Dyadic decomposition of a regular space.

Suppose (X,ρ) is ~~a~~ q-regular space ($q>0$). We 1st dyadically decompose X, following David and Christ. This decomposition will lead to a (bi-Lipschitz) identification ~~with~~ of X with ∂G, the (Gromov) boundary of a graph. Set $m = \mathcal{H}^q$.

Theorem (Christ, after David). For $k \in \mathbb{Z}$ ($k \geq k(X)$) there is $\mathcal{F}_k = \{Q^k_\alpha : \alpha \in I_k\}$ countable, where each Q^k_k is open, and there are $\delta \in (0,1)$; $a_0 > 0$; $\eta > 0$; $C_1, C_2 < \infty$ so that

(1) $m(X \setminus \bigcup_\alpha Q^k_\alpha) = 0$

(2) $l \geq k \Rightarrow Q^l_\beta \subseteq Q^k_\alpha$ or $Q^l_\beta \cap Q^k_\alpha = \emptyset$

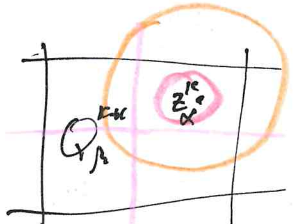
(3) for (k,α) and $l < k \Rightarrow \exists!\,(l,\beta)$: $Q^k_\alpha \subseteq Q^l_\beta$.

(4) $\operatorname{diam}(Q^k_\alpha) \leq C_1 \cdot \delta^k$

(5) $Q^k_\alpha \supseteq B(z^k_\alpha, a_0\delta^k)$ $\exists\, z^k_\alpha \in X$

(6)* $m(x \in Q^k_\alpha : \rho(x, X \setminus Q^k_\alpha) \leq t \cdot \delta^k) \leq C_2 \cdot t^\eta \, m(Q^k_\alpha)$.

$k(X)$ is smallest s.t. $\delta^{k(X)} \leq \operatorname{diam}(X)$



Property (6) is an estimate of the volume near the boundary of Q^k_α.

(2-3) give a tree structure, with "levels" $k \geq k(X)$.

(4,-5) say that Q^k_α is an approximate ball of radius δ^k.

The proof works with little changes for homogeneous spaces of Coifman-Weiss type.

(2-5) work for all metric spaces.

Pf. $\{z^k_\alpha\}_\alpha$: maximal s.t. $\rho(z^k_\alpha, z^k_\beta) \geq \delta^k$

so that $\forall z\, \exists \alpha$: $\rho(z^k_\alpha, z) < \delta^k$

We construct a <u>tree ordering</u>:

$$\begin{tikzcd} & (k-1,\beta) \arrow[dl] \arrow[d] \\ (k,\alpha') & (k,\alpha) \end{tikzcd}$$

where each (k,α) has a unique parent ~~auto~~ $(k-1,\beta)$ and

(i) $\rho(z_\alpha^k, z_\beta^{k-1}) < \delta^{k-1}$

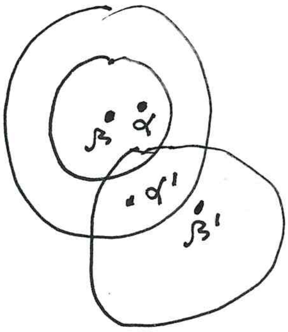
(ii) $\rho(z_\alpha^k, z_\gamma^{k-1}) < \frac{1}{2}\delta^{k-1} \Rightarrow (k,\alpha) \leftarrow (k-1,\gamma)$

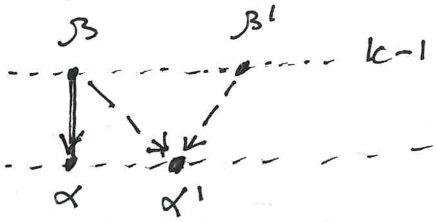
- By maximality $\exists\, z_\beta^{k-1} : \rho(z_\alpha^k, z_\beta^{k-1}) < \delta^{k-1}$

and at most one $z_\gamma^{k-1} : \rho(z_\alpha^k, z_\gamma^{k-1}) < \frac{1}{2}\delta^{k-1}$.

If such z_γ^{k-1} exists: $(k,\alpha) \leftarrow (k-1,\gamma)$.

If not, select one of the β's: $(k,\alpha) \leftarrow (k-1,\beta)$.





Set $Q_\alpha^k = \bigcup_{\substack{(k,\alpha)\\ \downarrow \\ (l,\beta)}} B(z_\beta^l, a_0\delta^l)$ (so that (3) holds)

Also: $(l,\beta) \leftarrow (k,\alpha) \Rightarrow \rho(z_\beta^l, z_\alpha^k) < \delta^k + \dots + \delta^{l-1} < \dfrac{\delta^k}{1-\delta} \leq 2\cdot\delta^k$ if $\boxed{\delta \leq 1/2}$

This gives (4) with $C_1 = 2$.

We prove $Q_\alpha^k \cap Q_\beta^k \neq \phi \Rightarrow \alpha = \beta$. If $x \in Q_\alpha^k \cap Q_\beta^k$:

$$\begin{tikzcd} (k,\alpha) \arrow[d] & (k,\beta) \arrow[d] \\ (m,\gamma) \arrow[dr] & (n,\sigma) \arrow[d] \\ & x \end{tikzcd}$$

then $\rho(z_\gamma^m, z_\sigma^n) < a_0\delta^m + a_0\delta^n \leq 2a_0\delta^n$ $(m \geq n)$

There are two cases:

- $m = n$, then choose $\boxed{2a_0 \leq 1}$ so we have $z_\gamma^n = z_\sigma^n$, and so $(k,\alpha) = (k,\beta)$.

- $m > n$:

$$\begin{tikzcd} (k,\alpha) \arrow[d] & (k,\beta) \arrow[d] \\ (n+1,\lambda) \arrow[d] & (n,\sigma) \arrow[d] \\ (m,\gamma) & x \end{tikzcd}$$

Then $\rho(z_\lambda^{n+1}, z_\sigma^n) \leq \rho(z_\lambda^{n+1}, z_\gamma^m) + \rho(z_\gamma^m, z_\sigma^n)$

$< 2\delta^{n+1} a_0 + 2\delta^n a_0 = 2(\delta + 1) a_0 \cdot \delta^n$

$\leq \frac{1}{2}\delta^n$ if $\boxed{2(1+\delta)a_0 \leq \frac{1}{2}}$

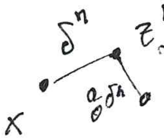
But in this case $(n+1,\lambda) \leftarrow (n,\sigma)$ and again $(k,\alpha) = (k,\beta)$

(2) now follows. If $l \geq k$ and $Q^l_\beta \cap Q^k_\alpha \neq \emptyset$ and $(l,\beta) \leftarrow (k,\gamma)$, then $Q^k_\gamma = Q^k_\alpha \Rightarrow Q^l_\beta \subseteq Q^k_\alpha$.

Property (1) depends on q-regularity.

Fix k and ~~choose~~ set $E = \bigcup_\alpha Q^k_\alpha$, pick $x \in X$, then select z^n_α: $\rho(x, z^n_\alpha) < \delta^n$. If $n \geq k \Rightarrow B(z^n_\alpha, a_0 \delta^n) \subset Q^k_\beta \subseteq E$ $\exists \beta$

and $B(z^n_\alpha, a_0\delta^n) \subset B(x, (1+a_0)\delta^n) = B$



By regularity $m\left(B(z^n_\alpha, a_0\delta^n)\right) \geq c \cdot m(B)$

Thus $$\frac{m(E \cap B)}{m(B)} \geq \frac{m(B(z^n_\alpha, a_0\delta^n))}{m(B)} \geq c > 0$$

As $n \to \infty$: $$\varliminf_{r \to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))} \geq c > 0 \quad \forall x \in X.$$

By Lebesgue differentiation theorem, $\chi_E = \chi_X$ a.e. ∎

<u>Lebesgue diff. T.</u> $E \subseteq X$: a.e. $x \in E \Rightarrow \lim_{r \to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))} = 1$

and a.e. $x \notin E \Rightarrow \lim_{r \to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))} = 0$

~~Theorem~~ We do not need (6) at the moment and its proof is postponed.

Let T be the tree of the qubes Q^k_α, ordered by inclusion as above. To Q^k_α we associate k, its level. We write $|\alpha| = k$.

<u>Theorem</u> T is q-regular for the metric with parameter δ.

<u>Pf.</u> Pick α with $|\alpha| = k$ $m(Q_\alpha) \approx \delta^{qk}$

For $n \geq 0$, consider β with $|\beta| = k+n$. Then:

$$\delta^{qk} \approx m(Q_\alpha) = \sum_{\substack{\beta \subseteq \alpha \\ |\beta| = |\alpha| + n}} m(Q_\beta) \approx \#\{\beta \subseteq \alpha : |\beta| = k+n\} \cdot \delta^{q(n+k)} = \# S_n(\alpha) \cdot \delta^{q(n+k)}$$

Thus $\# S_n(\alpha) \approx \delta^{-qn}$, and the characterization of q regular tree boundaries does the job.