Hilbert Spaces and H^2.

V: a vector space

$\langle \cdot, \cdot \rangle$ a Hermitian product

$\langle x, y \rangle = \overline{\langle y, x \rangle}$

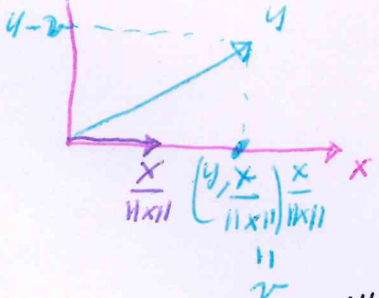
$\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0$ Set $\|x\| = \langle x, x \rangle^{1/2}$

$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

Cauchy – Schwarz inequality

$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$ with $= \iff x, y$ are lin. dip.

Pf.



Geometric intuition (Pythagoras Theorem) suggests that

$\|y\|^2 = \|v\|^2 + \|y - v\|^2$

with $v = \left(y, \frac{x}{\|x\|}\right) \frac{x}{\|x\|}$

(pf. exercise)

and this gives $\|y\|^2 \geq \|v\|^2 = \frac{|(y, x)|^2}{\|x\|^2}$ ∎

The map $V \longrightarrow \mathbb{R}^+$, $x \longmapsto \|x\|$ is a norm:

$\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$

$\|\alpha x\| = |\alpha| \cdot \|x\|$

$\|x + y\| \leq \|x\| + \|y\|$

Coming from an inner product, it enjoys an extra property:

Parallelogram Law

$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

A distance on V is defined by

$d(x, y) = \|x - y\|$:

$d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = 0$

$d(x, y) = d(y, x)$

$d(x, y) \leq d(x, y) + d(y, x)$

$(V, (\cdot,\cdot))$ is a Hilbert space if (V, d) is complete.

Examples.

(i) $\mathbb{C}^n \ni z = (z_1, \dots, z_n)$

$(z, w) = \sum z_j \overline{w_j}$

(ii) A a set $V = \{ A \xrightarrow{\varphi} \mathbb{C} : \|\varphi\|^2_{\ell^2(A)} = \sum_{i \in A} |\varphi(i)|^2 < \infty \}$ $\underset{\ell^2(A)}{\|}$

$(\varphi, \psi) = \sum_{i \in A} \varphi(i) \overline{\psi(i)}$

(iii) (M, μ) a measure space

$V = L^2(\mu) = \{ M \xrightarrow{\varphi} \mathbb{C}$ measurable s.t.

$\|\varphi\|^2_{L^2(\mu)} = \int_M |\varphi(x)|^2 d\mu(x) < +\infty \}$

$(\varphi, \psi) = \int_M \varphi(x) \overline{\psi(x)} d\mu(x)$

In this case, we identify $\varphi = \psi$ in $L^2(\mu)$ if $\varphi(x) = \psi(x)$ μ-a.e.

(iv) $W^{1,2}(\mathbb{R}^n) = \{ \mathbb{R}^n \xrightarrow{f} \mathbb{R} : f \in L^2(\mathbb{R}^n)$ and f has weak derivatives $\partial_j f \in L^2(\mathbb{R}^n)$, $j = 1 \dots n \}$

$\|f\|^2_{W^{1,2}} = \|f\|^2_{L^2} + \sum_{j=1}^{n} \|\partial_j f\|^2_{L^2}$ Sobolev Space

Weak derivative: $\exists \varphi_j \in L^2(\mathbb{R}^n)$:

$\int f \partial_j \psi \, dm = -\int \varphi_j \psi \, dm \quad \forall \psi \in C_0^\infty(\mathbb{R}^n)$

- Completeness is the issue

$H = (H, (\cdot,\cdot))$ is <u>separable</u> if it is so as a metric space. (i)–(iv) ~~Examples~~ are separable.
(ii) is sep. $\Leftrightarrow$ A is countable
(iii) is sep. $\Leftrightarrow$ (M, μ) is σ-finite.
We only deal with separable H.S.

(v) <u>Hardy space</u> H^2.

$a = \{a_n\}_{n=0}^\infty \in \ell^2(\mathbb{N})$

Form $\quad f(z) = \sum_{n=0}^\infty a_n z^n \quad : z \in \mathbb{C},\ |z|<1$

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

f converges in $\mathbb{D}$ to a holomorphic function:

$$\left|\sum_{m+1}^n a_k z^k\right| \le \left(\sum_{m+1}^n |a_k|^2\right)^{1/2} \cdot \left(\sum_{m+1}^n |z|^{2k}\right)^{1/2}$$

$$\le \left(\sum_{m+1}^n |a_k|^2\right)^{1/2} \cdot \frac{|z|^{2(m+1)}}{1-|z|^2} \xrightarrow[m,n\to\infty]{} 0$$

$$H^2(\mathbb{D}) = \{\mathbb{D} \xrightarrow{f} \mathbb{C} : f(z) = \sum_0^\infty a_n z^n \text{ with } \{a_n\}_{n=0}^\infty \in \ell^2\}$$

$$\|f\|_{H^2} = \|\{a_n\}\|_{\ell^2}$$

$$\left\langle \sum_0^\infty a_n z^n, \sum_0^\infty b_n z^n \right\rangle_{H^2} = \sum_0^\infty a_n \overline{b_n}$$

Reproducing kernel.

$f(z) = \langle f, k_z \rangle_{H^2}$ where

$k_z(w) = \sum_0^\infty \bar{z}^n w^n = \dfrac{1}{1-\bar{z}w}$ and

$\|k_z\|^2_{H^2} = k_z(z) = \dfrac{1}{1-|z|^2}$. Also $k_z(w) = \langle k_z, k_w \rangle_{H^2}$

<u>Exercise</u>: Bergman space.

$$A^2 = \left\{ f(z) = \sum_0^\infty a_n z^n : \|f\|^2_{A^2} = \sum_0^\infty \frac{|a_n|^2}{n+1} < +\infty \right\}.$$

Show that f converges for $|z|<1$; $A^2 \supsetneq H^2$; and find a "reproducing kernel" for A^2: $f(z) = \langle f, h_z \rangle_{A_2}$; $h_z \in A^2$.

<u>Exercise</u>: Dirichlet space.

$$\mathcal{D} = \left\{ f(z) = \sum_0^\infty a_n z^n : \|f\|^2_{\mathcal{D}} = \sum_0^\infty (n+1)|a_n|^2 < +\infty \right\}.$$

$\mathcal{D} \subsetneq H^2$: analyze it as above.

- Completeness of $H^2, A^2, \mathcal{D}$ comes from that of $L^2(\mathbb{N}, \mu)$ with $\mu(n) = \begin{cases} 1 & \text{for } H^2 \\ 1/(n+1) & \text{for } A^2 \\ n+1 & \text{for } \mathcal{D} \end{cases}$

- So far, adding the complex variable was just ornament. We will see that it is more significative later on.

<u>Def.</u> A <u>unitary operator</u> $H_1 \xrightarrow{U} H_2$ between Hilbert spaces is a linear bijection s.t.

$$(Ux, Uy)_{H_2} = (x, y)_{H_1} \quad \forall x, y \in H_1.$$

If such a map exists, H_1 and H_2 are <u>unitarily equivalent</u>.

e.g. $H^2(\mathbb{D}) \cong \ell^2(\mathbb{N})$ are unitarily equivalent,

$$\{a_n\}_{n=0}^\infty \overset{U}{\longmapsto} \sum_{n=0}^\infty a_n z^n$$

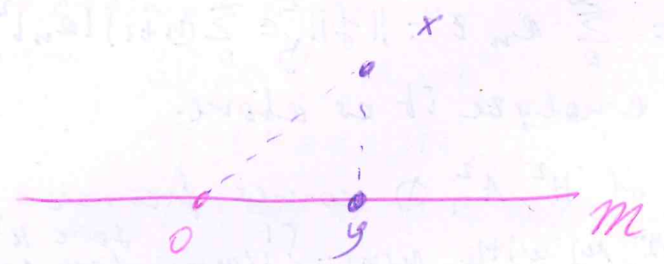
Orthogonality in H, Hilbert.

$x \perp y \iff (x, y) = 0$

Suppose $M \subseteq H$ is a closed subspace.

$M^\perp = \{ y \in H : y \perp x \}$

Lemma. M closed in H, $x \in H$.
Then $\exists!\, y \in M : \|y - x\| = \inf\{ \|z - x\| : z \in M \}$



Pf. Existence. $M \ni y_n : \|y_n - x\| \to d = \inf\{\|z - x\| : z \in M\}$

$\|y_m - y_n\|^2 = \|(y_m - x) - (y_n - x)\|^2$

$= 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - \|\frac{y_m + y_n}{2} - x\|^2 \cdot 4$

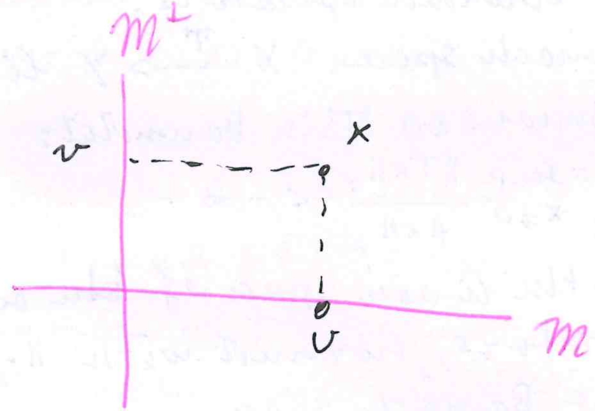
$\le 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - 4d^2 \underset{m,n \to 0}{\longrightarrow} 0$

Hence, $\{y_n\}$ is Cauchy: $y_n \underset{n\to\infty}{\longrightarrow} y \in M$

and $\|y_n - x\| \underset{n\to\infty}{\longrightarrow} d = \|y - x\|$.

Exercise: uniqueness.

Projection Theorem. $M \subseteq H$ closed; $x \in H$.
$\exists!\, v \in M, \; v \in M^\perp : \; x = v + v$.



Pf. Set v: $\|v - x\| = dist(x, M) = d$
and $v = x - v$. If $y \in M$, $t \in \mathbb{C}$, then:

$d^2 \le \|x - (v + ty)\|^2 = \|v - ty\|^2$

$= \|v\|^2 - 2Re[t(y, v)] + |t|^2 \cdot \|y\|^2$

$= d^2 - 2Re[t(y, v)] + |t|^2 \cdot \|y\|^2$.

$t \in \mathbb{R} \Rightarrow 2t\, Re(y, v) \le t^2\|y\|^2 \Rightarrow Re(y, v) = 0$

is $= t \in i\mathbb{R} \Rightarrow 2s\, Im(y, v) \le s^2\|y\|^2 \Rightarrow Im(y, v) = 0$

i.e. $(y, v) = 0 \;\; \forall y \in M$. ∎

We write $H = M \oplus M^\perp$ or $M^\perp = H \ominus M$.

e.g. $M_z \subseteq H^2(\mathbb{D})$: $M_z = \{ f \in H^2 ;\; f(z) = 0 \}$.

(a) M_z is linear

(b) $M_z = \{ f : (f, k_z) = 0 \}$ is closed:
$f_n \to f \Rightarrow |f_n(z) - f(z)| \le \|f_n - f\| \cdot \|k_z\| \to 0$

(c) $M_z^\perp = span(k_z) = \{ c \cdot k_z : c \in \mathbb{C} \}$
(Exercise).

Basics on bounded operators.

X, Y Banach spaces $X \xrightarrow{T} Y$ linear.

T is continuous $\Leftrightarrow$ T is bounded:

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} < +\infty .$$

$\mathcal{B}(X,Y)$ is the linear space of ~~the~~ bounded operators, normed with $\|\cdot\|$

It is a Banach space.

Examples. (i) $H_1 \xrightarrow{U} H_2$ unitary

(ii) $\ell^2(\mathbb{N}) \xrightarrow{S} \ell^2(\mathbb{N})$ $\quad S\varphi(n) = \varphi(n-1)$ or $S\delta_n = \delta_{n+1}$ | Shift operator

Isometry:

$\|S\varphi\|_{\ell^2} = \|\varphi\|_{\ell^2}$ but S is not surjective.

(iii) $\ell^2(\mathbb{Z}) \xrightarrow{S} \ell^2(\mathbb{Z})$: it also is unitary.

(iv)
$$\begin{tikzcd} \ell^2(\mathbb{N}) \arrow[r, "S"] \arrow[d, "U"'] & \ell^2(\mathbb{N}) \arrow[d, "U"] \\ H^2(\mathbb{D}) \arrow[r, "M_z"] & H^2(\mathbb{D}) \end{tikzcd}$$
$U\{a_n\}_{n=0}^{\infty} = \sum_0^{\infty} a_n z^n$

$M_z f(z) = z f(z)$

(v) ~~Projections~~. Orthogonal projections.

$M \subseteq H$ closed; $\quad H \xrightarrow{\pi} H$

$x \longmapsto v \in M : \|v - x\| = \operatorname{dist}(x, M)$

$\pi = \pi_M$ satisfies

(i) $\pi^2 = \pi$

(ii) $(\pi x, y) = (x, \pi y)$ $\quad \forall x, y \in H$.

Pf. (i) obvious (ii) $x = a + b \in M + M^\perp$; $y = c + d \in M + M^\perp$

$\Rightarrow (\pi x, y) = (a, c + d) = (a, c) = (a + b, c) = (x, \pi y)$.

The converse holds. If (i) and (ii) hold for $\pi \in \mathcal{B}(H) := \mathcal{B}(H, H)$, then

$\exists M \subseteq H$ closed s.t. $\pi = \pi_M$.

- An example of projection.

$A \subseteq M$ measurable in $L^2(M)$.

Define $L^2(M) \xrightarrow{\pi} L^2(M)$

$\varphi \longmapsto \chi_A \cdot \varphi$

π is orthogonal projection onto $L^2(A)$

- Conditional expectation.

$(\Omega, \mathcal{F}, \mathbb{P})$ a measure space, $\mathbb{P}(\Omega) = 1$,

$\mathcal{G} \subseteq \mathcal{F}$: σ-algebras.

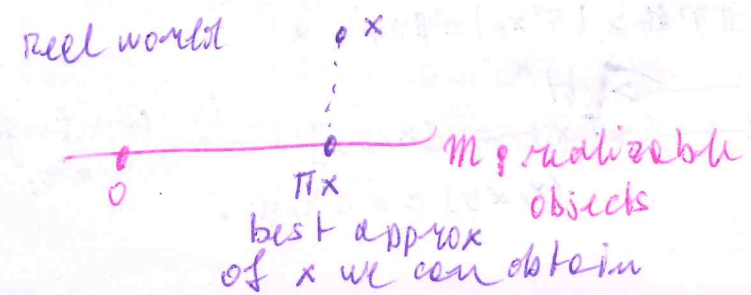
$L^2(\mathcal{G}) = \{\varphi \in L^2(\mathcal{F}) := L^2(\Omega, \mathcal{F}, \mathbb{P}) : \varphi \text{ is } \mathcal{G}\text{-meas.}\}$

is a closed subspace of $L^2(\mathcal{F})$.

$L^2(\mathcal{F}) \xrightarrow{\pi} L^2(\mathcal{G})$ is called conditional expectation:

$\pi\varphi := \mathbb{E}[\varphi \mid \mathcal{G}]$

- In applications



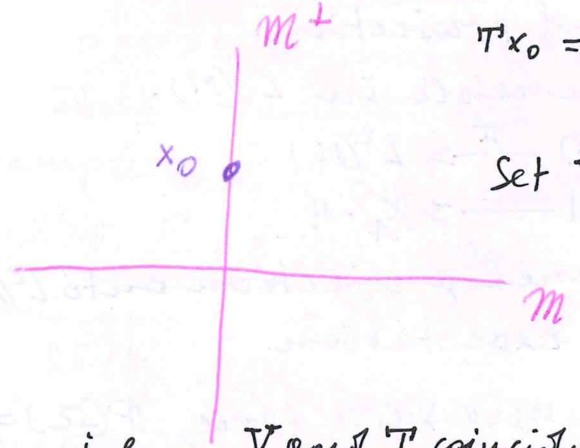
$\mathcal{B}(H, \mathbb{C}) = H^*$: dual space of H.

<u>Riesz Lemma</u>. $\forall T \in H^* \; \exists! \, y_T : Tx = (x, y_T)$.

Moreover: $\|T\| = \|y_T\|_H$.

<u>Pf.</u> $M := \ker T$. If $M = H \Rightarrow y_T = 0$.

$M \neq H \Rightarrow \exists \, x_0 \in M^\perp : \|x_0\| = 1$. We want:



$Tx_0 = (x_0, c x_0) = \bar{c}$:

$y_T = \overline{(Tx_0)} \, x_0$

Set $Vx = (x, \overline{(Tx_0)} x_0)$

Then $Vx_0 = Tx_0$

and $y \in M$

$\Downarrow$

$Vy = 0 = Ty$

i.e. V and T coincide on $\mathrm{span}(x_0) \oplus M$.

But $H = \mathrm{span}(x_0) \oplus M$:

$$y = \left(y - \frac{T(y)}{T(x_0)} x_0 \right) + \frac{T(y)}{T(x_0)} x_0 \text{ with:}$$

$$T\left(y - \frac{T(y)}{T(x_0)} x_0 \right) = 0.$$

About norms we have $|Tx| \leq \|x\| \cdot \|y_T\|$:

$\|T\| \leq \|y_T\|$.

Also $\|T\| \geq |Tx_0| = \|y_T\|$ ∎

$$\begin{aligned} H &\longrightarrow H^* \\ y &\longmapsto [x \mapsto (x, y)] \end{aligned} \quad \text{is conjugate linear:}$$

$$(x, \alpha y) = \bar{\alpha} (x, y).$$

<u>Reproducing</u> <u>Kernel Hilbert</u> <u>Spaces</u>.

H: a Hilbert space of functions $X \xrightarrow{f} \mathbb{C}$ defined on X, a set.

Suppose H has <u>bounded point evaluation</u> at each $x \in X$:

$$\begin{aligned} H &\xrightarrow{\eta_x} \mathbb{C} \\ f &\longmapsto f(x) \end{aligned} \quad \text{is bdd.}$$

Then $\forall x \in X \; \exists k_x \in H : f(x) = (f, k_x)$

Obs. Here the <u>structure</u> is not just a vector space and a inner product: we have <u>points</u>.

$H^2, A^2, \mathcal{D}$ are R-K.H.S.

$L^2(\mathbb{R}^n)$ is not $\quad W^{1,2}(\mathbb{R})$ is $\quad W^{1,2}(\mathbb{R}^n)$ $n \geq 2$ is not

Obs. that
$$\begin{aligned} X \times X &\xrightarrow{k} \mathbb{C} \\ (x, y) &\longmapsto k(x, y) := (k_x, k_y) \end{aligned}$$
is positive semidefinite:

$$\sum c_i k(x_i, x_j) \bar{c}_j = \sum c_i (k_{x_i}, k_{x_j}) \bar{c}_j$$
$$= \left(\sum c_i k_{x_i}, \sum c_j k_{x_j} \right) = \left\| \sum c_i k_{x_i} \right\|^2 \geq 0$$

Moreover it is not ~~strict~~ positive definite $\Leftrightarrow \{k_x\}_{x \in X}$ is not an independent set:

$\exists \; k_x = \sum_i^n c_i k_{x_i} \quad (c_1, \dots, c_n) \neq (0, \dots, 0)$.

$\mathrm{Span}\{k_x : x \in X\}$ is dense in H:

$f \in \mathrm{span}\{k_x : x \in X\}^\perp \Rightarrow f(x) = \langle f, k_x \rangle = 0 \; \forall x$

- This suggests a different presentation of ends:

$X \times X \xrightarrow{k} \mathbb{C}$ positive semidef.

$H' = \operatorname{span}\{k_x : x \in X\}$ where

$k_x : X \to \mathbb{C}$ $\quad k_x(y) = k(x,y)$

Define $(k_x, k_y) = k(x,y)$ and extend to H':

$$\left(\sum c_i k_{x_i}, \sum d_j k_{y_j} \right) = \sum c_i \bar{d}_j \, k(x_i, x_j).$$

To show that it is a inner product, we only need $0 = \| \sum c_i k_{x_i} \|^2 = \sum c_i \bar{c}_j \, k(x_i, x_j)$

$\Rightarrow$ ~~[illegible]~~ $\sum c_i k_{x_i} = 0$

$\Rightarrow$ $(\sum c_i k_{x_i})(x) = \langle \sum c_i k_{x_i}, k_x \rangle = 0.$

We consider cases in which $\{k_x\}_{x \in X}$ are lin. ind., but in some cases they are not.

e.g. $H = \{ [0,1] \xrightarrow{p} \mathbb{C} :$ polynomials of degree $\leq n \}$ with L^2 norm. $\dim(H) = n$, so $k_{x_1}, \dots, k_{x_{n+1}}$ are lin. dep. if $x_1, \dots, x_{n+1} \in [0,1]$ are distinct.

- Yet a third presentation:

H a fixed Hilbert space; k_x vectors ($x \in X$) s.t. $H = \overline{\operatorname{span}(k_x : x \in X)}$.

Define $k : X \times X \to \mathbb{C}$ by

$k(x,y) = (k_x, k_y)$

Identify then $h_x \equiv k_x : y \mapsto k(x,y)$.

<u>Orthonormal basis.</u>

$\{e_\alpha\}_{\alpha \in I} \subseteq H : (e_\alpha, e_\beta) = \delta_\alpha(\beta)$

is a <u>orthonormal system</u>.

It is a basis if $\overline{\operatorname{span}(e_\alpha : \alpha \in I)} = H$.

H is separable $\Leftrightarrow$ it has a countable basis.

In the countable case, given a countable, dense set $\{v_n\}_{n=1}^{\infty}$ in H, the <u>Gram-Schmidt algorithm</u> provides a o.n.b. for H.

Assuming (which is not possible! Why?) that the set is linearly indep., we have:

$e_1 = \frac{v_1}{\|v_1\|}$

$f_2 = v_2 - (v_2, e_1) e_1 \qquad e_2 = \frac{f_2}{\|f_2\|}$

$f_3 = v_3 - \sum_{j=1}^{2} (v_3, e_j) e_j \qquad e_3 = f_3 / \|f_3\|$

$\vdots$

$f_n = v_n - \sum_{j=1}^{n-1} (v_n, e_j) e_j \qquad e_n = f_n / \|f_n\|$

etc.. (Ex. fix details).

It suffices to have $\operatorname{span}\{v_n\}_{n=1}^{\infty}$ dense in H, hence linear independence is not an issue.

In concrete app. the <u>problem is</u> to know <u>a priori</u> that $\overline{\operatorname{span}\{v_n\}_{n=1}^{\infty}} = H$.

e.g. On $L^2\left([0,2\pi], \frac{dt}{2\pi}\right)$ a o.n.b. is given by $v_n(t) = e^{int}$ $(n \in \mathbb{Z})$.

Ex. Polynomials are dense in $C[0,1]$ and $C[0,1]$ is dense in $L^2[0,1]$. Thus, $v_n(x) = x^n$ satisfies

$$\overline{\text{span}\{u_n\}_0^\infty} = L^2[0,1].$$

G.-S. algorithm gives us a o.n.b. for $L^2[0,1]$, say $\{e_n\}_{n=0}^{+\infty}$.

e_n is the n^{th} Legendre polynomial. Fix details.

Thm. (1) Bessel's inequality.

$\{e_n\}_{n=1}^N$ o.n. system $\Rightarrow$

(a) $\|x\|^2 = \sum_{n=1}^N |(x,e_n)|^2 + \|x - \sum_1^N (x,e_n)e_n\|^2$

(Pythagorean Thm.)

(b) $\|x\|^2 \geq \sum_{n=1}^N |(x,e_n)|^2$.

(2) $x = \sum_{n=1}^\infty (x,e_n)e_n$ if $\{e_n\}_1^\infty$ is o.n.b.

(3) $\|x\|^2 = \sum_{n=1}^\infty |(x,e_n)|^2$.

Pf. (1) $x - \sum_1^n (x,e_j)e_j \perp \sum_1^n (x,e_j)e_j \Rightarrow$ (a) $\Rightarrow$ (b)

$\Rightarrow$ (2, 3) By (1): R.H.S. $\leq$ L.H.S.

If $\sum_1^\infty |(x,e_n)|^2$ converges then

$$y_N = \sum_1^N (x,e_n)e_n \xrightarrow[N\to\infty]{} x' \in H$$

Now: $(x-x', e_n) = \lim_{N\to\infty} \left(x - \sum_1^N (x,e_j)e_j, e_n\right)$

$= (x,e_n) - (x,e_n) = 0 \quad \forall n \Rightarrow x - x' \perp e_n \; \forall n$

$\Rightarrow x = x'$.

Also: $0 = \lim_{N\to\infty} \|x - \sum_1^N (x,e_n)e_n\|^2$

$\overset{\text{orth.}}{=} \lim_{N\to\infty} \left(\|x\|^2 - \sum_1^N |(x,e_n)|^2\right)$ ∎

Hardy space and Fourier series.

Suppose $f(z) = \sum_0^\infty a_n z^n \in H^2(\mathbb{D})$

~~Then~~ and set $f_r(t) = f(re^{it})$

Then $\hat{f}_r(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f_r(t) e^{-int}\,dt$

$$= \begin{cases} a_n r^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

So that $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f_r(t)|^2 dt = \sum_0^\infty |a_n|^2 r^{2n} \nearrow_{r\to 1} \|f\|_{H^2}^2$

i.e. $\|f\|_{H^2}^2 = \lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^2 dt =$

$= \sup_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^2 dt.$

e. The family $\{f_r\}_{0<r<1}$ is Cauchy in L^2:

$$\|f_r - f_q\|^2_{L^2} = \sum_0^\infty |a_n|^2 (r^{2n} - q^{2n})$$

$0<q<r<1$

$$\leq \sum_0^N |a_n|^2 (r^{2n} - q^{2n}) + 2\sum_{N+1}^\infty |a_n|^2$$

$$\leq (r-q)\cdot 2N \cdot \sum_0^\infty |a_n|^2 + 2\sum_{N+1}^\infty |a_n|^2$$

Choose first N, then $r-q$.

Then $\exists \lim_{r\to 1} f(re^{it}) = f^*(e^{it}) \in L^2$ in the L^2 sense.

and (i) $\|f^*(e^{it})\|^2_{L^2} = \sum_0^\infty |a_n|^2$

(ii) $\widehat{f^*}(n) = 0 \quad \forall n < \infty$.

We will see that $\lim_{r\to 1} f(re^{it}) = f^*(e^{it})$ converges pointwise a.e.

<u>An example of projection</u>.

$$\begin{array}{ccc} L^2\left([0,2\pi], \frac{dt}{2\pi}\right) & \xrightarrow{\ \Pi\ } & H^2(\mathbb{D}) \\ \mathcal{F}\downarrow & & \downarrow\mathcal{F} \\ \ell^2(\mathbb{Z}) & \xrightarrow{\ \hat{\Pi}\ } & \ell^2(\mathbb{N}) \end{array}$$

$$\varphi(t) = \sum_{-\infty}^{+\infty} a_n e^{int} \longmapsto \sum_0^\infty a_n e^{int} = \Pi\varphi(t) = f^*(e^{it})$$

Is there a better way to write Π?

<u>Proposition</u> Let $\varphi \in L^2\left([0,2\pi], \frac{dt}{2\pi}\right)$

Then $\Pi\varphi(z) = \frac{1}{2\pi i}\int_{|w|=1} \frac{\varphi(w)}{w - z}\, dw$

(CAUCHY INTEGRAL) $= \frac{1}{2\pi i}\int_{|w|=1} \frac{\varphi(w)}{1-\bar{w}z}\, dw \quad \forall z \in \mathbb{D}$

<u>Pf.</u> The integral converges if $\varphi \in L^1 \supseteq L^2$.

We can develop, with $w = e^{it}$, $z = e^{i\tau}$:

$$\frac{1}{2\pi i}\int_{|w|=1} \frac{\varphi(w)}{1-\bar{w}z}\, dw = \frac{1}{2\pi i}\int_{|w|=1} \sum_0^\infty \bar{w}^n z^n \cdot \sum_{-\infty}^{\infty} \hat{\varphi}(m) w^m\, dw$$

$$= \sum_{m,n} z^n \hat{\varphi}(m) \frac{1}{2\pi i}\int_{|w|=1} w^{m-n}\, dw$$

$$= \sum_{n=0}^\infty \hat{\varphi}(n) z^n = \Pi\varphi(z), \text{ at least formally.}$$

The issue of convergence can be dealt with in at least two ways. (a) Setting $\varphi_r(w) = \sum_{-\infty}^{+\infty} \hat{\varphi}(m) r^{|m|} e^{int}$ for $w = re^{it}$ (harmonic extension of φ to $\mathbb{D}$) or

(b) $\sum_{m=-\infty}^{+\infty} \hat{\varphi}(m) e^{imt}$ converges in L^2, hence, L^1, then we can switch $\int\sum = \sum\int$, using the fact that $t \longmapsto \frac{1}{1-\bar{z}e^{it}}$ is continuous on $[0,2\pi]$ ∎