# Chapter 1. De Branges spaces.

## I. Hermite–Biehler class

Charles Biehler (1845–1906)

$E$ is HB if $|E(z)| > |E(\bar z)| \quad \forall z \in \mathbb{C}^+$.

Examples: 1. $E(z) = e^{-iaz}$, $a>0$,
$|E(z)| = e^{ay}$, $z = x+iy$.

2. $P(z)$ – polynomial with zeros in $\mathbb{C}^-$

3. $\prod_{n=1}^{\infty}(1 - z/z_n) \quad z_n \in \mathbb{C}^- \quad \sum \frac{1}{|z_n|} < \infty$

It is not always included into definition but $E \neq 0$ on $\mathbb{R}$.

In fact, functions that we will need are more or less of this form:

$$E(z) = e^{-iaz} \lim_{R\to\infty} \prod_{|z_n|<R} (1 - z/z_n)$$

since $E$ belongs to the so-called Cartwright class.

Not. $f^*(z) := \overline{f(\bar z)}$ for entire $f$

Then $|E^*(z)| < |E(z)|$, $z \in \mathbb{C}_+$

$$\Theta(z) := \frac{E^*(z)}{E(z)} \qquad |\Theta(z)| < 1, \ z \in \mathbb{C}^+$$
and $|\Theta(t)| = 1$, $t \in \mathbb{R}$.

Thus, $\Theta$ is an inner function in $\mathbb{C}^+$.

Corollary. $E \in HB$, $z_n = x_n + iy_n$ its zeros.
Then $z_n$ satisfy the Blaschke condition
$$\sum \frac{|y_n|}{|z_n|^2} = \sum \operatorname{Im} \frac{1}{z_n} < \infty.$$

Ex 4. $E(z) = \prod_{n=1}^{\infty}\left(1 - z/z_n\right) e^{\mathrm{Re}\, z/z_n} e^{-iaz+bz}$

$z_n \in \mathbb{C}^+ \quad \sum \frac{1}{|z_n|^2} < \infty.$

This is the general form of HB function of order less than 2.

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In what follows we will need only HB functions of order 1 (of finite exp. type)!

Rem. $\Theta(z) = E^*/E$ is an inner function meromorphic on $\mathbb{C}$ $\Rightarrow$ $\Theta(z) = e^{ihz} B(z)$, $B$ - Bl. product with zeros $\bar{z}_n$: $|z_n| \to \infty$!

Ex. Conversely for any $\Theta$ of this form $\exists E \in HB$ such that $\Theta = E^*/E$.

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$$A := \frac{E+E^*}{2}, \quad B := \frac{E^*-E}{2i}$$

By construction, $A, B$ - entire functions, real on $\mathbb{R}$ (i.e. $A = A^*$, $B = B^*$) and $E = A - iB$.

Th. Let $E = A - iB$, $A, B$ real. TFAE:

1. $E \in HB$ 2. $\mathrm{Im}\, B/A > 0$ on $\mathbb{C}^+$.

Moreover, the zeros of $A$ and $B$ interlace.

Proof: $\frac{B}{A} = \frac{E^*-E}{i(E+E^*)} = i\frac{E-E^*}{E+E^*} = i\frac{1-\Theta}{1+\Theta}$

$$\mathrm{Re}\frac{1-\zeta}{1+\zeta} = \frac{1-|\zeta|^2}{|1+\zeta|^2} > 0 \Longleftrightarrow |\zeta| < 1$$

$E \in HB \Longleftrightarrow |\Theta| < 1$ in $\mathbb{C}^+ \Longleftrightarrow \mathrm{Im}\, B/A > 0$ on $\mathbb{C}^+$.

Since $\operatorname{Im} \frac{B}{A} > 0$ in $\mathbb{C}^+$, we have, by Herglotz theorem

$$\frac{B(z)}{A(z)} = pz + C + \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{t^2+1} \right) d\mu(t), \text{ where}$$

$p \geq 0$, $\operatorname{Im} C \geq 0$, $\mu$ – positive measure on $\mathbb{R}$ with $\int \frac{d\mu(t)}{t^2+1} < \infty$.

Since $A, B$ are real on $\mathbb{R}$, $C$ must be real.

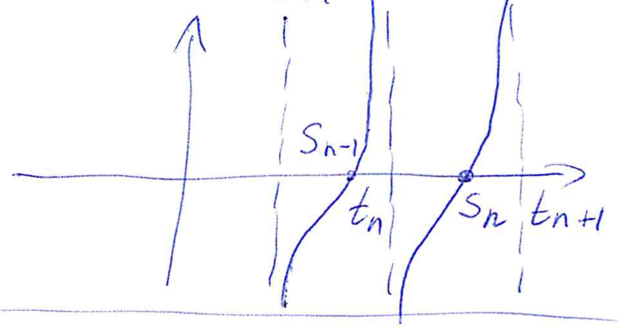
Since $\operatorname{Im} \frac{B}{A} = 0$ on $\mathbb{R} \setminus Z_A$, we conclude that $\mu$ is supported by $\{t_n\} = Z_A$:

$$\frac{B(z)}{A(z)} = pz + C + \sum_n \left( \frac{1}{t_n - z} - \frac{t_n}{t_n^2+1} \right) p_n,$$

where $p_n = \mu(\{t_n\}) = B(t_n)/A'(t_n) > 0$.

It follows $B$ has exactly one zero $s_n$ between two neighbor zeros of $A$:

$$\left(\frac{B}{A}\right)'(t) = p + \sum_n \frac{p_n}{(t-t_n)^2} > 0$$

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# II. De Branges spaces

**Def.** Let $E \in HB$. $\mathcal{H}(E) := \left\{ f\text{-entire} : \begin{array}{l} f/E \in H^2(\mathbb{C}^+) \\ \underbrace{f^*/E \in H^2(\mathbb{C}^+)}_{\Updownarrow} \\ f/E^* \in H^2(\mathbb{C}^-) \end{array} \right\}$

$$\|f\|_E^2 := \int_{\mathbb{R}} \left| \frac{f(t)}{E(t)} \right|^2 dt = \|f/E\|_{H^2}^2$$

Examples. 1. $E(z) = e^{-iaz}$

$$\mathcal{H}(E) = \left\{ f\text{-entire}, f \in L^2(\mathbb{R}), \begin{array}{l} f(z) e^{iaz} \in H^2(\mathbb{C}^+) \\ f(z) e^{-iaz} \in H^2(\mathbb{C}^-) \end{array} \right\}$$

Recall that $H^2 = \{ f \in L^2(\mathbb{R}) : \mathcal{F}f = 0 \text{ on } (-\infty; 0) \}$

Thus, $\mathcal{H}(E) = \{f : (\mathcal{F}f)(x) = 0,\ x \in (-\infty; -a) \cup (a; +\infty)\} =$
$= \mathcal{F}(L^2(-a,a)) = PW_a$ – Paley-Wiener space.

2. $E$ – polynomial of degree $n$ (with zeros in $\mathbb{C}^-$).
Then $\mathcal{H}(E) = \mathcal{P}_{n-1}$ – the space of polyn-s of deg $\le n-1$

<u>Rem.</u> $\mathcal{H}(E)$ depends on $\Theta$. If $E^*/E = E_1^*/E_1$,
then $E$ and $E_1$ have the same zeros. Put $E_1 = SE$.
Then $S^*/S = 1$, $S$ has no zeros. Then the map
$f \longmapsto Sf$ is the unitary map from $\mathcal{H}(E)$ onto $\mathcal{H}(E_1)$.

<u>Th.</u> $\mathcal{H}(E)$ is a reproducing kernel Hilbert space.

<u>Proof.</u> a) for $w \in \mathbb{C}$ put

$$K_w(z) = \frac{i}{2\pi} \frac{E(z)\overline{E(w)} - E^*(z)\overline{E^*(w)}}{z - \overline{w}}$$

Let us show that $(f, K_w)_{\mathcal{H}(E)} = f(w)$, $f \in \mathcal{H}(E)$.
WLOG let $w \in \mathbb{C}^+$.

$$(f, K_w)_E = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)\overline{E(t)}E(w)}{|E(t)|^2 (t-w)} dt -$$

$$- \frac{1}{2\pi i} \int \frac{f(t)\overline{E^*(t)}E^*(w)}{|E(t)|^2 (t-w)} dt = \frac{E(w)}{2\pi i} \underbrace{\int \frac{f(t)}{E(t)(t-w)} dt}_{= \frac{f(w)}{E(w)}} -$$

$$- \frac{E^*(w)}{2\pi i} \underbrace{\int \frac{\overline{f(t)}}{\overline{E(t)}} \frac{1}{t - \overline{w}} dt}_{=0}$$

We use that $f/E \in H^2$, $f^*/E \in H^2$ and $w \in \mathbb{C}^+$, so $\overline{w} \in \mathbb{C}^-$.

b) We show that $\sup_{w \in D} \|K_w\| < \infty$ for any bounded set $D \subset \mathbb{C}$. Clearly the problems can occur only near the real axis:

$$\|K_w\|^2 = K_w(w) = \frac{|E(w)|^2 - |E(\overline{w})|^2}{4\pi \operatorname{Im} w} = |E(w)|^2 \frac{1 - |\Theta(w)|^2}{4\pi \operatorname{Im} w}.$$

Let $\Theta = \prod_{n=1} b_n$, $b_n(z) = e^{\alpha_n} \frac{z - z_n}{z - \bar{z}_n}$, $z_n = x_n + iy_n$

Then $1 - |\Theta(w)|^2 = \sum_{n=1}^{\infty} \prod_{j=1}^{n-1} |b_j(w)|^2 (1 - |b_n(w)|^2) =$

$w = x + iy$

$$= \sum_{n=1}^{\infty} \prod_{j=1}^{n-1} |b_j(w)|^2 \frac{4y_n y}{|w - \bar{z}_n|^2}$$

Thus, $\frac{1 - |\Theta(w)|^2}{4\pi y} = \frac{1}{\pi} \sum_{n=1}^{\infty} |b_j(w)|^2 \frac{y_n}{|w - \bar{z}_n|^2} \xrightarrow[w \to x \in \mathbb{R}]{}$

$\to \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{y_n}{|x - \bar{z}_n|^2}$. Thus, $K_x(x) = \frac{|E(x)|^2}{\pi} \sum_{n=1}^{\infty} \frac{y_n}{|x - \bar{z}_n|^2}$.

In particular, $K_w(w)$ is continuous in $\mathbb{C}$.

c) Completeness of $\mathcal{H}(E)$.

Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{H}(E)$. Then $f_n/E$, $f_n^*/E$ are Cauchy sequences in $H^2(\mathbb{C}^+)$, $f_n/E \to f/E$, $f_n^*/E \to g/E$ for some $f, g$ analytic on $\mathbb{C}^+$. Also, $|f_n(w)|^2 \le \|f_n\|_E^2 K_w(w)$

So $|f_n|$ are uniformly bounded on any compact in $\mathbb{C}$

Thus, $\exists f_{n_k} \to f_0$ - entire uniformly on compacts.

Then $f_0 = f$ on $\mathbb{C}^+$, $f_0^* = g$ on $\mathbb{C}^+$. Thus $f$ is entire, $f^* = g$ and $f_{n_k} \to f$ in $\mathcal{H}(E)$.

<u>Rem.</u> 1. Another formula for the reproducing kernel of $\mathcal{H}(E)$ (follows by direct computation):

$$K_w(z) = \frac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{\pi(z - \bar{w})}.$$

2. $-\frac{\partial(\arg E(x))}{\partial x} = \frac{\partial}{\partial y} \log|E(x+iy)| \Big|_{y=0} > 0$

for any $E \in HB$.

Thus, there exists $\varphi \uparrow$ such that $e^{i\varphi(t)} E(t) = |E(t)|$

($\varphi' = -\arg E$ on $\mathbb{R}$), $\varphi$ is called phase function

Then $\Theta(t) = e^{2i\varphi(t)}$ $\varphi'(t) = \frac{1}{2}|\Theta'(t)| = \sum \frac{y_n}{|t - z_n|^2} + a$,

$K_x(x) = \frac{|E(x)|^2 \varphi'(x)}{\pi}$ and $\frac{B(t_n)}{A'(t_n)} = \frac{1}{\varphi'(t_n)}$.

# III. Orthogonal bases of reproducing kernels

Existence of such bases is one of the most interesting and striking properties of dB spaces.

Rem. $E \notin \mathcal{H}(E)$ but $\frac{E}{z-z_n} \in \mathcal{H}(E)$ $\forall z_n \in Z_E$.

Also, if $\{t_n\} = Z_A$, $\{s_n\} = Z_B$, then $\frac{A}{z-t_n}, \frac{B}{z-s_n} \in \mathcal{H}(E)$.

Th. Let $E = A - iB \in HB$, $Z_A = \{t_n\}$. Then $\{K_{t_n}\}$ is an orthogonal family in $\mathcal{H}(E)$ and either $\{K_{t_n}\}$ is an orth. basis, or $A \in \mathcal{H}(E)$ and $\{K_{t_n}\} \cup \{A\}$ is an OB in $\mathcal{H}(E)$.

Proof. $$h(z) := \frac{B(z)}{A(z)} = pz + c + \sum_n p_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right)$$
Herglotz representation.

Then $$\frac{h(z) - h(\bar{w})}{z - \bar{w}} = p + \sum_n \frac{p_n}{(z-t_n)(\bar{w}-t_n)}$$
$$= \frac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{A(z)\overline{A(w)}(z-\bar{w})}$$

$$\pi K_w(z) = p\, A(z)\overline{A(w)} + \sum_n p_n \frac{\overline{A(w)}}{\bar{w}-t_n} \underbrace{\frac{A(z)}{z-t_n}}_{\text{orthogonal}}.$$

First, check orthogonality:
$$K_{t_n}(z) = -\frac{A(z)B(t_n)}{\pi(z-t_n)}, \quad \left\|\frac{A(z)}{z-t_n}\right\|^2 = \frac{\pi^2}{|B(t_n)|^2} \cancel{\frac{|E(t_n)|^2 \varphi'(t_n)}{\pi}}.$$

$$(K_{t_n}, K_{t_m}) = \frac{B(t_n)}{\pi^2}\left(\frac{A(z)}{z-t_n}, K_{t_m}\right) = 0 \quad m \neq n$$
since $A(t_m) = 0$

Also, if $A \in \mathcal{H}(E)$, then $A \perp K_{t_n}$ for any $n$.

Put $z = w$: then
$$p|A(w)|^2 + \sum_n p_n \frac{|A(w)|^2}{|w-t_n|^2} = \pi K_w(w)$$

But $\sum_n p_n^2 \left|\frac{A(\omega)}{\omega - t_n}\right|^2 \left\|\frac{A(z)}{z-t_n}\right\|_E^2 =$

$$= \sum p_n^2 \left|\frac{A(\omega)}{\omega - t_n}\right|^2 \frac{\pi^2}{|B(t_n)|^2} \frac{|E(t_n)|^2 \varphi'(t_n)}{\pi} = \pi \sum p_n \frac{|A(\omega)|^2}{|\omega - t_n|^2}$$

We used that $|E(t_n)| = |B(t_n)|$ and $p_n = \frac{1}{\varphi'(t_n)}$.

Thus the orthogonal series converges in the norm of $\mathcal{H}(E)$, and so $pA(z) \in \mathcal{H}(E)$. Either $p=0$ or $p>0$ and $A \in \mathcal{H}(E)$.

If $p=0$, then $\forall \omega$ $K_\omega \in \overline{\operatorname{Span}}\{K_{t_n}\}$, so $\{K_{t_n}\}$ is OB. Otherwise, $\{K_{t_n}\} \cup \{A\}$ is OB.

<u>Cor.</u> If $\{K_{t_n}\}$ is OB, then $\forall f \in \mathcal{H}(E)$

$$\|f\|_E^2 = \sum \frac{|(f, K_{t_n})|^2}{\|K_{t_n}\|^2} = \pi \sum_n \frac{|f(t_n)|^2}{|E(t_n)|^2 \varphi'(t_n)}.$$

The embedding $\mathcal{H}(E) \hookrightarrow L^2\left(\pi \sum \frac{1}{|E(t_n)|^2 \varphi'(t_n)} \delta_{t_n}\right)$ is isometric.

<u>A more general result.</u> Let $\alpha \in [0, \pi)$, $S_\alpha = e^{i\alpha} E - e^{-i\alpha} E^*$. Then $S_\alpha$ is real on $\mathbb{R}$, has only real zeros $t_{n,\alpha}$ and $\{K_{t_{n,\alpha}}\}$ is an OB in $\mathcal{H}(E)$ unless $S_\alpha \in \mathcal{H}(E)$. Note that there is at most one $\alpha \in [0; \pi)$: $S_\alpha \in \mathcal{H}(E)$.

$$\varphi(t_{n,\alpha}) = \alpha \mod \pi, \quad \{t_{n,\alpha}\} = \{t : \Theta(t) = e^{2i\alpha}\}.$$

<u>Example:</u> $PW_\pi$, $\varphi(t) = \pi t$, $t_{n,\alpha} = \frac{\alpha}{\pi} + n$, $n \in \mathbb{Z}$

Then $f(z) = \sum_{n \in \mathbb{Z}} f\left(\frac{\alpha}{\pi} + n\right) \frac{\sin(\pi z - \pi n - \alpha)}{\pi z - \pi n - \alpha}$,

$$\|f\|_{PW_\pi}^2 = \sum_{n \in \mathbb{Z}} \left|f\left(\frac{\alpha}{\pi} + n\right)\right|^2.$$

Shannon – Kotelnikov – Whittacker sampling formula

Rem $f = A\sum \frac{c_n \mu_n^{1/2}}{z-t_n}$, $\mu_n = \frac{\pi}{\varphi'(t_n)}$, $\{c_n\}\in \ell^2$.

Rem. Clark measures for $\Theta$.

$$\operatorname{Re}\frac{e^{i\alpha}+\Theta(z)}{e^{i\alpha}-\Theta(z)} = \frac{y}{\pi}\int\frac{d\mu_\alpha}{|t-z|^2}, \quad \mu_\alpha = \sum\frac{\pi}{\varphi'(t_{n,\alpha})}\delta_{t_{n,\alpha}}$$

Exerc. $\mathcal{H}(E_1)$, $\mathcal{H}(E_2)$ coincide $\{t_n\}$ and $\{s_n\}$. Then

$\exists S$ without zeros $f\mapsto Sf$ - unitary $\mathcal{H}(E_1)$ on $\mathcal{H}(E_2)$

# IV Axiomatic approach

$\mathcal{H}=\mathcal{H}(E)$ has the following properties:

A1. $\mathcal{H}$ - RKHS, consisting of entire functions

A2. $f\in\mathcal{H} \Rightarrow f^*\in\mathcal{H}$ and $\|f^*\|=\|f\|$

3. $f\in\mathcal{H}$, $f(w)=0$, $w\in\mathbb{C}\setminus\mathbb{R} \Rightarrow \frac{z-\bar{w}}{z-w}f\in\mathcal{H}$

and $\left\|\frac{z-\bar w}{z-w}f\right\|=\|f\|$.

This is obvious $f/E\in H^2$, $w$ - zero on $\mathbb{C}^+ \Rightarrow \frac{1}{b_w}f/E\in H^2$

$\frac{z-w}{z-\bar w}f^*\in H^2$ - trivial. (division by Blaschke)

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**Th** Let $\mathcal{H}$ satisfy A1 - A3. Then $\exists$ $E\in HB$ such that $\mathcal{H}=\mathcal{H}(E)$. (Here $E$ can have real zeros)

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Proof. Let $K_w(z)$ be RK of $\mathcal{H}$.

We need to construct $E, A, B$.

Fix $\alpha\in\mathbb{C}^+$ put $E(z)=(z-\bar\alpha)K_\alpha(z)$. Clearly, $K_\alpha\neq 0$ (exists $f(\alpha)\neq 0$)

$A, B$ as usual.

We need to show, that $E\in HB$.

Let us use the properties.

$K_w(z)=\overline{K_z(w)}$ this is general property for RK

8.

$\forall w \quad K^*_w(z) \in \mathcal{H}$

$$(F, K^*_w)_E = \left(F/E, \frac{K_w}{E}\right)_{L^2} = \overline{\left(\frac{\overline{F}}{E}, \frac{K_w}{E}\right)_{L^2}} =$$

$$= \overline{(F^*, K_w)_E} = \overline{F^*(w)} = F(\overline{w}) = (F, K_{\overline{w}})$$

Thus, $\boxed{\overline{K_w(\overline{z})} = K_{\overline{w}}(z)}$

$$h(z) = K_w(z) - \frac{K_w(\alpha)}{K_\alpha(\alpha)} K_\alpha(z) \in \mathcal{H} \quad \text{u} \quad h(\alpha) = 0$$

$\frac{z-\overline{\alpha}}{z-\alpha} h \in \mathcal{H}$ Let $f \in \mathcal{H}, f(\overline{\alpha}) = 0$

$$\left(f, \frac{t-\overline{\alpha}}{t-\alpha} h\right)_E = \left(\frac{t-\alpha}{t-\overline{\alpha}} f, h\right) = f(w)\frac{w-\alpha}{w-\overline{\alpha}} =$$

$$= \left(f, \left(K_w - \frac{K_w(\overline{\alpha})}{K_{\overline{\alpha}}(\overline{\alpha})} K_{\overline{\alpha}}\right) \frac{\overline{w}-\overline{\alpha}}{\overline{w}-\alpha}\right)_E$$

true for any $f: f(\overline{\alpha}) = 0$ and for $K_{\overline{\alpha}}$ also. $\Rightarrow \forall f$

Thus,

$$\frac{z-\overline{\alpha}}{z-\alpha}\left(K_w - \frac{K_w(\alpha)}{K_\alpha(\alpha)} K_\alpha\right) = \frac{\overline{w}-\overline{\alpha}}{\overline{w}-\alpha}\left(K_w - \frac{K_w(\overline{\alpha})}{K_{\overline{\alpha}}(\overline{\alpha})} \cdot K_{\overline{\alpha}}\right)$$

Note that $K_{\overline{\alpha}}(\overline{\alpha}) = K_\alpha(\alpha)$

Express $K_w$:

$$K_w\left[\frac{(z-\overline{\alpha})(\overline{w}-\alpha) - (z-\alpha)(\overline{w}-\overline{\alpha})}{(z-\overline{w})(\alpha-\overline{\alpha})}\right] =$$

$$= \frac{1}{K_\alpha(\alpha)}\left[-(z-\overline{\alpha})(\overline{w}-\alpha) K_w(\alpha) K_\alpha + (z-\alpha)(\overline{w}-\overline{\alpha}) K_w(\overline{\alpha}) K_{\overline{\alpha}}\right]$$

Thus, $K_w(z) = \dfrac{-(z-\bar\alpha)(\bar w-\alpha)K_w(\alpha)K_\alpha + (z-\alpha)(\bar w-\bar\alpha)K_w(\bar\alpha)K_{\bar\alpha}}{(\alpha-\bar\alpha)K_\alpha(\alpha)(z-\bar w)}$.

Now, put $E = (z-\bar\alpha)K_\alpha$

$$A = \frac{E+E^*}{2} = \frac{1}{2}\left((z-\bar\alpha)K_\alpha + (z-\alpha)K_{\bar\alpha}\right)$$

(use plain frame).

$$B = \frac{E^*-E}{2i} = \frac{1}{2i}\left((z-\alpha)K_{\bar\alpha} - (z-\bar\alpha)K_\alpha\right)$$

Consider $\tilde K_w(z) = \dfrac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{\pi(z-\bar w)}$

We will show that $\tilde K_w = c_\alpha K_w$, where $c_\alpha > 0$ depends only on $\alpha$

then $\tilde K_w(w) > 0$ and so $E \in HB$.

$$\tilde K_w(z) = \frac{1}{4\pi i}\Bigg[\frac{\left((z-\alpha)K_{\bar\alpha} - (z-\bar\alpha)K_\alpha\right)\left((\bar w-\alpha)\overline{K_\alpha(w)} + (\bar w-\bar\alpha)\overline{K_{\bar\alpha}(w)}\right) + \left((z-\bar\alpha)K_\alpha + (z-\alpha)K_{\bar\alpha}\right)\left((\bar w-\bar\alpha)\overline{K_{\bar\alpha}(w)} - (\bar w-\alpha)\overline{K_\alpha(w)}\right)}{z-\bar w}\Bigg]$$

$$= \frac{1}{2\pi i}\left[\frac{(z-\alpha)(\bar w-\bar\alpha)\overline{K_{\bar\alpha}(w)}K_{\bar\alpha} - (z-\bar\alpha)(\bar w-\alpha)\overline{K_\alpha(w)}K_\alpha}{z-\bar w}\right]$$

$\overline{K_{\bar\alpha}(w)} = K_w(\bar\alpha)$, $\overline{K_\alpha(w)} = K_w(\alpha)$

$$= +\frac{K_\alpha(\alpha)(\alpha-\bar\alpha)}{2\pi i}K_w(z)$$

$$\alpha \in \mathbb{C}^+ \Rightarrow c_\alpha = \frac{K_\alpha(\alpha)(+\operatorname{Im}\alpha)}{\pi} > 0.$$

Т.о., нашу ф-ю $E$ надо умн. на $\frac{1}{\sqrt{c_\alpha}}E$. So $\tilde K_w(z) = K_w(z)$ △

We have two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}(E)$

$K_w \in \mathcal{H}, \mathcal{H}(E)$, Span $K_w$ is dense.

$K_w$ are RK for $\mathcal{H}$ $\mathcal{H}(E) \Longrightarrow$

$$\|\sum c_j K_{w_j}\|_{\mathcal{H}} = \|\sum c_j K_{w_j}\|_{\mathcal{H}(E)}$$

Let $f \in \mathcal{H}$, $\exists f_n \in \text{Span}\{K_w\}$

$f_n \to f$ on $\mathcal{H}$ (in part pointwise)

but $f_n \to \tilde{f}$ in $\mathcal{H}(E)$ (and pointwise)

$\Longrightarrow f = \tilde{f}$, i.e. $f \in \mathcal{H}(E)$

Rem. Description of all E.

# V. De Branges' ordering theorem

From now on any $E \in HB$ has no real zeros.

**Def.** $\mathcal{H} = \mathcal{H}(E)$ a de Branges space. We say that $\mathcal{H}_1 \subset \mathcal{H}$ is a de Branges subspace, if $(\mathcal{H}_1, \|\ \|_{\mathcal{H}})$ is a dB space.

($\Longleftrightarrow$ $\mathcal{H}_1$ is closed under $f \mapsto f^*$ and $\forall f \in \mathcal{H}_1$ with $f(w) = 0 \Rightarrow \frac{f}{z-w} \in \mathcal{H}_1$)

division property

**Th.** $\mathcal{H}$ - dB, $\mathcal{H}_1$ $\mathcal{H}_2$ - two dB subspaces. (without real zeros)

Then either $\mathcal{H}_1 \subset \mathcal{H}_2$ or $\mathcal{H}_2 \subset \mathcal{H}_1$.

dB subspaces form a chain ordered by inclusion.

Ex. $PW_a$ $\quad PW_t$, $0 < t \le a$ - all dB subspaces.

## Preparation: functions of bounded type.

Let $f \in Hol(\mathbb{C}^+)$, $f$ is of bounded type in $\mathbb{C}^+$, if $f = g/h$, where $g, h \in H^\infty(\mathbb{C}^+)$.

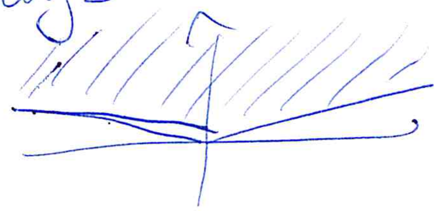
$$f = \frac{O_{|g|}}{O_{|h|}} \cdot B \cdot \underbrace{\frac{S_g}{S_h}}_{\text{singular inner f-s}}$$

$$O_{|g|} = \exp\left(\frac{1}{\pi i}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)\log|g(t)|\,dt\right).$$

1) Useful observation:

$$\log|O_g(z)| \le \frac{y}{\pi}\int \frac{|\log|g(t)||}{|t-z|^2}\,dt = o(|z|)$$

$|z| \to \infty$, $\delta < \arg z < \pi - \delta$



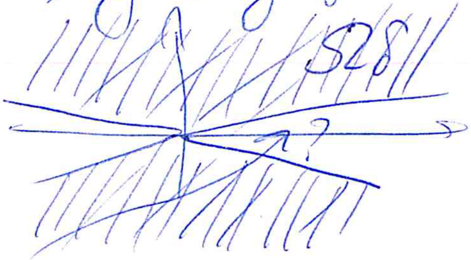
We will need the case $f$ is entire and of bdd type or $f$ is a ratio of two entire functions and of bdd type.

In this case $f = O_{|f|} \cdot B\, e^{ihz}$, $h \in \mathbb{R}$ (no singular measure on $\mathbb{R}$).

2) Th (Krein, 1940s) $f$ entire, $f$ is of bdd type in $\mathbb{C}^+$ and in $\mathbb{C}^-$ $\Longrightarrow$ $f$ is of finite exponential type.

$$|f(z)| \le C_1 e^{C_2|z|}$$

Sketch: $\log|f(z)| \lesssim |z|$ in




$\Omega_\delta = \{\delta < \arg z < \pi - \delta\} \cup \{\pi + \delta < \arg z < 2\pi - \delta\}$

How to estimate here

Use subharmonicity:

$$\log|f(x_0)| \le \frac{1}{\pi r^2} \int_{|z-x_0|\le r} \log|f(z)|\, dm_2(z),$$

plug in the estimate

$$\log|f(z)| \le \frac{y}{\pi} \int_{\mathbb{R}} \frac{t^2+1}{|t-z|^2} \cdot v(t)\, dt, \quad v \ge 0, \ v \in L^1(\mathbb{R})$$

to get $\log|f(x_0)| \lesssim |x_0|^{N}$ $(N=2)$

Thus, $f$ is of finite order and of exp type on $S_\delta$ $\forall \delta > 0$ $\Longrightarrow$ $f$ is of exp. type..

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c) If $f \in Hol(\mathbb{C}^+)$, $\operatorname{Im} f > 0 \Rightarrow f$ is of bdd type

$\frac{1}{f+i}$ bdd $\Rightarrow f+i$ of bdd type.

Thus $\int \frac{v(t)}{t-z}\, dt$ is of bdd type $\forall\, v \ge 0$ $v \in L^1(\mathbb{R})$ and thus $\forall\, v \in L^1(\mathbb{R})$.

d) De Branges' Lemma (a version of Phragmén-Lindelöf, a deep result).

Th. $f, g$ are entire of zero exp. type and

$$\min(|f(z)|, |g(z)|) \le \frac{C}{|\operatorname{Im} z|}, \quad z \in \mathbb{C}.$$

Then $f \equiv 0$, or $g \equiv 0$.

/equiv form: $\min(|f(z)|, |g(z)|) \le 1 \Rightarrow f = \text{const}$ or $g = \text{const}$/

Th. $\mu$ - measure on $\mathbb{R}$, $\int \frac{d\mu(t)}{t^2+1} < \infty$.

$\mathcal{H}_1 = \mathcal{H}(E_1)$, $\mathcal{H}_2 = \mathcal{H}(E_2) \subset L^2(\mu)$ isometrically.

If $E_1/E_2$ is of bdd type in $\mathbb{C}^+$, then

$\mathcal{H}_1 \subset \mathcal{H}_2$ or $\mathcal{H}_2 \subset \mathcal{H}_1$.

$E_1, E_2$ have no real zeros.

Why this theorem is stronger?

$d\mu(t) = \frac{dt}{|E(t)|^2}$, $\mathcal{H}_1\ \mathcal{H}_2 \subset L^2(\mu)$ isometr.

$\frac{E_1}{\cancel{z}-z_1}, \frac{E_2}{z-z_2} \in \mathcal{H} \Rightarrow \frac{E_1}{E(z-z_1)} \in H^2, \frac{E_2}{E(z-z_2)} \in H^2 \Rightarrow$

$\Rightarrow E_1/E_2$ - Cdd type

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<u>Proof:</u> Assume that $\mathcal{H}_1 \not\subset \mathcal{H}_2$, $\mathcal{H}_2 \not\subset \mathcal{H}_1$.

Choose: $F_1 \in L^2(\mu)$ : $F_1 \perp \mathcal{H}_2$, $F_1 \not\perp \mathcal{H}_1$

$F_2 \in L^2(\mu)$ : $F_2 \perp \mathcal{H}_1$, $F_2 \not\perp \mathcal{H}_2$.

Let $F \in \mathcal{H}_1$, $G \in \mathcal{H}_2$.

$$f(w) := \left(\frac{F - \frac{F(w)}{G(w)}G}{z-w}, F_1\right)_{L^2(\mu)} = \int_{\mathbb{R}} \frac{\left(F(t) - \frac{F(w)}{G(w)}G(t)\right)\overline{F_1(t)}}{t-w}\, d\mu(t)$$

$$g(w) = \left(\frac{G - \frac{G(w)}{F(w)}F}{z-w}, F_2\right)_{L^2(\mu)}$$

$f, g$ are correctly defined and analytic on $\{w: G(w) \neq 0\}$ and $\{w: F(w) \neq 0\}$ respectively.

<u>Step 1</u>. $f, g$ are entire, $f$ does not depend on $G$, $g$ does not depend on $F$.

Let $f_1(w) = \left(\frac{F - \frac{F(w)}{G_1(w)}G_1}{z-w}, F_1\right)$

$$f(w) - f_1(w) = F(w)\Bigg(\underbrace{\frac{G(w)G_1 - G_1(w)G}{G_1(w)G(w)}}_{\in \mathcal{H}_2}, F_1\Bigg) = 0$$

Thus, $f = f_1$ on $\{w : G(w) \neq 0, G_1(w) \neq 0\}$

$\left.\begin{array}{l} f \text{ anal on } \{G_1(w) = 0\} \\ f_1 \text{ anal on } \{G(w) = 0\} \end{array}\right\} \Longrightarrow f$ has entire extension.

(if $G(w) = 0$ choose $G_1(w) \neq 0 \Rightarrow f$ anal on $w$ also).

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<u>Step 2</u>. $f, g$ are of exp. type.

$$f(w) = \underbrace{\int \frac{F(t)\overline{F_1(t)}}{t-w}\, d\mu(t) - \frac{F(w)}{G(w)} \int \frac{G(t)\overline{F_1(t)}}{t-w}\, d\mu(t)}_{\text{Bdd type on } \mathbb{C}^+ \text{ and in } \mathbb{C}^-}$$

Take $G = K_i^{\mathcal{H}_2}$. Then $G = E_2 \times (\text{Bdd in } \mathbb{C}^+)$

$$\frac{F}{G} = \underbrace{\frac{F}{E_1}}_{H^2} \cdot \underbrace{\frac{E_1}{E_2}}_{\text{Bdd type}} \cdot \underbrace{\frac{E_2}{G}}_{\frac{1}{\text{Bdd on } \mathbb{C}^+}} \Longrightarrow \frac{F}{G} \text{ of Bdd type}$$

Thus $f$ is of Bdd type in $\mathbb{C}^+$ and in $\mathbb{C}^- \Longrightarrow$

$\Longrightarrow$ Krein th.

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<u>Step 3</u> Either $f$ or $g$ is $\equiv 0$.

<u>Case a</u>: $\quad E_1/E_2 = e^{ibz} \times \text{Outer}$ and $b > 0$.

$\log|O(z)| = o(|z|)$ on $S_{2\delta}$

Hence $E_1/E_2 \longrightarrow 0$ on $S_{2\delta}$

$$|f(w)| \lesssim \left(\left|\frac{F(w)}{G(w)}\right| + 1\right) \frac{1}{|\operatorname{Im} w|}$$

Choosing $G = K_i^{\mathcal{H}_2}$ we see that $\left|\frac{F}{G}\right| \to 0$ on $S_{2\delta}$

Thus, $|f| \to 0$ in $S_{2\delta}$ and $f$ of exp type

$\Longrightarrow f \equiv 0$.

Case 6: $E_1/E_2$ – outer. Then $f, g$ are of zero exp type.

$$|g(w)| \lesssim \left(1 + \left|\frac{G(w)}{F(w)}\right|\right)\frac{1}{|\operatorname{Im} w|}.$$

Thus, $\min\left(|f(w)|, |g(w)|\right) \lesssim \frac{1}{|\operatorname{Im} w|} \Longrightarrow f \equiv 0$ or $g \equiv 0$.

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Step 4. Let $f \equiv 0$. Hence

$$\int \frac{F(t)\overline{F_1(t)}}{t-z}\,d\mu(t) = \frac{F(w)}{G(w)} \int \frac{G(t)\overline{F_1(t)}}{t-z}\,d\mu(t)$$

Note that: if $v \in L^1(\mu)$, then $z \int_{\mathbb{R}} \frac{v(t)}{t-z}\,d\mu \longrightarrow \int_{\mathbb{R}} v\,d\mu$, $|z| \to \infty$ in $\Omega_\delta$

Now choose $F \in \mathcal{H}_1$ so that

$$\int F\overline{F_1}\,d\mu \neq 0$$

Then LHS $\sim -\frac{\int F\overline{F_1}\,d\mu}{z}$. But $\int G\overline{F_1}\,d\mu = 0$

$$\int \frac{G(t)\overline{F_1(t)}}{t-z}\,d\mu(t) = o\left(\frac{1}{z}\right) \Longrightarrow \frac{F(z)}{G(z)} \longrightarrow +\infty \quad |z| \to \infty,\ z \in \Omega_\delta$$

Choose $G \in \mathcal{H}_2$: $\int G\overline{F_2}\,d\mu \neq 0$

$$g(z) = \int \frac{G(t)\overline{F_2(t)}}{t-z}\,d\mu(t) - \underbrace{\frac{G(z)}{F(z)}}_{\to 0} \int \frac{(t)\overline{F_2(t)}}{t-z}\,d\mu(t)$$

$(F, F_2)_{L^2(\mu)} = 0$

Thus, $g \equiv 0$.

$$\underbrace{\int \frac{G(t)\overline{F_2(t)}}{t-z}\,d\mu(t)}_{\sim c/z} = \underbrace{\frac{G(z)}{F(z)}}_{\to 0} \underbrace{\int \frac{F(t)\overline{F_2(t)}}{t-z}\,d\mu(t)}_{= o(1/z)} \quad !!!$$

when $|z| \to \infty$, $z \in \Omega_\delta$

This contradiction proves the Ordering Theorem.