AHLFORS REGULAR SPACES 1

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§1 - Hausdorff measures and regular spaces.

(X, ρ): metric space; $Q \geq 0$. $E \subseteq X$; $\delta > 0$

$$\mathcal{H}^Q_\delta(E) = \inf\Big\{ \sum_j \mathrm{diam}(U_j)^Q : \bigcup_j U_j \supseteq E;\ \mathrm{diam}(U_j) \leq \delta \Big\}$$

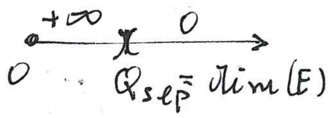
$\mathcal{H}^Q_\delta(E) \nearrow$ as $\delta \downarrow$

$$\mathcal{H}^Q(E) = \lim_{\delta \to 0} \mathcal{H}^Q_\delta(E) = \sup_{\delta > 0} \mathcal{H}^Q_\delta(E)$$ is the Q-<u>Hausdorff measure</u> of E w.r.t. the distance ρ.

(a) $\mathcal{H}^Q$ is a outer measure

(b) Borel sets in X are measurable

$$\dim(E) = \inf\{Q \geq 0 : \mathcal{H}^Q(E) = 0\}$$
$$= \sup\big(\{Q \geq 0 : \mathcal{H}^Q(E) = +\infty\} \cup \{0\}\big)$$



$B(x, r) \subseteq X$: metric ball
$= \{y \in X : \rho(x, y) < r\}$

<u>Def.</u> (X, ρ) is Q-<u>regular</u> if $c_0 \cdot r^Q \leq \mathcal{H}^Q(B(x,r)) \leq c_{00} \cdot r^Q$ for $0 < r \leq \mathrm{diam}(X) \leq +\infty$ (with $Q > 0$).

- Euclidean $\mathbb{R}^n$ is n-regular
- $\mathbb{S}^n$ with the spherical distance is n-regular
- The hyperbolic disc $\left(\{|z| < 1\} \subseteq \mathbb{C};\ ds^2 = \frac{|dz|^2}{(1-|z|^2)^2}\right)$ <u>is not</u> Q-regular for any $Q > 0$
- $\mathbb{R}$ with $\rho(x, y) = |x - y|^{1/2}$ is 2-regular.

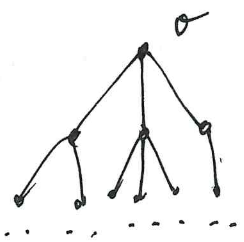
__Snowflaking:__ If (X,ρ) is a metric space and $0<\alpha<1$, then (X,ρ^α) is a metric space. For $E\subseteq X$:

$$\dim_{\rho^\alpha}(E) = \frac{1}{\alpha}\cdot \dim_\rho(E)$$

__Exercise.__ If (X,ρ) is a metric space, $[0,+\infty)\xrightarrow{\Phi}[0,+\infty)$ is concave and $\Phi(0)=0$, then $\Phi(\rho)$ is a distance on X

§2. __Trees__, __their boundaries__, ~~conditions for their regularity~~

$\mathbb{T} = (V(T), E(T))$ is a __tree__ with vertex set $V(T)\equiv T$ and edge set $E(T)$. For each $x\in \mathbb{T}$:



$\#\{e\in E(T) : x \text{ is an endpoint of } e\}<\infty$.

$\mathbb{T}$ is an __infinite tree__ if $\#\{e\in E(T) : x$ is an ~~edge~~ endpoint of $e\}\geq 2$.

$(\mathbb{T}, o)$ is a __rooted tree__ if $o\in \mathbb{T}$ is a fixed __root__.

A __path__ is a function $\{0,1,\dots,N\}\xrightarrow{\gamma}\mathbb{T}$

or $\mathbb{N}\xrightarrow{\gamma}\mathbb{T}$

or $\mathbb{Z}\xrightarrow{\gamma}\mathbb{T}$

or $\{0,-1,-2,\dots\}\xrightarrow{\gamma}\mathbb{T}$

such that $\gamma(n-1)$ and $\gamma(n)$ are endpoints of an edge.

$\{0\}\xrightarrow{\gamma}\mathbb{T}$ is a degenerate path.

The __edge-counting__ (__hyperbolic__) distance

$\mathbb{T}\times\mathbb{T}\xrightarrow{d}[0,+\infty)$ is

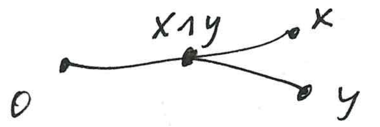
$$d(x,y) = \min\{N : \text{there is a path } \{0,\dots,N\}\xrightarrow{\gamma}\mathbb{T} \text{ with } \gamma(0)=x \text{ and } \gamma(N)=y\}$$

The unique injective path joining x and y is also the unique path realizing the minimum and it is denoted by $[x,y]$: the __geodesic__ joining x and y. If we consider it unoriented, we might identify $[x,y] = \{\gamma(0)=x, \gamma(1), \dots, \gamma(N)=y\}$.

$d : T \times T \to \mathbb{N}$ is a distance on T.

The definition of geodesic extends to injective maps $\mathbb{N} \xrightarrow{\gamma} T$ (half-infinite geodesics) or $\mathbb{Z} \xrightarrow{\gamma} T$ (doubly infinite geodesics).

Given (T,o) rooted, the <u>confluent</u> $x \wedge y \in T$ of $x,y \in T$ is $z = x\wedge y$ s.t. $[o,x] \cap [o,y] = [o,z]$.

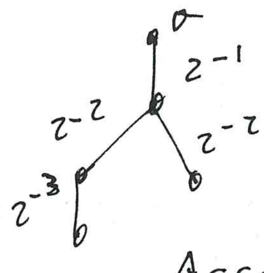


Denote $|x| := d(o,x)$

<u>Exercise.</u> Find a formula showing how $d(o, x\wedge y)$ changes w.r.t. the choice of the root.

<u>Visual metric on T and metric boundary.</u>

To an edge $e = \{x,y\}$ with $|x| = n-1$, $|y| = n$, associate the weight $\rho(e) = 2^{-n}$. Suppose T is an infinite tree (but for the root o, for which $\#\{e \in E(T) : o \text{ is an endpoint of } e\} \geq 1$).

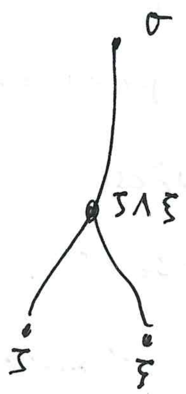


Associate to each path $\{0,1,\dots,N\} \xrightarrow{\gamma} T$ its ρ-length: $\rho\text{-length}(\gamma) = \sum_{i=1}^{N} \rho(\gamma(i), \gamma(i-1))$.

Define $\rho(x,y) = \min\{\rho\text{-length}(\gamma) : \gamma(0)=x, \gamma(N)=y\}$
$= \rho\text{-length}([x,y])$.

(T,ρ) is a metric space; $\operatorname{diam}_\rho(T) < 2$; each $x \in T$ is isolated in (T,ρ).

Let $(\overline{T}, \rho)$ be the completion of T w.r.t. ρ and $\partial T := \overline{T} \setminus T$ be the <u>metric boundary</u> of T.



To each geodesic $\mathbb{N} \xrightarrow{\gamma} T$ s.t. $\gamma(0) = \sigma$ we can associate a point $\zeta \in \partial T$ (since $\{\gamma(n)\}_{n=0}^{\infty}$ is Cauchy in $(\overline{T}, \rho)$) and the map $\gamma \longmapsto \zeta$ is a bijection, in the sense that each point in ∂T has a unique geodesic of this kind representing it (<u>Exercise</u>)

We use the same symbol (e.g. ζ) for the ~~geodesic~~ point $\zeta \in \partial T$ and for the geodesic, which we call $[\sigma, \zeta)$.

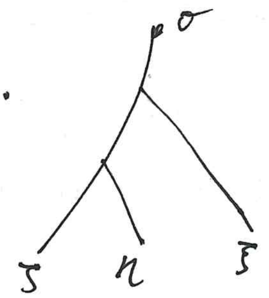
If $\zeta \neq \xi$, then $[\sigma, \zeta) \cap [\sigma, \xi) = [\sigma, \zeta \wedge \xi]$ for a well specified $\zeta \wedge \xi \in T$.

$$\rho(\zeta, \xi) = 2^{1 - |\zeta \wedge \xi|}$$

<u>Proposition.</u> $(\partial T, \rho)$ is a ultrametric space,

$$\rho(\zeta, \xi) \leq \max\{\rho(\zeta, \eta), \rho(\eta, \xi)\} \text{ for } \zeta, \eta, \xi \in \partial T.$$

<u>Pf.</u>



Easy exercise.

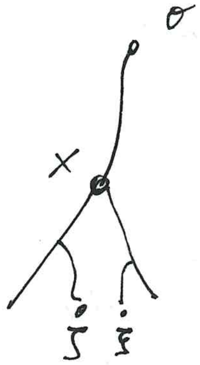
<u>Corollary</u>: If $[0, +\infty) \xrightarrow{\Phi} [0, +\infty)$ is increasing and $\Phi(t) = 0 \Leftrightarrow t = 0$, then $(\partial T, \Phi(\rho))$ is an ultrametric spaa.

§3 – <u>Trees having</u> <u>regular boundary</u>:

We look for manageable criteria to establish whether $(\partial T, \rho)$ is α-regular.

For $x, y \in T$, set $x \leq y$ if $y \in [\sigma, x]$

Let $T(x) = \{z : z \leq x\} \subseteq T$ be the subtree of T having root x. $\partial T(x)$ is a clopen subset of ∂T (Exercise), which can be identified with a metric ball of ∂T: for any $\zeta \in \partial T(x)$,



$$\partial T(x) = \{\xi \in \partial T : \rho(\zeta,\xi) \le 2^{1-|x|}\} = \overline{B(\zeta, 2^{1-|x|})} = B(\zeta, 2^{1-|x|+1/2}) \subseteq \partial T.$$

<u>Exercise</u>. Show that $(\partial T, \rho)$ is compact.

A <u>stopping time</u> below x is $S \subseteq T(x)$ such that (i) $\bigcup_{z\in S} \partial T(z) = \partial T(x)$; (ii) $z \neq w \Rightarrow \partial T(z) \cap \partial T(w) = \emptyset$.

By compactness, stopping times are finite.

Set $\mathrm{Stop}(x) = \{S \subseteq T(x) : S \text{ is a s.t.}\}$.

<u>Theorem</u>. T.F.A.E.

(I) $(\partial T, \rho)$ is Q-regular;

(II) $\exists\, 0 < c_1 < c_2$ s.t. $\forall x \in T$ $\forall S \in \mathrm{Stop}(x)$:

$$c_1 \cdot 2^{-|x|Q} \le \sum_{y\in S} 2^{-|y|Q} \le c_2 \cdot 2^{-|x|Q}$$

(III) $\exists\, 0 < c_3 < c_4$: $\forall x \in \partial T$ $\forall k \ge 0$

$$c_3 \cdot 2^{Qk} \le \# S_k(x) := \#\{y \in x : d(x,y) = k\} \le c_4 \cdot 2^{Qk}$$

These characterizations of Q-regularity are in Ao, MONGUZZI, SALVATORI (2019)

<u>Pf.</u> (I) $\Rightarrow$ (II) If $S \in \mathrm{Stop}(x)$, then

$$2 \cdot c_0 \cdot 2^{-Q|x|} \le \mathcal{H}^Q(\partial T(x)) = \sum_{y \in S} \mathcal{H}^Q(\partial T(y)) \le 2 \cdot c_{00} \cdot \sum_{y\in S} 2^{-Q|y|}$$

(II) $\Rightarrow$ (I) Fix $M > 0$ and cover $\partial T(x) \subseteq \bigcup_n U_n$, $\mathrm{diam}(U_n) \le 2^{-M}$

Replace U_n by $\partial T(w_n)$, $w_n \in T$, with $U_n \subseteq \partial T(w_n)$ and $2^{-|w_n|} < \mathrm{diam}(U_n) \le 2^{1-|w_n|}$

By compactness and the structure of ∂T, after removing some sets, we can assume $\partial T(x) = \sum_{n=1}^{N} \partial T(w_n)$, ~~hence~~ i.e. $\{w_n\}_1^N$ is a stopping set below x and

$\boxed{\text{Obs. } \mathrm{diam}(\partial T(w_n)) = \mathrm{radius}(\partial T(w_n))}$

$$2^{+Q} \sum_1^N \mathrm{diam}(U_n)^Q \ge \sum_1^N \mathrm{diam}(\partial T(w_n))^Q = \sum_1^N 2^{(1-|w_n|)Q} \ge 2^Q c_1 \cdot 2^{-|x|Q}.$$

This shows that $\mathcal{H}^Q(\partial T(x)) \geq c_1 \cdot 2^{-Q} \cdot 2^{Q(1-|x|)} = c_1 \cdot 2^{-Q} \cdot \mathrm{diam}(\partial T(x))$

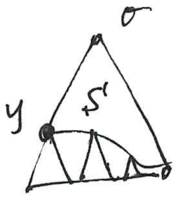
~~In the other direction $\{x\} \in$ ...~~

In the other direction $\{\partial T(w_n)\}_{n=1}^{\infty}$ is a cover with $\mathrm{diam}(\partial T(w_n)) \leq 2^{1-M}$ and

$$\sum_{n=1}^{N} \mathrm{diam}(\partial T(w_n))^Q \leq \sum_{n=1}^{N} 2^{Q(1-|w_n|)} \leq c_2 \cdot 2^{Q(1-|x|)}$$

i.e. $\mathcal{H}^Q(\partial T(x)) \leq c_2 \cdot \mathrm{diam}(\partial T(x))^Q$.

(III) $\Rightarrow$ (II). By rescaling, S.T.P. it for $x = o$.



Let $S' \in \mathrm{Stop}(o)$ and for $y \in S'$ consider $S_{N-|y|}(y)$ with $N = \max\{|z| : z \in S'\}$.

Then,

$$\sum_{v \in S_N(o)} 2^{-Q|v|} = \sum_{y \in S'} \sum_{v \in S_{N-|y|}(y)} 2^{-Q|v|} = \sum_{y \in S'} \sum_{v \in S_{N-|y|}(y)} 2^{-Q(|y|+N-|y|)}$$

$$2^{-QN} \cdot \# S_N(o) \geq c_3 \qquad c_3 \cdot \sum_{y \in S'} 2^{-Q|y|} \leq \sum_{y \in S'} 2^{-Q|y|} \cdot 2^{-(N-|y|)Q} \cdot \# S_{N-|y|}(y)$$

$$c_4 \qquad\qquad c_4 \cdot \sum_{y \in S'} 2^{-Q|y|}$$

$$\Downarrow$$

$$\frac{c_3}{c_4} \leq \sum_{y \in S'} 2^{-Q|y|} \leq \frac{c_4}{c_3}$$

(II) $\Rightarrow$ (III). If $S' \in \mathrm{Stop}(x)$, then

$$c_1 \cdot 2^{-Q|x|} \leq \sum_{y \in S_k(x)} 2^{-Q|y|} \leq c_2 \cdot 2^{-Q|x|}$$

$$= \# S_k(x) \cdot 2^{-Q(|x|+k)}$$

i.e. $$c_1 \leq \# S_k(x) \cdot 2^{-Qk} \leq c_2 \quad \blacksquare$$

e.g. a homogeneous tree, $\#\{e \in E(T) : x \text{ is an endpoint of } e\} = q+1 \ \forall x$, then $\# S_k(x) = q^k = 2^{\log_2 q \cdot k}$ has regular boundary with dimension $Q = \log_2 q$.