Chapter 2. Direct spectral theory – regular case.

$JY' = zHY$ $\quad Y = Y(x,z)$, H – Hamilt. on $(0,L)$

$M = \begin{pmatrix} \Theta_+ & \varphi_+ \\ \Theta_- & \varphi_- \end{pmatrix}$ $\quad JM' = zHM$

$M(0,z) = I$

$M(x,z)$ – fundamental solution.

$E_L(z) := \Theta_+(L,z) + i\Theta_-(L,z)$

We will see that $E_L \in HB$

$(H,L) \rightsquigarrow E_L.$

First of all why $E_L \in HB.$

I. J-contractivity of $M(x,z)$.

<u>Important formula:</u>

$$M^*(x,\lambda) J M(x,z) - J = (z-\bar\lambda)\int_0^x M^*(t,\lambda)H(t)M(t,z)\,dt$$

<u>Proof.</u> $M^*(x,\lambda)JM(x,z) - M^*(0,\lambda)JM(0,z) =$

$$= \int_0^x \left(M^*(t,\lambda)JM(t,z)\right)' dt =$$

$$= \int_0^x \left((M^*)'(t,\lambda)\, J M(t,z) + M^*(t,\lambda)\underbrace{JM'(t,z)}_{zHM(t,z)}\right)dt =$$

$$= \int_0^x \bar\lambda M^*(t,\lambda)H(t)\underbrace{J^2}_{=-I}M(t,z) + zM^*(t,\lambda)\cancel{J}H(t)M(t,z)\,dt.$$

$M' = -\lambda JHM$

$(M')^* = \bar\lambda M^*(t,\lambda)HJ$

Cor. $\frac{1}{i}\left(M^*(x,\lambda)\,J\,M(x,\lambda)-J\right)=2\,\mathrm{Im}\,\lambda\int_0^x M^*(t,\lambda)H(t)M(t,\lambda)\,dt$

In part. $\frac{1}{i}\left(M^*(x,\lambda)\,J\,M(x,\lambda)-J\right)\geq 0 \quad \forall x\in(0,L)$
$\lambda\in\mathbb{C}^+$.

Proof: $(M^*HMf,f)=(HMf,Mf)\geq 0$, since $H\geq 0$.

Def. A matrix A satisfying
$\frac{1}{i}(A^*JA-J)\geq 0$ is said to be J-contractive.

/ analogy with $A^*A\leq I \iff A$ is a contraction /

Prop. $\forall x\in(0,L)$ $\quad E_x(z)=\Theta_+(x,z)+i\Theta_-(x,z)\in HB$
unless $H=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ a.e. on $(0,x)$.

Proof: $JM(x,z)=\begin{pmatrix}-\Theta_- & -\varphi_-\\ \Theta_+ & \varphi_+\end{pmatrix}$, $\quad M^*=\begin{pmatrix}\overline{\Theta_+(x,z)} & \overline{\Theta_-(x,z)}\\ \overline{\varphi_+(x,z)} & \overline{\varphi_-(x,z)}\end{pmatrix}$

First entry is:

$$\frac{1}{i}\left(-\overline{\Theta_+(x,z)}\,\Theta_-(x,z)+\overline{\Theta_-(x,z)}\,\Theta_+(x,z)\right)=$$
$$=2\,\mathrm{Im}\left(\Theta_+(x,z)\overline{\Theta_-(x,z)}\right)$$

and
$$2\,\mathrm{Im}\,z\int_0^x \overline{\Theta}^T(t,z)\,H\,\Theta(t,z)\,dt=$$
$$=2\,\mathrm{Im}\,z\int_0^x \left(H(t)\Theta(t,z),\Theta(t,z)\right)dt$$

Thus, $\mathrm{Im}\left(\Theta_+(x,z)\overline{\Theta_-(x,z)}\right)\geq 0$ on $\mathbb{C}^+$

$$|E_x(z)|^2-|E_x(\bar z)|^2=4\,\mathrm{Im}\left(\Theta_+(x,z)\overline{\Theta_-(x,z)}\right)$$

So $|E_x(z)|\geq|E_x(\bar z)|$, $z\in\mathbb{C}^+$.

But we need strict inequality:

$$\Im\left(\Theta_+(x,z)\overline{\Theta_-(x,z)}\right)=0 \iff (H(t)\Theta(t,z),\Theta(t,z))=0$$

a.e. $t\in(0,x)$

$\Theta(t,z)\in \operatorname{Ker} H(t)$

But $J\Theta' = zH\Theta \implies \Theta'=0$ t-a.e.

$\Theta(0)=\begin{pmatrix}1\\0\end{pmatrix}$. Hence $\Theta(t,z)=\begin{pmatrix}1\\0\end{pmatrix}$ on $(0,x)$

Then $H=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ $\qquad H=(\cdot,e)e,\ e=\begin{pmatrix}0\\1\end{pmatrix}$.

One more corollary of the formula. (Why do we need it?)

Put $\lambda=0$. Then $\quad JM(x,z)=z\int_0^x H(t)M(t,z)\,dt$

Differentiate w.r.p. to z at $z=0$ (denote with point!)

$$J\dot{M}(x,0)=\underbrace{\int_0^x H(t)\overset{\shortparallel}{M(t,0)}\,dt}_{I} \geq 0$$

has trace x

$$\shortparallel$$

$$\begin{pmatrix}-\dot{\Theta}_-(x,0) & -\dot{\varphi}_-(x,0)\\ \dot{\Theta}_+(x,0) & \dot{\varphi}_+(x,0)\end{pmatrix}$$

Corollary: $\dot{\Theta}_-(x,0)\leq 0,\ \dot{\varphi}_+(x,0)\geq 0,$

$\operatorname{tr} J\dot{M}(x,0)=\dot{\varphi}_+(x,0)-\dot{\Theta}_-(x,0)=x.$

Ch2. Direct spectral theory (Reg. case)

I. Construction of an operator.

We want to construct a certain (unbdd selfadjoint) operator on $L^2([0,L],H)$, associated with H

essentially if $Jf' = Hg$, then $f \mapsto g$

In part, solutions of our system become eigenvectors

$$L^2([0,L],H) = \left\{ f : [0,L] \to \mathbb{C}^2 : \underbrace{\int_0^L (H(t)f(t), f(t))\,dt}_{= \|f\|^2} < \infty \right\}$$

But H is not invertible. This leads to some technical problems.

E.g. assume that $\exists\, I \subset (0,L)$ s.t. H is a constant rank one matrix, $H(x) = (\cdot, e)e$ $\quad e = \begin{pmatrix} e_+ \\ e_- \end{pmatrix}$

for some $e \in \mathbb{R}^2$ ($\|e\|^2 = 1$) $\qquad H = \begin{pmatrix} e_+^2 & e_+e_- \\ e_+e_- & e_-^2 \end{pmatrix}$

$Jf' = Hg \in \mathrm{Lin}\{e\}$ on I. Multiply by $e^\perp = Je$

$(Jf', e^\perp) = 0$
$= -(f', e)$ Thus, $(f(x), e)$ is a constant on I.

which means that $f(x) = \text{const}$ a.e. on I as an element of $L^2([0,L],H)$.

Def. An interval $I \subset \mathbb{R}_+$ is called H-indivisible, if $\exists\, e \in \mathbb{R}^2$: $H(x) = (\cdot, e)e$, a.e. $x \in I$, and there is no larger interval $I' \supset I$ where this equality holds.

$\mathcal{H} := \{ f \in L^2([0,L], H) : \forall I - H\text{-indiv.}$
$f = c_I$ on I a.e.$\}$

$\mathcal{H}$ is chosen so that any solution $JY' = zHY$ is on $\mathcal{H}$.

$\mathcal{D} := \{ f \in \mathcal{H} :$ f abs. cont. on $[0,L]$
$\exists g \in \mathcal{H} : Jf' = Hg$
$f_-(0) = f_-(L) = 0 \}$

Then we define $D : \mathcal{D} \to \mathcal{H}$ by $\boxed{Df = g}$

There is still one technical problem. It may happen that $\mathcal{D}$ is not dense in $\mathcal{H}$.

Assume that $H = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = (\cdot, e)e$, $e = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ on $(0, \varepsilon)$. Let $Jf' = Hg$. Then $f \in \mathcal{D}$

(f, e) is constant, i.e. f_- is constant on $(0, \varepsilon)$

$f_-(0) = 0 \Rightarrow f_- \equiv 0$ on $(0, \varepsilon)$. Now $h = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & 0 < x < \varepsilon \\ 0, & x > \varepsilon \end{cases}$

is orthogonal to $\mathcal{D}$.

<u>Th</u>. $\mathcal{D}^\perp$ is spanned by the functions of the form $\varphi_1 = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & 0 < x < \varepsilon \\ 0, & x > \varepsilon \end{cases}$ and $\varphi_2 = \begin{cases} e, & L - \varepsilon < x < L \\ 0, & 0 < x < L - \varepsilon \end{cases}$

for those φ_1, φ_2 which belong to $\mathcal{H}$.

<u>Proof</u>. Let $\varphi \in \mathcal{H}$, $\varphi \perp \mathcal{D}$. Let $g \in \mathcal{H}$:

$\langle g, e^\perp \rangle_H = 0$ $\qquad f(x) := -J\int_0^x H(t) g(t)\, dt$

Then $f \in \mathcal{D}$

$f_-(L) = (f(L), e) = \left(J \int_0^L H(t) g(t)\, dt, e \right) =$

$$= \int^{L} (\mathcal{J} H(t) g(t), e)\,dt = -\int_0^{L} (H(t)g(t), \underset{\mathcal{J}e}{\underset{\shortparallel}{e^{\perp}}})\,dt =$$

Then

$$= \langle g, e^{\perp} \rangle_H .$$

$$0 = \langle \varphi, f \rangle_H = \int_0^L \left(H(x)\varphi(x), \mathcal{J}\int_0^x H(t) g(t)\,dt \right) dx =$$

$$= -\int_0^L \Big(\underbrace{\mathcal{J}\int_t^L H(x)\varphi(x)\,dx}_{\Phi(t)}, H(t) g(t) \Big)\,dt = -\langle \Phi, g \rangle_H$$

Note that $\Phi \in \mathcal{H}$. On an indivisible interval $\underset{=(a,b)}{I}$ $H(x) = (\cdot, e_I) e_I$

$$\Phi(t) = \mathcal{J}(\Phi(a) + C(t) e_I) = \mathcal{J}\Phi(a) + C(t) e_I^{\perp}$$

$$e_I^{\perp} = 0 \text{ a.e. on } I \quad (\text{as element of } L^2([0,L], H))$$

So $\Phi = \mathrm{const}$ a.e. on I.

$\Phi \in \mathcal{H}$, $\langle \Phi, g \rangle_H = 0$ $\forall g : g \perp e^{\perp}$ in $L^2([0,L], H)$

Hence $\Phi(t)$ is in the equivalence class of $e^{\perp}$,

$$\Phi(t) = c \underset{\mathcal{J}e}{\underset{\shortparallel}{e^{\perp}}} + \Omega(t), \quad \underset{\text{a.e. } t}{\Omega(t) \in \operatorname{Ker} H(t)}$$

Φ abs cont. $\Rightarrow$ Ω abs cont.

$\mathcal{M} := \{ t \in (0,L) : \Omega(t) \neq 0 \}$, open set, union of intervals

Rank $H = 1$ on $\mathcal{M}$, so $\exists\, \zeta : \mathcal{M} \to \mathbb{R}^2$:

$$H(x) = (\cdot, \zeta(x)) \zeta(x) \text{ for a.e. } x \in \mathcal{M}.$$

$$\|\zeta(x)\| = 0$$

$\Omega(x) \perp \zeta(x)$ a.e. x $\Rightarrow$ $\Omega(x) = \omega(x) \mathcal{J} \zeta(x)$

with $\omega(x) = \|\Omega(x)\|$ $\quad \|\Omega(x)\|$ abs cont $\Rightarrow \zeta(x)$

abs. cont. on $\mathcal{M}$.

$$\omega'(x)\mathcal{J}\zeta(x)+\omega(x)\mathcal{J}\zeta'(x)=\mathcal{P}'(x)=$$
$$=-\mathcal{J}H(x)\varphi(x)=-(\varphi(x),\zeta(x))\mathcal{J}\zeta(x)$$
$$\zeta'(x)\perp\zeta(x) \qquad (\zeta(x),\zeta(x))=1$$

$\Longrightarrow \omega(x)\zeta'(x)=0 \Rightarrow \zeta(x)=$const on each interval in $\mathcal{M}$ $\Rightarrow$ any interval of $\mathcal{M}$ is a subinterval in some H-indiv. interval

$\varphi\in\mathcal{H} \Rightarrow \varphi$ const on any subint of $\mathcal{M}$, hence ω' is const. on intervals in $\mathcal{M}$.

Thus, $\forall I\subset\mathcal{M}$ comp-t $\exists e_I\in\mathbb{R}^2$: $\Omega(x)=$(linear funct. of x)$\cdot\mathcal{J}e_I$

If $I\subset(0,L)$ is a comp. of $\mathcal{M}$ which has no 0 or L as endpoints, then $\Omega=0$ on endpoints $\Rightarrow \Omega\equiv 0$ on I, !!!

Thus, $\mathcal{M}=I_0\cup I_L$ at most, $I_0=(0,\varepsilon)$, $I_L=(L-\varepsilon, L)$

$$H(x)\varphi(x)=\begin{cases} c_0 e_0, & x\in I_0\\ c_L e_L, & x\in I_L\\ 0 & \text{otherwise}\end{cases}$$

Indeed, $-\mathcal{J}H\varphi=\omega'\mathcal{J}\zeta$ $\quad$ ω' and ζ are const on any interval in $\mathcal{M}$.

$\mathcal{P}(L)=0$. Hence, $\Omega(L)=-c\mathcal{J}\overset{=e^\perp}{e}=\Omega(x)$ $x\in I_L$

$\Omega\in\operatorname{Ker}H \Rightarrow H=(\cdot,e)e$ on I_L.

For the case of I_0 ($e_0=e$) one should consider $f(x)=\mathcal{J}\int_x^L H(t)g(t)\,dt$

Th. Assume that there is no $\varepsilon>0$ such that $H=(\cdot,e)e$ on $(0,\varepsilon)$ or on $(L-\varepsilon,L)$, $e=\begin{pmatrix}0\\1\end{pmatrix}$

Then $\mathcal{D}$ is dense on $\mathcal{H}$ and

$$D: f\mapsto g\ ,\ f\in\mathcal{D}$$

is a correctly defined selfadjoint operator.

Exerc. about resolvent.

D^* exists $\quad \mathcal{D}^*=\{h: f\mapsto (Df,h) \text{ is a continuous linear functional}\}$

Let $f\in\mathcal{D}$, $f_-(0)=f_-(L)=0$, g - corresp. $Jf'=Hg$

$$\langle Df,f\rangle_H=\int_0^L(\underbrace{HDf}_{g},f)=\int_0^L(Jf',f)=$$

$$=\int_0^L(-f_-'\overline{f_+}+f_+'\overline{f_-})=\underbrace{2i\,\mathrm{Im}(f_+\overline{f_-})\Big|_0^L}_{=0}-$$

$$-\int_0^L(f_+\overline{f_-'}-f_-\overline{f_+'})=\int_0^L(f,Jf')=\langle f,Df\rangle_H$$

So D is symmetric, $\mathcal{D}\subset\mathcal{D}^*$.

Let $f\in\mathcal{D}^*$, $g=D^*f$. Let $\tilde{f}$ be the solution of $J\tilde{f}'=Hg$, $\tilde{f}=-J\int_0^x Hg$

Then $\tilde{f}\in\mathcal{D}$.

We need to verify that $\tilde{f}_-(L)=0$

Const. vector $e^\perp\in\mathcal{D}$ (with $g\equiv 0$)

$$\langle g,e^\perp\rangle_H=\langle f,De^\perp\rangle_H=0$$

$$\int_0^L(Hg,e^\perp)=\int_0^L(J\tilde{f}',e^\perp)=-\int_0^L(\tilde{f}',e)=-\tilde{f}(L)+\tilde{f}(0)$$

Let $u \in D$

$$\langle Du, f \rangle_H = \langle u, g \rangle_H = \int_0^L (u, \mathcal{J} f') =$$

$$= (u, \mathcal{J} \tilde{f})\Big|_0^L + \int_0^L (\mathcal{J} u', \tilde{f}) = \langle Du, \tilde{f} \rangle_H$$

$$\mathcal{J} u' = H \underset{Du}{\overset{\shortparallel}{v}}$$

Let $h \in H : \langle h, e^{\perp} \rangle_H = 0$ and $u(x) = -\mathcal{J} \int_0^x H h$

Then $h = Du$ $\quad f - \tilde{f} \perp h \quad \forall h \ \langle h, e^{\perp} \rangle_H = 0$

Thus, $\tilde{f} - f = c e^{\perp}$, that is $\tilde{f} - c e^{\perp}$ is a representative of equivalence class of f.

$\tilde{f} - c e^{\perp} \in D$ (and so $\mathcal{D}^* = \mathcal{D}$) Why?

$\tilde{f} \in D$ and $e^{\perp} \in D$.

Th. Let H be a Hamiltonian on $[0; +\infty)$ such that $\exists \varepsilon > 0$ with $H = (\cdot, e) e$ and also $\nexists B : (B; +\infty)$ is indivisible.

$\mathcal{D}$ – the same, without condition on right ent

Then $\mathcal{D}$ is dense on $\mathcal{H}$ and $D : f \mapsto g$ is a selfadj operator.

In the regular case one can consider other selfadjoint conditions on the right end.

Th Let $e_\alpha = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$, $L < \infty$, $\alpha \in [0, \pi]$

Cond. on L: $\exists \varepsilon > 0$: $H = (\cdot, e_\alpha) e_\alpha$ on $(L - \varepsilon, L)$.

$\mathcal{D}_\alpha = \{ \ldots$ with $f_-(L)=0$ replaced by $f_+(L)\cos\alpha + f_-(L)\sin\alpha = 0 \}$

$D_\alpha : f \mapsto g$ $\quad f \in \mathcal{D}_\alpha$

$\mathcal{D}_\alpha$ is dense in $\mathcal{H}$ and D_α is selfadjoint.

II. Generalized Fourier transform

Important identity: $\quad M = \begin{pmatrix} \Theta_+ & \Phi_+ \\ \Theta_- & \Phi_- \end{pmatrix}$

$JM' = zHM$

$$M^*(x,\lambda) J M(x,z) - J = (z - \bar\lambda) \int_0^x M^*(t,\lambda) H(t) M(t,z)\, dt$$

$$M^*(x,\lambda) J M(x,z) - \underset{M^*(0,\lambda)\ M(0,z)}{J} = \int_0^x \left(M^*(t,\lambda) J M(t,z) \right)' dt =$$

$$= \int_0^x (M^*(t,\lambda))' J M(t,z) + M^*(t,\lambda) \underbrace{J M'(t,z)}_{= zHM(t,z)}\, dt$$

$$-(\lambda J H M(t,\lambda))^* = -\bar\lambda M^*(t,\lambda) H J^* \quad J^* = -J \quad \text{e.g.}$$

$$(M'(t,\lambda))^* = (-\lambda J H M)^* = -\bar\lambda M^* H J$$

$\boxed{\text{Case } \mathbb{C}\ \ z = \lambda}$

Prop. $\forall x \in (0,L)$ $\quad E_x(z) = \Theta_+(x,z) + i\Theta_-(x,z) \in HB$

unless $H = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ a.e. on $(0,x)$.

$$JM(x,z) = \begin{pmatrix} -\Theta_- & -\Phi_- \\ \Theta_+ & \Phi_+ \end{pmatrix} \qquad -\Theta_+(x,\bar\lambda)\Theta_-(x,z) + \Theta(x,\bar\lambda)\Theta_+(x,z)$$

$$= (z - \bar\lambda) \int_0^x \Theta^T(t,\bar\lambda) H(t) \Theta(t,z)\, dt$$

$\lambda := z$

$$\operatorname{Im}\left[\Theta_+(x,z)\overline{\Theta_-(x,z)}\right] = \operatorname{Im} z \int_0^x (H\Theta, \Theta)\, dt$$

$$|E(z)|^2 - |E(\bar z)|^2 = 4\operatorname{Im}\left(A(z)\overline{B(z)}\right)$$

$\Longrightarrow |E_x(z)|^2 \geq |E_x(\bar z)|^2 \geq 0$. We need a strict ineq.!

$\circledcirc \iff \Theta(t,z) \in \operatorname{Ker} H(t)$ a.e. $t \in (0,X)$

$J\Theta' = zH\Theta$

Hence, $\Theta' = 0$ a.e. on $(0,X)$. Since $\Theta(0) = \begin{pmatrix}1\\0\end{pmatrix}$, we see that $\Theta(t) \equiv \begin{pmatrix}1\\0\end{pmatrix}$, $H = (\cdot\,; e)e$, $e = \begin{pmatrix}0\\1\end{pmatrix}$

Assume that $L < \infty$, $H \not\equiv \begin{pmatrix}0&0\\0&1\end{pmatrix}$ on $(0,L)$

Then $E_L \in HB$, the space $\mathcal{H}(E_L)$ exists.

$\Theta(x,z) = \Theta_z(x)$

$$K_\lambda(z) = \frac{1}{\pi}\langle \Theta_z, \Theta_\lambda \rangle_H =$$

$$= \frac{1}{\pi}\int_0^L \overline{\Theta(t,\lambda)}^T H\Theta(t,z)\,dt =$$

$$= \frac{1}{\pi(z-\bar\lambda)}\left[\Theta_+(x,z)\Theta_-(x,\bar\lambda) - \Theta_-(x,z)\cdot\Theta_+(x,\bar\lambda)\right]$$

It is correct since $\boxed{\Theta_- = -B}$

Th. Assume that (L) is satisfied. Then $\int_0^L (Hf, \Theta_{\bar w})\,dt = \int (f, H\Theta_{\bar w}) = \int f^T H\Theta_w$

$\mathcal{U}: f \longmapsto \overset{F(w)}{\frac{1}{\sqrt{\pi}}}\langle f, \Theta_w \rangle_H$ is an isomorphism of H onto $\mathcal{H}(E)$. $\int_0^L$

<u>Proof</u>: $(\mathcal{U}\Theta_\lambda)(w) = \sqrt{\pi}\,\overline{K_\lambda(\bar w)} = \sqrt{\pi}\,K^*_\lambda(w)$

$\mathcal{U}\Theta_\lambda \in \mathcal{H}(E)$ $\forall \lambda$

$(\mathcal{U}\Theta_\lambda, \mathcal{U}\Theta_z)_{\mathcal{H}(E)} = \pi\langle \overline{K^*_\lambda}, \overline{K^*_z}\rangle_{\mathcal{H}(E)} = \pi(K_z, K_\lambda) = \pi K_z(\lambda) =$

$= \langle \Theta_\lambda, \Theta_z \rangle_H$

$\mathcal{U}$ isometr. on $\overline{\operatorname{Span}\{\Theta_\lambda\}}$

<u>Problem</u> to show that $\{\Theta_\lambda\}$ are complete in H.

Here we need use oper-s D_α. Choose α: cond. (R_α) with α is satisfied on the right end.

Then $\{\Theta(x, \lambda_k)\}$ with λ_k:

$$\Theta_+(L,\lambda_k)\cos\alpha + \Theta_-(L,\lambda_k)\sin\alpha = 0$$

satisfy $J\Theta'(x,\lambda_k) = \lambda_k H\Theta(x,\lambda_k)$
and $\Theta_-(0,\lambda_k)=0$, so $\Theta_1(x,\lambda_k)\in D_\alpha$
and $D_\alpha\Theta(\cdot,\lambda_k) = \lambda_k\Theta(\cdot,\lambda_k)$

Clearly there are no other eigenvectors for λ

D_α is selfadj. It also has discrete spectrum. Its resolvent is an integral operator with a Hilbert-Schmidt kernel.

Почему нет других с.в.?

$$J f'(x) = \lambda_k H f \qquad f_-(0)=0$$
$$f_+(L)\cos\alpha + f_-(L)\sin\alpha = 0$$
$$f = \beta\Theta + \gamma\varphi \qquad \varphi_-(0)=1 \Rightarrow \gamma = 0$$

<u>Def</u>. $E\in HB$ is regular, if $\frac{1}{(z+i)E(z)}\in H^2(\mathbb{C}^+)$

(Then in particular $E\in Cart$ and E of finite exp. type by Krein)

<u>Th</u>. E_L is regular.

<u>Proof</u>. $U\varphi_z\in\mathcal{H}(E_L)$. Let us compute it

$$U\varphi_z(w) = \int_0^L \varphi_z^T H\Theta_w = \frac{1}{w}\int_0^L \varphi_z^T J\Theta_w' = \frac{1}{w}\Big(\varphi_z^T J\Theta_w\Big|_0^L -$$
$$- \int_0^L (\varphi_z^T)' J\Theta_w\Big) =$$
$$= \frac{1}{w}\Big(\varphi_z^T(L) J\Theta_w(L) - 1 + \int_0^L (zH\varphi_z)^T J\Theta_w\Big) -$$
$$\int_0^L \varphi_z^T H \underbrace{J^T J}_{I}\Theta_w\Big)$$

$$U\varphi_z(w) = \frac{1}{\sqrt{\pi}}\,\frac{\varphi_z^T(L)J\Theta_w(L) - 1}{w-z}\in\mathcal{H}(E_L)$$

$$\frac{\varphi_\pm(L,z)}{E_L}\in H^2 \qquad \text{Take } w=-i \Rightarrow \frac{1}{(z+i)E_L(z)}\in H^2.$$

<u>Rem</u>. $\mathcal{H}(E)$ is regular $\Longleftrightarrow$ $\forall f\in\mathcal{H}(E)$ $\frac{f-f(0)}{z}\in\mathcal{H}(E)$

<u>Proof</u>: $\frac{f-f(0)}{zE}\in H^2$.