Infinite products and Weierstrass factorization

Infinite products in $\mathbb{C}$.

$\{P_n\}_{n=1}^{+\infty}$ in $\mathbb{C}$

$$P = \prod_{n=1}^{\infty} P_n = \lim_{n\to\infty} \prod_{j=1}^{n} P_j \quad \underline{\text{converges}} \text{ if}$$

the limit $\in \mathbb{C}\setminus\{0\}$.

Sometimes we relax to: there might be

$P_{n_1} = \dots = P_{n_m} = 0$, and we ask $\prod_{n \neq n_1 \dots n_m} P_n \in \mathbb{C}\setminus\{0\}$.

$$\prod_{n=1}^{\infty} P_n \text{ converges} \Rightarrow \lim_{n\to\infty} P_n = 1$$

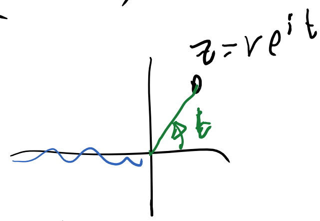
$$\frac{\prod_{n=1}^{N} P_n}{\prod_{n=1}^{N-1} P_n} \xrightarrow[N\to\infty]{} \frac{P}{P} = 1.$$

<u>Exercise</u>: viceversa fails.

<u>Hint</u>. $\prod\left(1-\frac{1}{n}\right)$ diverges

$\prod\left(1-\frac{1}{n^2}\right)$ converges

$$Log(re^{it}) = \log r + it; \quad -\pi < t \le \pi$$



<u>Lemma</u> $\Pi = \prod_{n=1}^{\infty} P_n = \prod_{n=1}^{\infty}(1+a_n)$; $\Sigma = \sum_{n=1}^{\infty} Log(1+a_n)$;

Π converges $\Leftrightarrow$ Σ converges.

<u>Proof</u>. $(\Leftarrow)$ $(1+a_1)\cdot\ldots\cdot(1+a_N) = e^{\underbrace{Log(1+a_1)+\dots+Log(1+a_N)}_{\Sigma_N \to}}$

$P_n = 1 + a_n$

$\Pi_N \to \Longleftarrow$

$(\Rightarrow)$ <u>obs.</u> $\Pi_N = \prod_{n=1}^{N} P_n \to \Pi_\infty \not\Rightarrow Log(\Pi_N) \to Log(\Pi_\infty)$

$(\Rightarrow)$ obs. $\Pi_N = \prod_{n=1} p_n \to \Pi_\infty \not\Rightarrow \mathrm{Log}(\Pi_N) \to \mathrm{Log}(\Pi_\infty)$

$$p_n = e^{iA/n^2} \Rightarrow \Pi_\infty = e^{iA\sum_{n=1}^\infty \frac{1}{n^2}}$$

$$\mathrm{Log}(\Pi_\infty) = i\underbrace{\left[A\sum_{n=1}^\infty \frac{1}{n^2} - 2k\pi\right]}_{(-\pi,\pi]}$$

We know $\frac{\Pi_N}{\Pi_\infty} \xrightarrow[N\to\infty]{} 1 \Rightarrow \mathrm{Log}\left(\frac{\Pi_N}{\Pi_\infty}\right) \to 0$

and $\mathrm{Log}\left(\frac{\Pi_N}{\Pi_\infty}\right) = \mathrm{Log}\left(e^{\mathrm{Log}(p_1)+\cdots+\mathrm{Log}(p_N)-\mathrm{Log}(\Pi_\infty)}\right)$

$$= \mathrm{Log}(p_1)+\cdots+\mathrm{Log}(p_N)-\mathrm{Log}(\Pi_\infty) + h_N 2\pi i$$

$\exists h_N \in \mathbb{Z}$

$$\Rightarrow 2\pi i(h_{N+1}-h_N) = \mathrm{Log}\left(\frac{\Pi_{N+1}}{\Pi_\infty}\right) - \mathrm{Log}\left(\frac{\Pi_N}{\Pi_\infty}\right) - \mathrm{Log}(p_{N+1})$$

$\xrightarrow[N\to\infty]{}$ $0-0-0$: h_N is definitely constant

$h_N = h_\infty \quad N \geq \bar{N}$.

Consider

$$\mathrm{Log}(p_1)+\cdots+\mathrm{Log}(p_N) = \mathrm{Log}(\Pi_\infty) + \mathrm{Log}\left(\frac{\Pi_N}{\Pi_\infty}\right) - h_N 2\pi i$$

$\downarrow N\to\infty$

$$\mathrm{Log}(\Pi_\infty) + 0 - h_\infty 2\pi i$$

Compare $\sum \mathrm{Log}(1+a_n)$ and $\sum a_n$

Exercise. Find $\{a_n\}$ s.t.

$\sum \operatorname{Log}(1+a_n)$ conv. but $\sum a_n$ div.

" div. but " conv.

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$\sum_{n=1} \log(p_n)$ converges absolutely.

From calculus: $\sum_{n=1}^{\infty} |\operatorname{Log}(1+a_n)|$ converges

$\iff \sum_{n=1}^{\infty} |a_n|$ converges. Pf. Exercise.

Products of holomorphic functions.

Lemma. $\Omega \subseteq \mathbb{C}$ open, simply connected and $f \in \operatorname{Hol}(\Omega)$, $f(z) \neq 0$ $\forall z$

$\Rightarrow \exists g \in \operatorname{Hol}(\Omega)$: $f = e^{g}$

Pf. $\frac{f'}{f} \in \operatorname{Hol}(\Omega) \underset{\Omega \text{ is s.c.}}{\Rightarrow} \exists g \in \operatorname{Hol}(\Omega)$: $g' = \frac{f'}{f}$

$\Rightarrow (f \cdot e^{-g})' = f' e^{-g} - f e^{-g} g' = f e^{-g}\left(\frac{f'}{f} - g'\right) = 0$

so $f e^{-g} = \text{const.} = 1$

add a const. to g. ∎

Corollary. If $f \in H(\mathbb{C})$ (entire) and $f(z) = 0 \iff z = z_1, \dots, z_N \neq 0$ (counting multiplicities) and $z = 0$ with multiplicity m

$$\Rightarrow \boxed{f(z) = z^m \cdot \prod_{n=1}^{N} \left(1 - \frac{z}{z_n}\right) e^{+g(z)}}$$

with $g \in \operatorname{Hol}(\mathbb{C})$

$$\Rightarrow \boxed{f(z) = z^{m} \prod_{n=1} \left(1 - \frac{z}{z_n}\right) e^{\cdots}} \quad \text{with } g \in \mathrm{Hol}(\mathbb{C}).$$

Wish: $f \in \mathrm{Hol}(\mathbb{C})$ and $\{a_n\}_{n=1}^{\infty}$ are its zeros

$$\Rightarrow f(z) = z^{m} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \cdot e^{g(z)}$$

This is asking too much.

Example. $f(z) = \sin(2\pi z) = 0 \Leftrightarrow z = n \in \mathbb{N}$

and $\prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)$ does not converge absolutely

because $\sum \frac{|z|}{|n|}$ diverges.

Sequences of holomorphic functions.

Theorem (Weierstrass). If $f_n \in \mathrm{Hol}(\Omega_n)$, $\Omega \xrightarrow{f} \mathbb{C}$

and $f_n \xrightarrow[n \to \infty]{} f$ uniformly on compact sets

$\Rightarrow$ (a) $f \in \mathrm{Hol}(\Omega)$, (b) $f_n' \longrightarrow f'$ uniformly on compact sets.

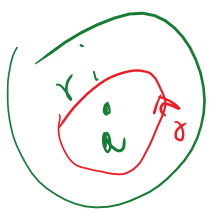
Def. $\forall K$ compact in Ω, $K \subseteq \Omega_n$ $\forall n \geq n(K)$

and $f_n \to f$ uniformly on K.

and $\forall z \in \Omega_n$ definitively $\Rightarrow z \in \Omega$

Pf. $\overline{\Delta(a,r)} \subseteq \Omega$, γ a closed curve in $\overline{\Delta(a,r)}$ $\qquad \Omega = \bigcup_m \bigcap_{n \geq m} \Omega_n$

$$\Downarrow$$



$$\int_\gamma f(z)\,dz = \lim_n \int_\gamma f_n(z)\,dz = 0$$

$\Downarrow$ MORERA

$f \in \mathrm{Hol}(\Delta(a,r))$ $\qquad$ (a)

$f \in Hol(\Delta(a,r))$ (a)

(b) $|f'(z) - f_n'(z)| \underset{\text{Cauchy formula}}{=} \left|\frac{1}{2\pi i}\int_{|w-a|=r} \frac{f(w)-f_n(w)}{(w-z)^2}\,dw\right|$

$$\le \sup_{|w-a|=r} |f(w)-f_n(w)| \cdot \frac{2\pi r}{2\pi}\,\frac{1}{(r/2)^2} \xrightarrow[n\to\infty]{} 0$$

if $|z-a|\le r/2$

Theorem (HURWITZ). Suppose $f_n \in Hol(\Omega)$, $f_n(z)\neq 0\ \forall z$, Ω connected

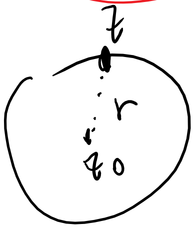
$f_n \to f$ u.c. Then:

(a) $f \equiv 0$ in Ω

(b) $f(z) \neq 0 \quad \forall z \in \Omega$.

Pf.: Suppose $f \not\equiv 0$ and $f(z_0)=0$, $f(z)\neq 0$ for $0<|z-z_0|\le r$

$$\left|\frac{1}{f_n(z)} - \frac{1}{f(z)}\right| = \frac{|f(z)-f_n(z)|}{|f_n(z)|\cdot|f(z)|} \underset{\substack{|z-z_0|=r \\ n\ge n_0}}{\le} \frac{|f(z)-f_n(z)|}{\min_{|w-z_0|=r}|f(w)|^2/2}$$



$\xrightarrow[n\to\infty]{} 0$ uniformly as $|z-z_0|=r$.

Also: $f_n'(z) \xrightarrow[n\to\infty]{} f'(z)$ u.c. uniformly for $|z-z_0|=r$.

$$\Rightarrow \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\gamma=|z-z_0|} \frac{f_n'(z)}{f_n(z)}\,dz \overset{\text{check}}{=} \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f'(z)}{f(z)}\,dz$$

|| (left) $\lim_{n\to\infty}$ # roots of $f_n(z)=0$ inside $|z-z_0|=r$

|| (right) # roots of $f(z)=0$ inside $|z-z_0|=r$

|| since $f(z_0)=0$

$n \to \infty$

inside
$|z - z_0| = r$

$|z - z_0| = r$

$\|$ since $f(z_0) = 0$

1

$\| \not\ll \mathrm{Hp.}$

0

contradiction!

What does it say about infinite products?

Suppose $\Omega \subseteq \mathbb{C}$ open connected, $f_n \in \mathrm{Hol}(\Omega)$,

$p(z) = \prod_{n=1}^{\infty} f_n(z)$. If p converges u.c.

then $p(z) = 0 \iff \exists n : f_n(z) = 0$.

Weierstrass factorization

Suppose $\{a_n\}_{n=1}^{\infty}$ are the zeros of $f(z) \in \mathrm{Hol}(\mathbb{C})$.
$a_n \neq 0$

We'd like $p(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)$ to converge. We know
$a_n \to \infty$
$n \to \infty$

i.e. $\sum_{n=1}^{\infty} \mathrm{Log}\left(1 - \frac{z}{a_n}\right)$ converges
absolutely

i.e. $\sum_{n=1}^{\infty} \frac{1}{|a_n|}$ converges absolutely

which is not true in general.

Idea: $\mathrm{Log}(1 - w) = w + o(w) = w + \frac{w^2}{2} + \dots + \frac{w^m}{m} + o(w^m)$

$\mathrm{Log}(1 - w) - \left(w + \dots + \frac{w^m}{m}\right) = P_m(w) = o(w^m)$

$(1 - w) \cdot e^{-\left(w + \dots + \frac{w^m}{m}\right)} = e^{P_m(w)}$ is very close to 1.

$\left(1 - \frac{z}{a_n}\right) \rightsquigarrow \left(1 - \frac{z}{a_n}\right) \cdot e^{-\left(\frac{z}{a_n} + \dots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}\right)}$

$$\left(1-\frac{z}{a_n}\right) \leftrightsquigarrow \left(1-\frac{z}{a_n}\right)\cdot e^{-\left(\frac{z}{a_n}+\dots+\frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}\right)} = E_{m_n}\left(\frac{z}{a_n}\right)$$

and we'll consider

$$\prod\left(1-\frac{z}{a_n}\right) \leftrightsquigarrow \prod_{n=1}^{\infty} E_{m_n}\left(\frac{z}{a_n}\right)$$

with m_n properly chosen.