Linear Systems (SISO) and $H^2(\mathbb{D})$

Signals in discrete time $\{a_n\}_{n=0}^{+\infty} \subseteq \mathbb{C}$

$$\xrightarrow{I} \boxed{T} \xrightarrow{O}$$

T is **linear** $T(\lambda a + \mu b) = \lambda T(a) + \mu T(b)$

stable e.g. $\|T(a)\|_{\ell^2} \le C\|a\|_{\ell^2}$

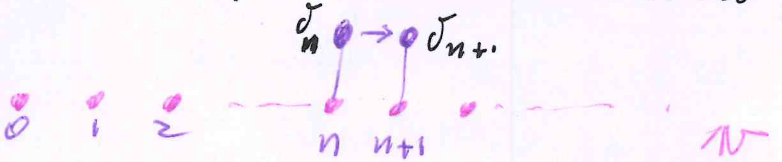
time-invariant $Sa(n) = a(n-1)$: $TS = ST$

causal $a(n) = 0 \ \forall n < 0 \Rightarrow Ta(n) = 0 \ \forall n < 0$

- Linearity is a feature of most (not all) systems in engineering
- Stability has many versions, not equivalent. e.g. $\|T(a)\|_{\ell^\infty} \le C \cdot \|a\|_{\ell^\infty}$ is very common.
- Time invariance is a basic requirement
- Causality is encoded in the restriction $n \in \mathbb{N}$ (instead of $n \in \mathbb{Z}$).
- A further requirement is that the system does not encode too much info:

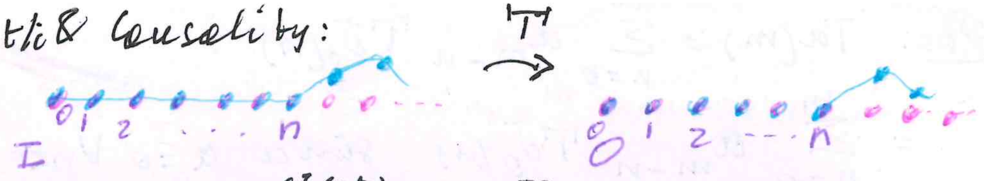
(*) $\boxed{\exists D > 0: \ a_m = 0 \ \forall m \le n - D \Rightarrow Ta(n) = 0.}$

- Linear systems are also called "plants".



$S\delta_n = \delta_{n+1}$

tid & Causality:



For $a \in \ell^2(\mathbb{N})$, $Za = \sum_0^\infty a_n z^n \in H^2(\mathbb{D})$ is its z-transform (engineers use $1/z^n$ inst.)

Theorem 1 The system $\ell^2(\mathbb{N}) \xrightarrow{T} \ell^2(\mathbb{N})$ is linear, stable, time invariant, causal $\Leftrightarrow \exists b \in H^\infty(\mathbb{D})$ s.t.

(**) $Z(Ta)(z) = b(z) Za(z) \quad \forall z \in \mathbb{D}$.

Moreover, $\|T\| := \sup_{a \in \ell^2} \frac{\|Ta\|_{\ell^2}}{\|a\|_{\ell^2}} = \|b\|_{H^\infty}$.

Proof. Write $a = \sum_{n=0}^{+\infty} a_n \delta_n$; $\delta_n(m) = \begin{cases} 1 & m = n \\ 0 & m \ne n \end{cases}$

The series converges in $\ell^2(\mathbb{N})$, then (linearity + stability)

$$Ta = \sum_{n=0}^\infty a_n T\delta_n = \sum_{n=0}^\infty a_n T(S^n \delta_0) =$$

$$= \sum_{n=0}^\infty a_n S^n(T\delta_0) \quad \text{by time invariance}$$

hence, $Ta(m) = \sum_{n=0}^\infty a_n T\delta_0(m-n) = a * T\delta_0(m)$.

$\boxed{Ta = a * T\delta_0.}$ $T\delta_0$: unit impulse response

Obs. A priori: $\|Ta\|_{\ell^2} \le \|T\delta_0\|_{\ell^1} \cdot \|a\|_{\ell^2}$

$$\|Ta\|_{\ell^2} = \Big(\sum_m \Big|\sum_n a(m-n) T\delta_0(n)\Big|^2\Big)^{1/2}$$

$$\le \sum_n |T\delta_0(n)| \cdot \Big(\sum_m |a(m-n)|^2\Big)^{1/2} \quad \text{(Minkowsky)}$$

$\le \|T\delta_0\|_{\ell^1} \cdot \|a\|_{\ell^2}$: we can do better.

Obs. $Ta(m) = \sum_{n=0}^{\infty} a_{m-n} T'\delta_0(n)$

$= \sum_{n=0}^{m} a_{m-n} T'\delta_0(n)$ since $a_n = 0 \ \forall n<0$

$= \sum_{n=0}^{m} a_n T'\delta_0(m-n)$ since $T'\delta_0(n) = 0 \ \forall n<0$

If we add (*), we have:

values T takes into account when computing $Ta(n)$

$$T'a(m) = \sum_{n=0}^{\min(m, D-1)} a_{m-n} \cdot T'\delta_0(n),$$

because only $a_m, a_{m-1}, \ldots, a_{m-D+1}$ are considered.

i.e. $T'\delta_0(n) = 0 \begin{cases} \forall n<0 \\ \forall n \geq D \end{cases}$

In this case, $Z(T'\delta_0)$ is a polynomial of degree $\leq D-1$ (or a rational function if we use $1/z$): this justify engineers' interest in polynomials/rational functions.

Stability $\Rightarrow Ta \in \ell^2 \subseteq \ell^\infty$, hence,

$Z(Ta)(z) = \sum_{m=0}^{\infty} Ta(m) z^m$ converges for $|z|<1$:

$$Z(Ta)(z) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} a_{m-n} T'\delta_0(n) z^m$$

$$= \sum_{n=0}^{\infty} T'\delta_0(n) z^n \sum_{m=n}^{+\infty} a_{m-n} z^{m-n}$$

$= Z(T'\delta_0)(z) \cdot Za(z)$ i.e. we have (**) with $b = Z(T'\delta_0)$

A priori $b \in H^2(\mathbb{D})$.

~~The direction~~ We will use the following results on multiplication of holomorphic functions.

Th. 2 Let $b \in Hol(\mathbb{D})$ and define

$H^2(\mathbb{D}) \xrightarrow{M_b} H^2(\mathbb{D})$ by $M_b f = b \cdot f$

Then M_b is bounded $\Leftrightarrow b \in H^\infty(\mathbb{D})$, and, moreover

$$\|M_b\|_{\mathcal{B}(H^2(\mathbb{D}))} = \|b\|_{H^\infty(\mathbb{D})}$$

Pf. (T.2). It suffices to prove the equality of norms (Exercise).

In one direction: ~~since $k \in H^2(\mathbb{D})$~~.

$|b(z) k_z(z)| = |\langle M_b k_z, k_z \rangle| \leq \|M_b\| \cdot \|k_z\|^2$

$= \|M_b\| \, k_z(z) \Rightarrow |b(z)| \leq \|M_b\| \ \forall |z|<1.$

That reasoning works for all R.K.H.S.
The other inequality is more specific:
for $\varphi \in H^2(\mathbb{D})$, $0 \le r < 1$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |M_b \varphi(e^{it})|^2 dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} |b(re^{it})|^2 \cdot |\varphi(re^{it})|^2 dt$$

$$\le \|b\|^2_{H^\infty} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} |\varphi(re^{it})|^2 dt \le \|b\|^2_{H^2} \cdot \|\varphi\|^2_{H^2}$$

$$\Rightarrow \|M_b \varphi\|_{H^2} \le \|b\|_{H^\infty} \cdot \|\varphi\|_{H^2} \quad \#$$

Back to the proof of T.1.:

$\|T\| = \|M_b\|$ because Z is unitary
$= \|b\|_{H^\infty}$ by T.2.

Exercise. Verify that, if $b \in H^\infty(\mathbb{D})$, then $\exists\ \ell^2(\mathbb{N}) \xrightarrow{T} \ell^2(\mathbb{N})$ as in the statement of T.1 s.t.

$Z(Ta) = b \cdot Za \quad \forall a \in \ell^2(\mathbb{N})$ and
$\|T\| = \|b\|_{H^\infty}$.

Hint. $a \mapsto Ta = Z^{-1} M_b Z a$
has the required properties ∎

Obs. $H^\infty(\mathbb{D})$ is an algebra: $\varphi, \psi \in H^\infty \Rightarrow \varphi \cdot \psi \in H^\infty$
it is a Banach algebra: $\|\varphi\psi\|_{H^\infty} \le \|\varphi\|_{H^\infty} \cdot \|\psi\|_{H^\infty}$
it has unit: $1 \in H^\infty(\mathbb{D})$
it is commutative.

$$\begin{tikzcd}
\ell^2(\mathbb{N}) \arrow[r, "T"] \arrow[d, "Z"'] & \ell^2(\mathbb{N}) \arrow[r, "U"] \arrow[d, "Z"] & \ell^2(\mathbb{N}) \arrow[d, "Z"] \\
H^2(\mathbb{D}) \arrow[r, "M_b"] & H^2(\mathbb{D}) \arrow[r, "M_c"] & H^2(\mathbb{D})
\end{tikzcd}$$

$b = Z(T\delta_0) \qquad c = Z(U\delta_0)$

$$M_c M_b = M_{bc}$$

Obs. $\|Z(T\delta_0)\|_{H^\infty} \le \|T\delta_0\|_{\ell^1}$
is an inequality following from Thm 1 and its proof:

$$|Z(T\delta_0)(z)| = \left|\sum (T\delta_0)(n)\, z^n\right| \le \sum |T\delta_0(n)|.$$

The estimate can be much better:
there are $b = Z(T\delta_0)$ s.t.
$\|b\|_{H^\infty} < +\infty$ and $\|T\delta_0\|_{\ell^1} = +\infty$.

- If $T\delta_0 \in \ell^1(\mathbb{N})$, then $\sum T\delta_0(n) z^n$ converges absolutely on $\overline{\mathbb{D}}$ to a continuous function. But $H^\infty(\mathbb{D}) \setminus C(\overline{\mathbb{D}}) \ne \emptyset$, hence $\|T\delta_0\|_{\ell^1} = +\infty$ can happen.
- Even for ~~polynomial~~ rational b it can happen that $\|b\|_{H^\infty} < \|T\delta_0\|_{\ell^1}$. Let $|a|<1$.

$$b(z) = \frac{a-z}{1-\bar{a}z} = (a-z)\sum \bar{a}^n z^n = a + \sum_1^\infty |a|^2 \bar{a}^{n-1} z^n$$
$$- \sum_0^\infty \bar{a}^n z^{n+1} = a + \sum_1^\infty (1-|a|^2)\, \bar{a}^{n-1} z^n.$$

$$\|T\delta_0\|_{\ell^1} = |a| + \frac{1-|a|^2}{1-|a|} = |a| + 1 + |a| = 1 + 2|a|$$

For an example we have

$\|T\bar{o}\|_{\ell^1} = 1 + 2|a|$, while $\left\|\frac{a-z}{1-\bar{a}z}\right\|_{H^\infty} = 1$

- ~~We can make things worse with a Blaschke product.~~

~~Suppose $\sum(1-|a_n|) < \infty$, so that~~

~~$b(z) = \prod \frac{z-a_n}{1-\bar{a}_n z}$ converges~~

~~and $\|b\|_{H^\infty} = 1$.~~

~~But $b = Z(k_1 * \cdots * k_n * \cdots) = Z(k)$,~~

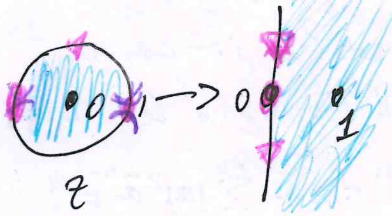
~~with $\frac{z-a_n}{1-\bar{a}_n z} = Z(k_n)(z)$~~

~~hence $\|k\|_{\ell^1} = \|k_1 * \cdots * k_n * \cdots\|_{\ell^1}$~~

~~and we can find $\{a_n\}$ so that the latter is $+\infty$.~~

- $g(z) = e^{-\frac{1+z}{1-z}}$

$w = \frac{1+z}{1-z}$



(Special case of singular inner function:

$\mu = \delta_0/2\pi$ in

$g(z) = \exp\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\right)$

with $d\mu \perp dt$)

Calculation: $g(e^{it}) = e^{-i\cot(t/2)}$

which is discontinuous at $t=0$.

Hence, $\|Z^{-1}g\|_{\ell^1} = +\infty$, yet $\|g\|_{H^\infty} = 1$.

After passing through the system, information is lost ("filtered").

$$\ell^2 \longrightarrow \boxed{T} \xrightarrow{T(\ell^2)}$$

What can we say on $T(\ell^2)$?

- $T(\ell^2)$ is time-invariant:

$\varphi \in T(\ell^2) \Rightarrow \tau\varphi \in T(\ell^2)$

Pf. $M_z(b\varphi)(z) = z\, b(z)\varphi(z) = b(z)\, z\varphi(z)$,

- $T(\ell^2) \subseteq \ell^2$ might not $M_b^{\mathcal{D}}(H^2)$ be closed:

$(1-z)H^2 \neq H^2$ since $(1-z)\varphi(z) \neq 1$

but $\overline{(1-z)H^2} = H^2$ because

$1 \in \overline{(1-z)H^2}$: $(1-z)\varphi(z) = 1 \Leftrightarrow \varphi(z) = \frac{1}{1-z}$

but $\varphi_r(z) = \frac{1}{1-rz} \in H^2$ and

$\lim_{r\to 1} \frac{1-z}{1-rz} = 1$ in H^2 (Ex. fill the gaps).

Problem. Identify the closed, shift invariant subspaces of H^2:

$E \subseteq H^2$ linear, $E \neq 0, H^2$,

$M_z E \subseteq E$.

Exercise Which notion of invariance do we have for functions in $\ell^2(\mathbb{N})$.

Obs. $zE \neq E$. Pf. Let $k_E = \min\{n \geq 0 :$ all φ's in E are divisible by $n\}$. Then, $k_{zE} = k_E + 1$.

Thm. (Beurling). E closed, $E \subseteq H^2$, $zE \subseteq E$. Then, $\exists!\, \theta$ inner s.t. $E = \theta H^2$.

Obs. Viceversa, θ inner $\Rightarrow \theta H^2$ is closed, since $f \longmapsto \theta f$ is an isometry on H^2.

Pf. $zE \subsetneq E$: pick $\theta \in E \ominus zE$, $\|\theta\| = 1$.

$\theta \in E$ and $\theta \perp zE$, then

$z^n\theta \in z^n E \subseteq z(z^{n-1}E) \subseteq z^{n-1}E \subseteq \cdots \subseteq zE$

satisfies: $z^n\theta \perp \theta$.

i.e. $\int_{\mathbb{T}} \theta z^n \bar{\theta}\, dt = 0 \quad \forall n \geq 1$, hence $\forall n \in \mathbb{Z}, n \neq 0$;

i.e. $|\theta|^2 = \text{const} = 1$ by normalization.

We want to show $E = \theta H^2$.

$z^n\theta \in E \quad \forall n \geq 0 \Rightarrow \theta H^2 \subseteq E$ (exercise).

Viceversa:

Suppose $f \in E \ominus \theta H^2$

$f \perp \theta H^2 \Rightarrow f \perp \theta z^n \quad \forall n \geq 0$

and $f \in E \Rightarrow z^n f \in zE_{\perp\theta} \; \forall n \geq 1$

$\Rightarrow z^n f \perp \theta \quad \forall n \geq 1$.

$$\Rightarrow \left.\begin{array}{l} \int_{\mathbb{T}} f\bar{\theta}\bar{z}^n\, dt = 0 \quad \forall n \geq 0 \\ \int_{\mathbb{T}} f\bar{\theta} z^n\, dt = 0 \quad \forall n \geq 1 \end{array}\right| \Rightarrow f \equiv 0.$$

$\#$

Uniqueness: $\theta_1 \theta_2$

$$\theta_1 H^2 = \theta_2 H^2 \Rightarrow \theta_2 \bar{\theta}_1 = \alpha \in \mathbb{C}, \quad |\alpha| = 1,$$

Exercise.

Model Spaces. Let θ be a inner function and define

$$K_\theta = H^2 \ominus \theta H^2$$

to be the corresponding model space.

Property: K_θ is invariant under the backshift operator M_z^*.

Pf. The backshift is the adjoint of the shift: if $\varphi \in H^2$

$$M_z^* \varphi(z) = \langle M_z^* \varphi, k_z \rangle_{H^2}$$

$$= \langle \varphi, M_z k_z \rangle_{H^2} =$$

$$= \sum_{n=0}^{\infty} \hat{\varphi}(n) \overline{\hat{k}_z(n-1)}$$

$$= \sum_{m=0}^{+\infty} \hat{\varphi}(m+1) \hat{k}_z(m)$$

$$= \frac{\varphi(z) - \varphi(0)}{z} \quad \left(\text{Exercise: fill in the details}\right)$$

That is, $M_z^* \angle(a) = \angle(\tau^* a)$

where $\tau^* a(n) = a(n+1)$ (a shift to the left).

Suppose $\varphi \in K_\theta$. Then $\varphi \perp f\theta$ $\forall f \in H^2$

Hence: $\langle M_z^* \varphi, \theta f \rangle_{H^2} = \langle \varphi, M_z \theta f \rangle_{H^2}$

$$= \langle \varphi, \theta (z f) \rangle_{H^2} = 0 \quad \forall f \in H^2$$

i.e. $M_z^* \varphi \in K_\theta$. ∎

Theorem. K_θ is a R.K.H.S. with reproducing kernel

$$k_z^\theta(w) = k^\theta(z,w) = \frac{1 - \overline{\theta(z)}\, \theta(w)}{1 - \bar{z} w}.$$

Pf. Suppose $\varphi \in K_\theta$. Then

$$\varphi(z) = \left\langle \varphi(w), \frac{1}{1 - \bar{z} w} \right\rangle_{H^2} =$$

$$= \left\langle \varphi(w), \frac{1 - \overline{\theta(z)}\, \theta(w)}{1 - \bar{z} w} \right\rangle_{H^2} + ~~\langle \ldots \rangle~~$$

$$+ \left\langle \varphi(w), \frac{\overline{\theta(z)}\, \theta(w)}{1 - \bar{z} w} \right\rangle_{H^2} =$$

$$= \left\langle \varphi(w), \frac{1 - \overline{\theta(z)}\, \theta(w)}{1 - \bar{z} w} \right\rangle_{H^2}$$

since $\left\langle \varphi(w), \frac{\overline{\theta(z)}\, \theta(w)}{1 - \bar{z} w} \right\rangle_{H^2} = \theta(z) \cdot \langle \varphi, \theta k_z \rangle_{H^2} = 0$ since $\theta k_z \in \theta H^2$

Thus (i) $\varphi(z) = \langle \varphi, k_z^\theta \rangle_{H^2} \quad \forall \varphi \in K_\theta$

and it only remains to show that

(ii) $k_z^\theta \in K_\theta$.

Suppose $f \in H^2$, then

$\langle \theta f, k_z^\theta \rangle_{H^2} =$

$= \langle \theta f, (1 - \overline{\theta(z)}\,\theta) k_z \rangle_{H^2}$

$= \theta(z) f(z) - \overline{\overline{\theta(z)}} \langle \theta f, \theta k_z \rangle_{H^2}$

$= \theta(z) [f(z) - \langle f, k_z \rangle_{H^2}]$

because $|\theta|^2 = 1$ on $\partial \mathbb{D}$

$= 0$. i.e. $k_z^\theta \perp \theta H^2$ $\blacksquare$

<u>Exercise</u>. Show that θH^2 is a R.K.H.S. with kernel

$$h_z^\theta(w) = \frac{\overline{\theta(z)}\,\theta(w)}{1 - \bar{z} w}$$