell$. We can always assume this, since we can always add on the left an interval of any length where $H = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. This does not change entire function corresp. to H_N.

Now let $F_N(x) = \int_{L_N - h}^{L_N - h + x} H_N(s)\,ds$

If $h \in [0, \ell]$, then $F_N \in C[0, h]$, $\|F_N(x)\| \leq x$ ($\|H_N\| \leq 1$).

$\|F_N(x+\delta) - F_N(x)\| \leq \delta$).

Thus, $\exists\, F_N$ such that $F_N \rightrightarrows F$ on $[0, h]$
equicontinuous

F_N are uniform Lipschitz $\Rightarrow$ F is ~~unif~~ Lipschitz (abs. cont.)

for a.e. $x \in (0, h]$ $\exists\, F'(x) \geq 0$ (each of F_N is monotone w.r.p. to x in matrix sense)

$\mathrm{tr}\, F'(x) = 1$. Thus F' will be our Hamiltonian!

We have $\|M_N^h(z)\| \le e^{a|z|}$, $0<h<L_N$
(proved in first lecture).

By the Montel theorem, $\exists\ M_{N_k}$:

$M^h_{N_k} \rightrightarrows M_h$ uniformly on compacts on $\mathbb{C}$.
M_h – entire matrix function

<u>Proposition</u>

Let us show that M_h is the monodromy matrix of a canonical system on $(0,h)$ with Hamiltonian F!

<u>Proof</u>.
$$JM_N(L_N-h, x+\delta) - JM_N(L_N-h, x) =$$
$$= z\int_{L_N-h+x}^{L_N-h+x+\delta} H_N(s)M_N^h(s)\,ds$$

$$JM_N^h(x+\delta, z) - JM_N^h(x,z) = z\int_{L_N-h+x}^{L_N-h+x+\delta} H_N(s)M_N^h(s)\,ds$$

$$\Rightarrow \|M_N^h(x+\delta, z) - M_N^h(x,z)\| \le |z|\int_{L_N-h+x}^{L_N-h+x+\delta} \|\ldots\| \le \le |z|\, e^{h|z|}\delta$$

Thus, $M_N^h(x,z)$ is equicontinuous and so compact on (L_N-h, L_N)

$M_N^h(x+(L_N-h), z)$ is equicont. on $[0,h]$

So $\exists\ \widetilde{M}(x,z)$: $M_N^h(x+(L_N-h), z) \rightrightarrows \widetilde{M}(x,z)$

<u>$\widetilde{M}$ also is Lipschitz in x</u>

$$JM_N^h(x+L_N-h, z) - J = z\int_0^x H_N(s+L_N-h)\,M_N^h(L_N-h+s, z)\,ds$$
$x\in(0,h)$

Letting $N\to\infty$ (for fixed z)

$$J\widetilde{M}(x,z) - J = z\int_0^x H_N(s+L_N-h)\,\widetilde{M}(s)\,ds + r_N(x)$$

where $\varepsilon_N \rightrightarrows 0$ on $[0,h]$.

$$\int_0^x H_N(s+L_N-h)\tilde{M}(s,z)ds =$$
$$= F_N(x)\tilde{M}(x,z) - \int_0^x F_N(s)\tilde{M}'(s,z)ds \to$$

Now we pass to the limit and integrate back:

$$\to F(x)\tilde{M}(x) - \int_0^x F(s)\tilde{M}'(s,z)ds = \int_0^x F'(x)\tilde{M}(s)ds$$

$$J\tilde{M}(x,z) = z\int_0^x F'(s)\tilde{M}(s,z)ds$$

$\tilde{M}(x,z)$ is the fundamental solution with canonical system (F', h). Note that

$$\boxed{\tilde{M}(h,z) = M_h(z).}$$

$$M_N^h(z) = M_N^h(L_N, z) = M_N^h(h + (L_N - h), z) \to$$

$$M_h(z) \leftarrow \quad \to \tilde{M}(h,z).$$

Thus M_h is the monodromy matrix for (F', h).

$$M_N(z) = M_N^h(z)\, M_N(L_N - h, z)$$

Consider the first column, mult by S_N

$$S_N(z)\Theta_N(z) = M_N^h(z)\, \underbrace{\underbrace{M_N(L_N-h, z)}_{S_N(z)}\begin{pmatrix}1\\0\end{pmatrix}}_{G_N^h(z)}$$

$\Theta_N(z) \xrightarrow{\text{unif on } \mathbb{C}} \Theta(z)$ $\qquad M_N^h(z) \to M_h(z)$ - entire

$\det M_N^h \equiv 1 \Rightarrow (M_N^h)^{-1} \rightrightarrows M_h^{-1}$

Thus, G_N^h converge unif on $\mathbb{C}$ to a real entire function $G_0 = M_0^{-1}\Theta$

Lem. Either $G_h^+ + iG_h^- \in HB$, or $G_h^- \equiv 0$.

Proof. We need a slightly more general formula:

$$M_N(x,z) = M_N^h(x,z)\, M_N(L_N - h, z)$$

Put $x = L_N - h$

$$\frac{\Theta_N^+(L_N - h, z)}{\Theta_N^-(L_N - h, z)} = \frac{[M_N(L_N - h, z)]_{11}}{[M_N(L_N - h, z)]_{12}} = \frac{(G_N^h)^+(z)}{(G_N^h)^-(z)}$$

If $G_h^- \not\equiv 0$, then $\frac{(G_N^h)^+}{(G_N^h)^-} \rightrightarrows \frac{G_h^+}{G_h^-}$ in $\mathbb{C}^+$ (on compacts)

Since $\frac{\Theta_N^+(L_N - h, z)}{\Theta_N^-(L_N - h, z)}$ is Herglotz, we conclude that G_h^+/G_h^- is Herglotz.

Thus, we have shown:

Prop. $\forall\, E \in HB$ without real zeros, $E(0) = 1$ regular, we have seen that $\forall\, h > 0$ $\exists\, (H_h, h)$:

$$\Theta(z) = M_h(z)\, G_h(z)$$

M_h – monodromy matrix for (H_h, h), and either $E_h = G_h^+ + iG_h^- \in HB$ or $E_h = G_h^+$ is real entire without zeros.

We would like to show that $\exists\, h > 0$ such that $E_h \equiv 1$. Then Θ is the first column of M_h and the inverse problem is solved.

Lem. If E_h is real without zeros, then $E_h \equiv 1$.

Proof. E-regular $\Rightarrow \Theta_+, \Theta_-$ are of finite exp type (Krein). Thus, $E_h(z) = Ae^{bz}$, $A=1$.

Let us show that $b=0$

$$\frac{1}{(z+i)E_h} \in H^2 \Rightarrow \int_{-\infty}^{\infty} \frac{|\log|E_h(t)||}{t^2+1}\,dt \asymp \int \frac{|\log\|\Theta(t)\||}{t^2+1}\,dt < \infty$$

M_h - monodromy matrix $\Rightarrow \int \frac{|\log \|M_h^{(t)}\binom{1}{0}\||}{t^2+1}\,dt < \infty$

all functions are regular

If $G_h^+ = e^{bz}$, $G_h^- = 0$!!!

Idea of the proof: Assume that $E_h \in HB$ $\forall h > 0$.

We will show:

1. $\mathcal{H}(E_h)$ is embedded to $\mathcal{H}(E)$ isometr.
2. E_h is regular.
3. G_h is a first column of some monodromy matrix
4. This will lead to a contradiction when h is suff. large.

Each of this is a deep and difficult theorem!

V. Isometric embeddings of de Branges spaces.

Th. Let $E \in HB$, μ a measure on $\mathbb{R}$.

a) Then $\mathcal{H}(E)$ is isometrically embedded onto $L^2(|E|^{-2}d\mu)$ (i.e. $\int \left|\frac{f}{E}(t)\right|^2 d\mu(t) = \int \left|\frac{f}{E}\right|^2 dt$)

iff $\exists A \in H^\infty(\mathbb{C}^+)$, $\|A\|_\infty \le 1$, such that

$$\operatorname{Re} \frac{E + E^* A}{E - E^* A}(z) = \frac{\operatorname{Im} z}{\pi} \int_{\mathbb{R}} \frac{d\mu(t)}{|t - z|^2}, \quad z \in \mathbb{C}^+.$$

Note that if we put $\Theta = E^*/E$, then

$$\operatorname{Re} \frac{1 + \Theta A}{1 - \Theta A} = \frac{1 - |\Theta A|^2}{|1 - \Theta A|^2} > 0 \text{ on } \mathbb{C}^+$$ and so $\exists \mu$ on $\mathbb{R}$ and $p \ge 0$ such that

$$\operatorname{Re} \frac{1 + \Theta A}{1 - \Theta A}(z) = p \operatorname{Im} z + \frac{\operatorname{Im} z}{\pi} \int_{\mathbb{R}} \frac{d\mu(t)}{|t - z|^2}.$$

---

b) If $p > 0$, then $\exists \alpha \in \mathbb{R}$: $\underbrace{e^{i\alpha} E - e^{-i\alpha} E^*}_{S_\alpha} \in \mathcal{H}(E)$

and $\mathcal{H}(E) \ominus \operatorname{Lin}\{S_\alpha\}$ is isometrically contained in $L^2(\mathbb{R}, \frac{d\mu}{|E|^2})$.

This theorem is due to de Branges. However, it has a long history (development).

Exerc. (Relation between $\mathcal{H}(E)$ and K_{Θ_E}).

The mapping $f \longmapsto f/E$ is a unitary map $\mathcal{H}(E)$ onto K_Θ.

$K_\Theta = H^2 \ominus \Theta H^2 = \{ f \in H^2 : \bar{f}\Theta$ is the boundary values of some other H^2 function $\}$.

Note that One can ask: when $K_\Theta \subset L^2(\mu)$ isometrically?

Special case: Let $A = e^{-2i\alpha}$

$$\underbrace{\frac{e^{i\alpha}E + e^{-i\alpha}E^\#}{e^{i\alpha}E - e^{-i\alpha}E^\#}}_{\tilde{S}_\alpha / S_\alpha} = pz + c + \sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1} \right) p_n$$

$S_\alpha \qquad e^{i\alpha}E = \tilde{S}_\alpha + iS_\alpha \qquad \frac{\tilde{S}_\alpha}{S_\alpha} \qquad \{t_n\} = Z_{S_\alpha}$

If $p = 0$, then $\{K_{t_{n,\alpha}}\}$ - OB, $\quad p_n = \frac{1}{\varphi'(t_{n,\alpha})}$

and $\|f\|_E^2 = \pi \sum_n \frac{|f(t_n)|^2}{|E(t_n)|^2 \varphi'(t_n)}$. $\qquad \mu = \sum_n \frac{\pi}{\varphi'(t_n)} \delta_{t_n}$.

The same is true for K_Θ in general

Let $\alpha \in [0, 2\pi)$ $\quad \operatorname{Re} \frac{e^{i\alpha} + \Theta}{e^{i\alpha} - \Theta} = p_\alpha \operatorname{Im} z + \frac{y}{\pi} \int \frac{d\mu_\alpha(t)}{|t - z|^2}$

$p_\alpha = 0$. then $K_\Theta \subset L^2(\mu_\alpha)$ $\quad \mu_\alpha$ - Clark measures (Clark, 1970).

Interesting operator theory relations

μ_α are exactly spectral measures of all unitary rank one perturb. of the restr. shift on K_Θ.

Sarason: general Θ with $\Theta(0)=0$

Alexandrov. general Θ (Alexandrov-Clark measures).

Why all such measure are natural?

Let A be inner then $K_\Theta \subset K_{A\Theta}$

ve Bruuor.

$$\mathrm{Re}\,\frac{1+A\Theta}{1-A\Theta} = \frac{y}{\pi}\int\frac{d\mu(t)}{|t-z|^2} \Longrightarrow \mu \text{ is a Clark}$$

measure for $A\Theta$, $K_{A\Theta} \underset{\text{isom}}{\subset} L^2(\mu) \Longrightarrow K_\Theta$ also.

Now one can take convex union of inner A? But they will not correspond to convex combinati of measures.

$\underline{\mathrm{IV}}$ $\mathcal{H}(E_h) \subset \mathcal{H}(E)$ isometically

We need to apply this theorem with $d\mu = \left|\frac{E_h}{E}\right|^2 dt$.

Thus, we need to find

$$\int\left|\frac{f}{E_h}\right|^2 = \int\left|\frac{f}{E}\right|^2 = \int\left|\frac{f}{E_h}\right|\left|\frac{E_h}{E}\right|^2 d\mu$$

$A \in H^\infty$, $\|A\|_\infty \le 1$: $\mathrm{Re}\,\frac{1+SA}{1-SA} = \left|\frac{E_h}{E}\right|^2$ on $\mathbb{R}$.

$S = E^*/E$

We use the matrix M_h:

$E_1 := (M_h)_{11} + i(M_h)_{21}$, $E_2 := (M_h)_{12} + i(M_h)_{22}$

this are Herglotz functions.

m. $A(z) := \dfrac{E_2 - iE_1}{E_2 + iE_1}$ is bdd by 1.

Proof: $\Im(E_2\bar{E_1}) = -\frac{1}{2i}\left\langle J\begin{pmatrix}E_1\\E_2\end{pmatrix}, \begin{pmatrix}E_1\\E_2\end{pmatrix}\right\rangle =$

$= -\frac{1}{2i}\left(J M_h^T\begin{pmatrix}1\\i\end{pmatrix}, M_h^T\begin{pmatrix}1\\i\end{pmatrix}\right) =$

$$M_h^T \mathcal{J} M_h = \mathcal{J} \quad (\Longleftrightarrow \det M_h = 1)$$

$$(M_h^T)^* \mathcal{J} M_h^T = -\mathcal{J}(M_h^*)^{-1}\mathcal{J} M_h^{-1}\mathcal{J}$$

$$\left(M_h^T = -\mathcal{J}M_h^{-1}\mathcal{J},\ (M_h^T)^* = -\mathcal{J}(M_h^*)^{-1}\mathcal{J}\right)$$

$$= -\frac{1}{2i}\left((M_h^*)^{-1}\mathcal{J}M_h^{-1}\binom{-i}{1},\binom{-i}{1}\right) =$$ result is real – take conjugation

$$= \frac{1}{2i}\left((\overline{M}_h^*)^{-1}\mathcal{J}\,\overline{M}_h^{-1}\binom{i}{1},\binom{i}{1}\right) \geq$$

$$\geq \frac{1}{2i}\left(\mathcal{J}\binom{i}{1},\binom{i}{1}\right) = 1$$

<u>Exerc</u>. 1. $\frac{1}{i}\left(M_h^*\mathcal{J}M_h - \mathcal{J}\right) \geq 0 \Longrightarrow \frac{1}{i}\left((\overline{M}_h^{-1})^*\mathcal{J}\,\overline{M}_h^{-1} - \mathcal{J}\right) \geq 0$

One may B: $\det B = 1$ and

$\operatorname{Im}\left(\frac{B_{22}}{B_{21}}\right) \geq 0 \quad \operatorname{Im}(B_{12}/B_{11}) \geq 0.$

~~2. BC-Iront. ⇒ их правые тоже~~ $\quad \operatorname{Im}(B_{11}/B_{21}) \geq 0$

Thus, E_2/E_1 is Herglotz in $\mathbb{C}^+$ and so

$$A = \frac{E_2/E_1 - i}{E_2/E_1 + i} \text{ is bdd.}$$

<u>It remains to prove the identity:</u>

First, if $x \in \mathbb{R}$, $y \in \mathbb{C}$, then

$$\operatorname{Re}\frac{1 + \frac{x-i}{x+i}\frac{y-i}{y+i}}{1 - \ldots} = \operatorname{Re}\left(\frac{xy-1}{i(x+y)}\right) = \operatorname{Re}\left(i\frac{1+x^2}{x+y} - ix\right) =$$

$$= -\operatorname{Im}\frac{1+x^2}{x+y}$$

Now let $x = G_h^+/G_h^-$, $y = \frac{M_{11} + iM_{21}}{M_{12} + iM_{22}} = E_2/E_1$

$$\mathrm{Re}\ \dots = -\frac{1+\frac{(G_h^+)^2}{(G_h^-)^2}}{\frac{G_h^+}{G_h^-}+\frac{E_2}{E_1}} = +\frac{\big((G_h^+)^2+(G_h^-)^2\big)\,\mathrm{Im}\,\frac{E_2}{E_1}}{\cancel{(G_h^-)^2}\,\frac{|E_1G_h^+ + E_2G_h^-|^2}{\cancel{(G_h^-)^2}|E_1|^2}} =$$

$$= \frac{|E_h|^2}{|E_1G_h^+ + E_2G_h^-|^2} = \frac{|E_h|^2}{|M_{11}G_h^+ + M_{12}G_h^- + i(M_{21}G_h^+ + M_{22}G_h^-)|}$$

$$\mathrm{Im}\,\frac{M_{11}+iM_{21}}{M_{12}+iM_{22}} = \frac{\overbrace{M_{11}M_{22}-M_{21}M_{12}}^{=1}}{|M_{12}+iM_{22}|^2} = \frac{1}{|E_1|^2}$$

But $\Theta = M_h G_h$

So $\Theta_+ = M_{11}G_h^+ + M_{12}G_h^-$

$\Theta_- = M_{12}G_h^+ + M_{22}G_h^-$

and so this expression is $|E|^2$.

Lem. $\mathcal{H}(E_h) \subset \mathcal{H}(E)$.

use the following simple criterion:

Th. f-entire. $f \in \mathcal{H}(E) \iff \begin{cases} f/E \in L^2(\mathbb{R}) \\ |f(z)|^2 \le C\dfrac{|E(z)|^2-|E(\bar z)|^2}{\mathrm{Im}\, z} \\ z \in \mathbb{C}\setminus\mathbb{R} \end{cases}$

Proof: $h_\tau(z) := \dfrac{i\tau}{z+i\tau}\,\dfrac{f(z)}{E(z)}$

$h_\tau \in L^2(\mathbb{R})$ and $|h_\tau(z)| \le C/|z|(\mathrm{Im}\, z)^{1/2}$

Then $\displaystyle\int_{\mathbb{R}} \frac{h_\tau(t)}{t-z}\,dt = h_\tau(z)$, and so $h_\tau \in H^2$

Now let $\tau \to +\infty$ then $f/E \in H^2$.

<u>Lem.</u> $A = \begin{pmatrix} A_+ \\ A_- \end{pmatrix}$: $\Im(A_+ \bar{A}_-) \geq 0$

N: $\frac{1}{i}(N^* J N - J) \geq 0$, $B = NA$

Then $|B_+ + iB_-|^2 - |B_+ - iB_-|^2 \geq |A_+ + iA_-|^2 - |A_+ - iA_-|^2$.

<u>Proof.</u> $\Im(B_+ \bar{B}_-) = \frac{1}{2i}(JB, B) = \frac{1}{2i}(N^* J N A, A) \geq$

$\geq \frac{1}{2i}(JA, A) = \Im(A_+ \bar{A}_-)$

Thus, since $\Theta = M_h G_h$ we see that

$|\cancel{E}|^2 - |E^*|^2 \geq |G_h^+|^2 - |G_h^-|^2$ on $\mathbb{C}^+$

and so $f \in \mathcal{H}(E_h) \Longrightarrow$

$f \chi_{E_h} \in L^2(|E|^{-2} dt)$ and $|f(z)| \leq C \frac{|E(z)|^2 - |E^*(z)|^2}{\Im z}$

whence $f \in \mathcal{H}(E)$.

Thus, $\mathcal{H}(E_h) \subset \mathcal{H}(E)$ isometrically if $p = 0$.

If p (for our A) > 0 then $\exists X$ - a dB sulsp on $\mathcal{H}(E_h)$ of codim 1 such that $X \subset \mathcal{H}(E)$.

~~VII. End of the proof.~~

<u>Th.</u> E-регулар $\mathcal{H}(G) \subset \mathcal{H}(E)$ isometr., G, E have no real zeros. Then G is regular

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From this we conclude that E_h is regular.

Now we will need one more important theorem.

$\mathcal{T}h$. $E \in HB$-regular, $E(0)=1$, $E \neq 0$ on $\mathbb{R}$.
$E = \Theta_+ + i\Theta_-$. Then $\exists\ \Phi = \begin{pmatrix}\varphi_+ \\ \varphi_-\end{pmatrix}$-entire:
$N(\lambda) = [\Theta(\lambda)\ \Phi(\lambda)]$ satisfies (i) $N(0)=I$, $\det N(\lambda)=1$,
(ii) $\frac{1}{i}\left(N^* J N - J\right) \geq 0$, $\lambda \in \mathbb{C}^+$ (J-contractive)
(iii) φ_-/Θ_-, φ_+/Θ_+ are Herglotz in $\mathbb{C}^+$ and have no linear term.

(iii) следует из (ii) (кроме отсутствия линейного слагаемого).

We apply this theorem to E and E_h and get matrices M and N_h. Then M and $\tilde{M} = M_h N_h$ have the same first column Θ.
They have determinant 1: $\Rightarrow \det(\Theta, \tilde{\varphi} - \varphi) = 0$
$\tilde{\varphi} - \varphi = c\Theta$, $\tilde{\varphi} = \varphi + c\Theta$, $c = c(\lambda)$.
Θ_+ and Θ_- have no common zeros $\Rightarrow c(\lambda)$-entire
We will show that $c \equiv 0$.

$$c = \frac{\tilde{\varphi}_-}{\Theta_-} - \frac{\varphi_-}{\Theta_-}.$$ By Th., φ_-/Θ_- is Hergl.

Also $\tilde{\varphi}_-/\Theta_-$ is a Hergl. since $A = \tilde{M}$ satisfy
$\frac{1}{i}(A^* J A - J) \geq 0\ \forall z$. — npe $2\times$ J contr. $\Rightarrow$ J contr. itself.

Thus, c is entire, difference of two Herglotz $\Rightarrow c(\lambda) = a + b\lambda$, $a, b \in \mathbb{R}$.
$N_h(0) = I \Rightarrow a = 0$
Thus, $M(z) = \begin{pmatrix} 1 & bz \\ 0 & 1\end{pmatrix} = M_h(z) N_h(z)$
$\tilde{M} = [\Theta\ \ \varphi + b\lambda\Theta]$

$$\frac{\Theta_+ bz + P_+}{\Theta_+} = \frac{\Theta_h^+ N_{h1,2} + P_h^+ N_{h2,2}}{\Theta_h^+ G_h^+ + P_h^+ G} =$$

$$M_h = [\Theta_h \quad P_h]$$

$$= \frac{N_{h1,2}}{G_h^+}\left[\frac{\frac{N_{h2,2}}{N_{h1,2}} - \frac{G_h^-}{G_h^+}}{\frac{\Theta_h^+}{P_h^+} + \frac{G_h^-}{G_h^+}} + 1\right] = \text{ using that } \det N_h \equiv 1$$

$$= \frac{1}{(G_h^+)^2} \cdot \frac{1}{\frac{\Theta_h^+}{P_h^+} + \frac{G_h^-}{G_h^+}} + \frac{N_{h1,2}}{G_h^+}.$$

By (iii) of theorem $LHS = bz(1+o(1))$, P_+/Θ_+ has no linear term. By (iii) also $\frac{N_{h1,2}(it)}{G_h^+(it)} = o(t)$, $t \to \infty$.

G_h^+ - function of order 1 with real zeros τ_n, so

$$|G_h^+(it)|^2 = C \prod_n (1 + t^2/\tau_n^2).$$ So $G_h^+(it)$ grows faster than any polynomial.

Also, by (iii) $\frac{N_{h1,2}(it)}{G_h^+(it)} = o(t)$, $t \to +\infty$.

Thus, if $b \neq 0$, $\left|\frac{\Theta_h^+(it)}{P_h^+(it)} + \frac{G_h^-(it)}{G_h^+(it)}\right| = o\left(\frac{1}{t^2}\right)$

$\left(o\left(\frac{1}{t^N}\right) \text{ for any } N\right).$

But $-\frac{\Theta_h^+}{P_h^+}$ and $\frac{-G_h^-}{G_h^+}$ are Herglotz functions.

A Herglotz function of it can not decay faster than $1/t$; $\operatorname{Im}\left(pz + c + \sum p_n\left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right)\right)\Big|_{z=it} \geq$

$$\geq p_1 \frac{t}{t^2 + t_1^2}.$$

Thus, $b=0$. So, $M = M_h N_h$.

M_h is the monodromy matrix of some canonical system on $(0,h)$. Then $\operatorname{tr} J \dot{M}_h(0) = h$.

$$\operatorname{tr} J\dot{M}(0) = h + \operatorname{tr} J\dot{N}(0).$$

<u>Lem</u> $N(\lambda)$ is J-contractive on $\mathbb{C}^+$, $N(0)=I$ $\Longrightarrow$ $\operatorname{tr} J\dot{N}(0) > 0$.

<u>Proof.</u> $\frac{1}{i}\left(N^* J N - J\right) \geq 0$ $\quad N(\lambda) = I + \lambda \dot{N}(0) + o(\lambda)$

$$\frac{1}{i}\left(\left(I + \lambda\dot{N}(0) + o(\lambda)\right)^* J \left(I + \lambda\dot{N}(0) + o(\lambda)\right) - J\right) =$$

$$= \frac{1}{i}\left(\left(I + \bar{\lambda}\dot{N}^*(0) + o(\lambda)\right) J \left(I + \lambda\dot{N}(0) + o(\lambda)\right) - J\right) =$$

$$= \frac{1}{i}\left(\bar{\lambda}\dot{N}^*(0)J + \lambda J\dot{N}(0)\right) + o(\lambda) \geq 0 \quad \forall \lambda \in \mathbb{C}^+$$

$|\lambda|$ suff small

Thus $\frac{1}{i}\left(\bar{\lambda}\dot{N}^*(0)J + \lambda J\dot{N}(0)\right) \geq 0 \quad \forall \lambda \in \mathbb{C}^+$

Take $\lambda = i$ $\quad J\dot{N}(0) - \dot{N}^*(0)J = J\dot{N}(0) + (J\dot{N}(0))^* \geq 0$

$\det N(\lambda) \equiv 1 \quad a_{11}a_{22} - a_{12}a_{21} = 1$

$\quad a_{11}'a_{22} + a_{11}a_{22}' - a_{12}'a_{21} - a_{12}a_{21}' = 0$

at 0 $\quad a_{11} = a_{22} = 1,\ a_{12} = a_{21} = 0 \Rightarrow \operatorname{tr}\dot{N}(0) = 0$

Thus, $\dot{N} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}$ and so $J\dot{N}$ is selfadj (a_{ij} are real.)

Thus $J\dot{N}(0) \geq 0$ and so $\operatorname{tr} J\dot{N}(0) \geq 0$

We conclude that $\boxed{\operatorname{tr} J\dot{M}(0) \geq h.}$

But M does not depend on h. Hence for $h > \operatorname{tr} J\dot{M}(0)$ $E_h \equiv 1$. This proves

<u>Th</u> E-regular HB function without real zeros, $E(0)=1$. Then $\exists\ (H, L)$ such that $E = E_L$

VIII. dB subspaces of a dB space are regular.

Th. E-regular, $\mathcal{H}(G)\subset\mathcal{H}(E)$ isometrically.
Then G is regular.

Proof. We need to show that $\forall g\in\mathcal{H}(G)$
$f(z)=\frac{g(z)-g(0)}{z}\in\mathcal{H}(G)$ ($f\in\mathcal{H}(E)$ by regularity).

Let $h\in\mathcal{H}(E)\ominus\mathcal{H}(G)$. Put

$$\Psi_1(w)=\left(\frac{G(w)f(t)-f(w)G(t)}{t-w},h\right)_{\mathcal{H}(E)}$$

$$\Psi_2(w)=\left(\frac{G^*(w)f(t)-f(w)G^*(t)}{t-w},h\right)_{\mathcal{H}(E)}$$

Ψ_1, Ψ_2 – entire and

$$\Psi_1G^*-\Psi_2G=f(w)\Big(\underbrace{\frac{G(w)G^*-G^*(w)G}{t-w}}_{\in\mathcal{H}(G)},h\Big)_{\mathcal{H}(E)}=0$$

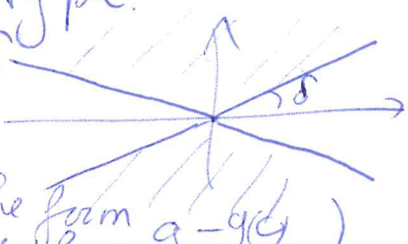
Thus, $\Psi_1=\Psi G$, $\Psi_2=\Psi G^*$, since G and G^* have no common zeros.

a) Let us show: $\Psi\equiv 0$.

$$\Psi(w)=\underbrace{\left(\frac{f(t)}{t-w},h\right)_{L^2\left(\frac{dt}{|E|^2}\right)}}_{\text{Cauchy of } L^1 \text{ funct.}}-\underbrace{\frac{f(w)}{G(w)}}_{\text{bdd type}}\underbrace{\left(\frac{G}{t-w},h\right)_{L^2\left(\frac{dt}{|E|^2}\right)}}_{(w-w_0)\times\text{Cauchy}+\text{const.}}$$

Ψ of bdd type in $\mathbb{C}^+$ and $\mathbb{C}^-$ $\Longrightarrow$ of exp. type.

$\log|\Psi(z)|\le C|z|$ in Ω_δ



$|G(iy)|\to\infty\Longrightarrow$
$|\Psi(iy)|\xrightarrow[y\to+\infty]{}0$ (use the form of $f=\frac{g-g(0)}{z}$)

Thus, $\Psi \equiv 0$. (for $\Psi(-i\tau)$ use the second representation);
Since $h \in \mathcal{H}(E) \ominus \mathcal{H}(G)$ - arbitrary, we have

$$\frac{G(w) f(z) - f(w) G(z)}{z-w} \in \mathcal{H}(G).$$

B) Let $P: \mathcal{H}(E) \to \mathcal{H}(G)$ - projection, $P^\perp$ - projection onto orth. complement.

$$\underbrace{\frac{G(w) f - f(w) G}{z-w}}_{\in \mathcal{H}(G)} = \underbrace{\frac{G(w) Pf - (Pf)(w) G}{z-w}}_{\in \mathcal{H}(G)} + \underbrace{\frac{G(w) P^\perp f - (P^\perp f)(w) G}{z-w}}_{\in \mathcal{H}(G)}$$

$\Longrightarrow$

Take the inner product with $P^\perp f$:

$$\left(\frac{G(w) P^\perp f - (P^\perp f)(w) G}{t-w}, P^\perp f \right)_{\mathcal{H}(E)} = 0$$

$$\int \left| \frac{P^\perp f(t)}{E(t)} \right|^2 \frac{1}{t-w} \, dt = \frac{P^\perp f(w)}{G(w)} \int \frac{G(t) \overline{P^\perp f(t)}}{|E(t)|^2 (t-w)} \, dt$$

$w := i\tau, \quad \tau \to +\infty$

$$LHS \sim \frac{1}{i\tau} \int_{\mathbb{R}} \left| \frac{P^\perp f(t)}{E(t)} \right|^2 dt$$

Choose $w_0 \in Z_G$. Then

$$\int \frac{G(t) \overline{P^\perp f(t)}}{|E(t)|^2 (t-w)} \, dt - \underbrace{\int \frac{G(t)}{t-w_0} \frac{\overline{P^\perp f(t)}}{|E(t)|^2} \, dt}_{\left(\frac{G}{t-w_0}, P^\perp f \right)_{\mathcal{H}(E)} = 0} =$$

$$= (w - w_0) \int \frac{G(t) \overline{P^\perp f(t)}}{(t-w_0)(t-w) |E(t)|^2} \, dt \xrightarrow[\text{Lebesgue thm}]{}$$

$$\xrightarrow[\substack{w = i\tau \\ \tau \to +\infty}]{} - \int \frac{G(t)}{t-w_0} \frac{\overline{P^\perp f(t)}}{|E(t)|^2} \, dt = 0$$

To prove theorem, we need to show: $\left| \frac{P^\perp f(i\tau)}{G(i\tau)} \right| = O\left(\frac{1}{\tau}\right)$.

For f this is true:

$$\frac{f(i\tau)}{G(i\tau)} = \frac{g(i\tau)-g(0)}{i\tau\, G(i\tau)} = O\left(\frac{1}{\tau}\right)$$

$$\left|\frac{Pf(i\tau)}{G(i\tau)}\right| \le C \sqrt{\frac{|G(i\tau)|-|G^*(i\tau)|}{\tau\, |G(i\tau)|}}$$

c) <u>We show that $P^\perp f$ is a linear combination of G, G^*</u>

Let $h \in \mathcal{H}(G)$, $w \in \mathbb{C}\setminus\mathbb{R}$, $h(\overline{w}) = 0$

We show that $\left(\frac{G(w)P^\perp f - (P^\perp f)(w)G}{t-w}, h\right)_{\mathcal{H}(G)} = 0$

Since $\mathcal{H}(G) \subset \mathcal{H}(E)$ isometrically (here we use it!) we can calculate the product on $\mathcal{H}(E)$.

We need to show:

$$I = \int (P^\perp f)(t) \underbrace{\overline{\left(\frac{h(t)}{t-\overline{w}}\right)}}_{\in \mathcal{H}(G)} \frac{dt}{|E(t)|^2} = 0 \text{ -obvious}$$

$$II = \int \frac{G(t)\,\overline{h(t)}}{(t-w)\,|E(t)|^2}\, dt = 0$$

Let $\tilde{h} \in \mathcal{H}(G)$, $\tilde{h}(\overline{w}) \ne 0$

$$II = \frac{1}{\overline{\tilde{h}(\overline{w})}} \left(\frac{\tilde{h}(w)\,G - G(w)\,\tilde{h}}{t-w}, h\right)_{L^2\left(\frac{dt}{|E|^2}\right)} +$$

$$+ \frac{G(w)}{\overline{\tilde{h}(\overline{w})}}\left(\tilde{h}, \frac{h}{t-\overline{w}}\right)_{L^2\left(\frac{dt}{|E|^2}\right)} \underset{\text{all functions on } \mathcal{H}(G)}{=}$$

$$= \int \frac{G(t)\,\overline{h(t)}}{(t-w)\,|G(t)|^2}\, dt = \int \frac{\overline{h(t)}}{\overline{G(t)}(t-w)}\, dt = 0.$$

$\frac{h}{G} \in H^2$, $\frac{h}{G(t-\overline{w})} \in H^1$

Thus, $\underbrace{\dfrac{G(\omega)\,P^{\perp}f(z)-(P^{\perp}f)(\omega)\,G(z)}{z-\omega}}_{\in \mathcal{H}(G),\ \text{orthogonal to any } h:\ h(\bar\omega)=0} = C K^{G}_{\bar\omega}(z)$

Thus, $P^{\perp}f=\alpha G+\beta G^{*}$.

WLOG f is real on $\mathbb{R}$ and so is $P^{\perp}f$.

Then $\beta=\bar\alpha$, $P^{\perp}f=\alpha G+\bar\alpha G^{*}$.

It follows that $\big|\,|G(z)|-|G^{*}(z)|\,\big|\le C|P^{\perp}f(z)|$, $z\in\mathbb{C}^{+}$.

Hence $\left|\dfrac{Pf(i\tau)}{G(i\tau)}\right|\le C\sqrt{\dfrac{|(P^{\perp}f)(i\tau)|}{\tau\,|G(i\tau)|}}$.

$$\left|\frac{P^{\perp}f(i\tau)}{G(i\tau)}\right| \le \underbrace{\left|\frac{f(i\tau)}{G(i\tau)}\right|}_{=O(1/\tau)} + \left|\frac{Pf(i\tau)}{G(i\tau)}\right| \le$$

$$\le C\left(\frac{1}{\tau}+\frac{1}{\sqrt{\tau}}\left|\frac{P^{\perp}f(i\tau)}{G(i\tau)}\right|^{1/2}\right) \Longrightarrow$$

$$\Longrightarrow \left|\frac{P^{\perp}f(i\tau)}{G(i\tau)}\right| \le \frac{C}{\tau}.$$