Weierstrass factorization and sinc.

$f(z) = \sin(\pi z) = 0 \Leftrightarrow z = n \in \mathbb{Z}$, then

$$\sin(\pi z) = z \cdot e^{g(z)} \cdot \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}$$

because (see Weierstrass factorization)

$$\sum_{n \neq 0} \frac{1}{n^{1+1}} = 1.$$

The product on the right converges u.c. on $\mathbb{C} \setminus \mathbb{Z}$, hence

$$\pi \cdot \cot(\pi z) = \frac{D \sin(\pi z)}{\sin(\pi z)} = \frac{1}{z} + g'(z) + \frac{D\left\{\prod_{n \neq 0}\left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}\right\}}{\prod_{n \neq 0}\left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}}$$

$$= \frac{1}{z} + g'(z) + \sum_{n \neq 0} \frac{D\left\{\left(1 - \frac{z}{n}\right) e^{z/n}\right\}}{\left(1 - \frac{z}{n}\right) e^{z/n}}$$

$$= \frac{1}{z} + g'(z) + \sum_{n \neq 0} \left(\frac{1}{n} - \frac{1/n}{1 - z/n}\right)$$

$$= \frac{1}{z} + g'(z) + \sum_{n \neq 0} \left(\frac{1}{n} + \frac{1}{z - n}\right)$$

Claim: $g'(z) = 0$

Pf.(1) By comparison of order and genus, $g(z)$ is a polynomial.
$\cot(\pi z)$ is 1-periodic and so is
$$h(z) = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{n} + \frac{1}{z - n}\right),$$ hence
g' is 1-periodic, hence constant.
Verify that $h(z) + h(1/2 - z) = 0$, ~~AS~~ $\cot(\pi x) \Rightarrow g' = 0$.

Finally,

$$\pi = \lim_{z \to 0} \frac{\sin(\pi z)}{z} = e^{g} \cdot \lim_{z \to 0} \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}$$

$= e^{g}$ (since the product converges)

$\Rightarrow e^{g} = \pi$.

Fact: $$\sin(\pi z) = \pi z \cdot \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) \cdot e^{z/n}$$
$$= \pi z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

~~Pf.(1)~~ Note on genus. $k(z) = \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}$ has genus 1 and $\sin(\pi z)$ has order 1. By Hadamard's Theorem,

genus $\leq$ order $\leq$ genus $+1$.
$\qquad\qquad\ \ \| $
$\qquad\qquad\ \ 1$

Hence $\deg(g) \leq 1$, so $g \equiv$ const.

Pf.(2) Via M.L. (or rather the ideas behind it).

$$\frac{\pi^2}{\sin^2(\pi z)} - \sum_{n=-\infty}^{+\infty} \frac{1}{(z - n)^2} = P(z) \in Hol(\mathbb{C}),$$

because it is free of poles, and it is 1-periodic. Also it is easy to see that $\lim_{y \to \pm\infty} P(x + iy) = 0$ uniformly w.r.t. x.

This and Liouville $\Rightarrow$ P is const.

~~As before~~ Since $P(\infty) = 0$, $P \equiv 0$ ■

GENUS and ORDER

$f \in \mathrm{Hol}(\mathbb{C})$, $\{a_n\}$ its zeros.

Suppose $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}} < +\infty$ $\exists h \geq 0$ integer.

Then
$$(*) \quad \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \dots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h}$$
converges u.c. because $\sum_{n=1}^{\infty} \frac{1}{h+1}\left(\frac{R}{|a_n|}\right)^{h+1} < \infty$. $\forall R > 0$

Consider the smallest h: $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}} < +\infty$.

Then $(*)$ is the <u>canonical product</u> associated to $\{a_n\}$, h is its <u>genus</u>.

$|a_n| \to \infty$ fast: low genus

$|a_n| \to \infty$ slow: high genus

If $f(z) = z^m \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \cdot e^{\frac{z}{a_n} + \dots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h} \cdot e^{g(z)}$

where the product is canonical and $g(z)$ is a polynomial, then

$$\text{genus}(f) = \text{MAX}(\text{genus}(\text{product}), \text{degree}(g)).$$

<u>Algorithm</u>. $f \in \mathrm{Hol}(\mathbb{C})$. Do its zeros lead to a canonical product? If yes, choose the one having the least genus.

In practice, choose h as smallest with $\sum \frac{1}{|a_n|^{h+1}} < \infty$.

compute $f/\text{canonical product} = e^g$.

Is g a polynomial? If yes, compute genus(f).

<u>Exercise</u>. We have computed
$$\sin(\pi z) = \pi z \cdot \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) \cdot e^{z/n}$$

By Weierstrass factorization we know that we can also write
$$\sin(\pi z) = \pi z \cdot \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) \cdot e^{\frac{z}{n} + \frac{1}{2}\left(\frac{z}{n}\right)^2} \cdot e^{g(z)}$$

What is $g(z)$?

<u>Solution</u>. $1 = \prod_{n \neq 0} e^{\frac{1}{2}\left(\frac{z}{n}\right)^2 + g(z)}$, i.e.

$$0 = g(z) + z^2 \cdot \sum_{1}^{\infty} \frac{1}{n^2} = g(z) + \frac{\pi^2}{6} \cdot z^2.$$

<u>Punchline</u>: finding first the genus of the canonical product is crucial.

<u>Genus = 0</u> $f(z) = c \cdot z^m \cdot \prod \left(1 - \frac{z}{a_n}\right)$ with $\sum \frac{1}{|a_n|} < +\infty$

<u>Genus = 1</u> $f(z) = c \cdot z^m \cdot \prod \left(1 - \frac{z}{a_n}\right) e^{z/a_n} \cdot e^{az}$

with $\sum \frac{1}{|a_n|} = +\infty$, $\sum \frac{1}{|a_n|^2} < +\infty$, $a \in \mathbb{C}$

or $f(z) = c \cdot z^m \cdot \prod \left(1 - z/a_n\right) \cdot e^{az}$

with $\sum \frac{1}{|a_n|} < +\infty$, $a \neq 0$.

Or, all together,
$$f(z) = c \cdot z^m \cdot \prod \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}} \cdot e^{az}$$

with either $\sum \frac{1}{|a_n|} = +\infty$, $\sum \frac{1}{|a_n|^2} < \infty$

or $\sum \frac{1}{|a_n|} < \infty$, $\sum \frac{1}{a_n} + a \neq 0$.

<u>Genus = 2</u>: <u>Exercise</u>.

The order of an entire $f \in \mathrm{Hol}(\mathbb{C})$ is

$$\lambda(f) := \sup_{r>0} \frac{\log\log M(r)}{\log r} \quad \text{with } M(r) = \max_{|z|=r} |f(z)|.$$

The basic inspiration comes from

$$\lambda(e^{z^m}) = m$$

In fact ~~(Exercise)~~ $\lambda(P(z)) = \deg(P(z))$ if $P(z)$ is a polynomial (see $\boxed{\square}$ below).

~~The canonical factorization of an entire function of finite genus provides a naive estimate on the order.~~

$$\cancel{f(z) = c \cdot z^m \cdot \prod \left(1 - \frac{z}{a_n}\right) \cdot e^{\frac{z}{a_n} + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h} \cdot e^{g(z)}}$$

<u>Hadamard Theorem</u>. Let $h(f)$ be the genus of $f \in \mathrm{Hol}(\mathbb{C})$. Then,

$$h(f) \le \lambda(f) \le h(f) + 1.$$

The proof that $\lambda(f) \le h(f)+1$ is "brute force". The estimate $h(f) \le \lambda(f)$ passes through Jensen's formula and it runs deeper.

<u>Proof that $\lambda(f) \le h(f)+1$.</u>

Let $E_h(z) = (1-z)\, e^{z + z^2/2 + \cdots + z^h/h}$

<u>Claim</u>. $\operatorname{Log}|E_h(z)| \le (2h+1)|z|^{h+1}$.

Pf. of the Claim. For $|z|<1$, $\operatorname{Log} E_h(z)$ defines a holomorphic function and

$$\operatorname{Log} E_h(z) = \sum_{n=h+1}^{\infty} \frac{z^n}{n}, \text{ hence}$$

$$\operatorname{Log}|E_h(z)| \le |\operatorname{Log}(E_h(z))| \le \sum_{n=h+1}^{\infty} \frac{|z|^n}{n} \le \frac{|z|^{h+1}}{(h+1)(1-|z|)}$$

Thus $(1-|z|)\operatorname{Log}|E_h(z)| \le \frac{|z|^{h+1}}{h+1}$ if $|z|<1$. $(*)$

Also, $\operatorname{Log}|E_h(z)| - \operatorname{Log}|E_{h-1}(z)| = \operatorname{Log}|e^{z^h/h}| \le |z|^h/h \quad \forall z$

We now verify the Claim by induction.

$h=0$: $\log|1-z| \le \log(1+|z|) \le |z|$.

$h-1 \to h$: $\operatorname{Log}|E_h(z)| \le \operatorname{Log}|E_{h-1}(z)| + |z|^h$

$\le (2h-1)|z|^h + |z|^h \le (2h+1)|z|^{h+1}$ if $|z| \ge 1$.

If $|z| \le 1$, by $(*)$

$$\operatorname{Log}|E_h(z)| \le |z| \cdot \operatorname{Log}|E_h(z)| + \frac{|z|^{h+1}}{h+1}$$

$$\le |z| \cdot \left\{ \operatorname{Log}|E_{h-1}(z)| + \frac{|z|^h}{h} \right\} + \frac{|z|^{h+1}}{h+1}$$

$$\le |z| \left\{ (2h-1)|z|^h + \frac{|z|^h}{h} \right\} + \frac{|z|^{h+1}}{h+1}$$

$$\le (2h+1)|z|^{h+1} \ \blacksquare$$

End of the proof that $\lambda(f) \le h(f)+1$.

If $P(z) = \prod_n E_h(z/a_n)$ is the canonical product associated with f, then

$$\log|P(z)| = \sum_n \log|E_h(z/a_n)| \le (2h+1)|z|^{h+1} \cdot \sum_n \frac{1}{|a_n|^{h+1}}$$

$$\Rightarrow \lambda(P(z)) \le h+1 = h(P(z))+1 \ \blacksquare$$

Given the claim if $P(z)$ is the canonical product,
then $\log|P(z)| = \sum_n \log\left|E_h\left(\frac{z}{a_n}\right)\right| \leq (2h+1)|z|^{h+1} \cdot \sum_n \frac{1}{|a_n|^{h+1}}$
with $\sum_n \frac{1}{|a_n|^{h+1}} < \infty$ since order $(P(z)) = h$.

$\Rightarrow \lambda(P(z)) \leq h(P(z)) + 1.$

If $f(z) = z^m \cdot P(z) \cdot e^{g(z)}$,

$\log|f(z)| = m \log|z| + \log|P(z)| + |\operatorname{Re} g(z)|$
$\leq 3 \cdot \max\{m \log|z|, \log|P(z)|, |g(z)|\}$

So $\dfrac{\log\log|M_f(R)|}{\log R} \leq \underset{R\to\infty}{o}(1) + \max\left\{\underset{R\to\infty}{o}(1), \dfrac{\log\log M_P(R)}{\log R}, \dfrac{\log M_g(R)}{\log R}\right\}$

$\Rightarrow \lambda(f) = \overline{\lim_{R\to\infty}} \dfrac{\log\log M_f(R)}{\log R} =$

$\leq \max\{\lambda(P(z)), \deg(g)\}$
$\leq \max(h(P(z))+1, \deg(g))$
$\leq h(f) + 1$ ∎

JENSEN'S FORMULA If $f \in \mathrm{Hol}(\{|z| \leq \rho\})$ and $a_1, \dots, a_n$ are its zeros in $\{|z| \leq \rho\}$ then:

$$\log|f(0)| = -\sum_{j=1}^{n} \log\left(\frac{\rho}{|a_j|}\right) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(\rho e^{it})|\,dt$$

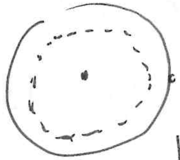
Lemma. $\int_0^{2\pi} \log|1-e^{it}|\,dt = 0.$

Pf. $\operatorname{Log}(1-z) \in \mathrm{Hol}(\mathbb{D})$, hence $0 = \operatorname{Log}(1-0) = \quad (0<\rho<1)$

$= \frac{1}{2\pi i}\int_{|w|=\rho} \operatorname{Log}(1-w)\frac{dw}{w} \underset{w=e^{it}}{=} \frac{1}{2\pi i}\int_0^{2\pi} \operatorname{Log}(1-\rho e^{it})\, i\,dt$

$= \frac{1}{2\pi}\left\{\int_0^{2\pi} \log|1-\rho e^{it}|\,dt + i\int_{-\pi}^{\pi} \arg(1-\rho e^{it})\,dt\right\}.$

Also $\log|1-\rho e^{it}| = \frac{1}{2}\log[(1-\rho\cos t)^2 + \rho^2\sin^2 t]$
$= \frac{1}{2}\log[1+\rho^2 - 2\rho\cos t] = \frac{1}{2}\log[(1-\rho)^2 + 2\rho(1-\cos t)]$



Since for $\frac{1}{2} \leq \rho < 1$

$1-\cos t \leq 1+\rho^2 - 2\rho\cos t \leq 2,$

$\big|\log|1-\rho e^{it}|\big| \leq \frac{1}{2}\max\{\log 2, |\log(1-\cos t)|\}$
$\in L^1(0, 2\pi)$

and we can use D.C.T. ∎

Proof of J.'s F. The map $z \mapsto \dfrac{a_j/\rho - z/\rho}{1 - \bar{a}_j/\rho \, z/\rho} =$
$= \rho\dfrac{a_j - z}{\rho^2 - \bar{a}_j z}$ maps $\{|z| \underset{(=)}{\leq} \rho\}$ onto $\{|z| \underset{(=)}{\leq} 1\}$

and $g(z) := f(z) \big/ \prod_{j=1}^{n} \dfrac{\rho(a_j - z)}{\rho^2 - \bar{a}_j z}$ has no zeros in $\{|z| \leq \rho\}$.

Hence $\log|g(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|g(\rho e^{it})|\,dt$

$\log|f(0)| - \sum_{j=1}^{n} \log\frac{|a_j|}{\rho} \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(\rho e^{it})|\,dt$

∎

~~So $f(z) = e^{g(z)} \cdot \prod_n E_h\left(\frac{z}{a_n}\right)$.~~

~~We want to show that $\deg(g(z)) \le h$, so proving that $h(f) \le \lambda(f)$.~~

We use a slight refinement of Jensen's formula.

<u>Poisson - Jensen</u> formula.

$$\log|f(z)| = -\sum_{j=1}^{n} \log\left|\frac{\rho^2 - \bar{a}_j z}{\rho(z-a_j)}\right| \qquad (\mathrm{PJ})$$

$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{\rho e^{it}+z}{\rho e^{it}-z}\right) \log|f(\rho e^{it})|\,dt$$

if $|z|<\rho$

where $a_1, \dots, a_n$ are the zeros of $f(z)$ with $|a_i| \le \rho$. (We assume $|a_i| < \rho$: the general case is as in Jensen's formula).

To keep the argument self-contained we also prove Poisson's formula for a function $v(z)$ which is harmonic ($\Delta v = 0$) in $|z| \le \rho$.

$$v(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{\rho e^{it}+z}{\rho e^{it}-z}\right) v(\rho e^{it})\,dt \quad (\mathrm{P})$$

for $|z|<\rho$.

Observe first that $w \mapsto \rho\,\dfrac{z/\rho - w/\rho}{1 - \bar{z}/\rho \cdot w/\rho} = \varphi_z(w) = \dfrac{\rho^2(z-w)}{\rho^2 - \bar{z}w}$

$\varphi_z(0) = z$, $\varphi_z(z) = 0$,

$\varphi_z^{-1} = \varphi_z$, φ_z is a automorphism of $|w| \le \rho$.

If $\Delta v(w) = 0$ in $|w| \le \rho$, then MVP holds:

$$(\mathrm{MVP}) \quad v(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(\rho e^{it})\,dt = \frac{1}{2\pi i}\int_{|w|=\rho} v(w)\,\frac{dw}{w},$$

since $w = \rho e^{it} \Rightarrow \dfrac{dw}{iw} = \dfrac{\rho i e^{it}\,dt}{i\rho e^{it}} = dt$.

After the change of variables $\zeta = \varphi_z(w)$,

$$\frac{d\zeta}{i\zeta} = \frac{\varphi_z'(w)\,dw}{i\varphi_z(w)} = \frac{-(\rho^2 - \bar{z}w) + (z-w)\bar{z}}{i(\rho^2 - \bar{z}w)^2}\,\frac{\rho^2 - \bar{z}w}{z-w}\,dw$$

$$= \frac{i(\rho^2 - |z|^2)}{(\rho^2 - \bar{z}w)(z-w)}\,dw \underset{w=\rho e^{it}}{=} \frac{i(\rho^2 - |z|^2)\rho i e^{it}\,dt}{(\rho^2 - \bar{z}\rho e^{it})(z - \rho e^{it})}$$

$$= \frac{(\rho^2 - |z|^2)\,dt}{(\rho - \bar{z}e^{it})(\rho - z e^{-it})} = \frac{\rho^2 - |z|^2}{|\rho - z e^{-it}|^2}\,dt$$

and $\mathrm{Re}\left(\dfrac{\rho e^{it}+z}{\rho e^{it}-z}\right) = \dfrac{\mathrm{Re}[(\rho e^{it}+z)(\rho e^{-it}-\bar{z})]}{|\rho e^{it} - z|^2}$

$= \dfrac{\rho^2 - |z|^2}{|\rho - e^{-it}z|^2}$. Finally,

$$v(z) = (v \circ \varphi_z)(0) = \frac{1}{2\pi}\int_{|\zeta|=\rho} v(\varphi_z(\zeta))\,\frac{d\zeta}{i\zeta}$$

$$= \frac{1}{2\pi}\int_{|w|=|\varphi_z(\zeta)|=\rho} v(w)\,\frac{d\varphi_z(w)}{i\varphi_z(w)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\rho^2 - |z|^2}{|\rho - z e^{-it}|^2}\,v(\rho e^{it})\,dt \quad \blacksquare$$

Poisson–Jensen formula follows:

$$F(w) = f(\varphi_z(w)) = f\left(\frac{\rho^2(z-w)}{\rho^2 - \bar{z}w}\right)$$

vanishes at $A_i = \varphi_z(a_i) = \dfrac{\rho^2(z-a_i)}{\rho^2 - \bar{z}a_i}$,

hence, for $|z| < \rho$,

$$\log|f(z)| = \log|F(0)|$$

$$= -\sum_{i=1}^{n} \log\frac{\rho}{|A_i|} + \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(w)|\,\frac{dw}{iw}$$

$$= -\sum_{i=1}^{n} \log\left|\frac{\rho^2 - \bar{z}a_i}{\rho(z-a_i)}\right| + \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(\rho e^{it})| \cdot \mathrm{Re}\left(\frac{\rho e^{it} + z}{\rho e^{it} - z}\right) dt$$

as wished.

Pf. that $h(f) \leq \lambda(f)$.

We start by showing that, with $h = [\lambda(f)]$, $\sum \frac{1}{|a_n|^{h+1}} < +\infty$, so that $f(z) = P(z)\, e^{g(z)}$ where $P(z)$ is a canonical product of genus h.

Let $\nu(\rho) = \#\{a_i : |a_i| \leq \rho\}$ and consider Jensen's formula with radius 2ρ:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(2\rho \cdot e^{it})|\,dt - \log|f(0)| =$$

$$= \sum_{|a_i| \leq 2\rho} \log\left|\frac{2\rho}{|a_i|}\right| \geq \sum_{|a_i| \leq \rho} \log 2 = \nu(\rho) \cdot \log(2).$$

On the other hand

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(2\rho e^{it})|\,dt \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log M(2\rho)\,dt$$

$\leq (2\rho)^{\lambda(f)+\varepsilon}$ ($\forall \varepsilon > 0$, if $\rho \geq \rho(\varepsilon)$ large enough)

since $\limsup_{r\to\infty} \dfrac{\log\log M(r)}{\log r} = \lambda(f)$

$\Longrightarrow \log\log M(r) \leq (\lambda(f)+\varepsilon)\log r$ ($\forall \varepsilon > 0\ \exists \rho(\varepsilon) \ldots$)

i.e. $\log M(r) \leq r^{\lambda(f)+\varepsilon}$

so that $\nu(\rho)\log(2) + \log|f(0)| \leq 2^{\lambda+\varepsilon} \cdot \rho^{\lambda+\varepsilon}$

Order zeros: $|a_1| \leq |a_2| \leq \cdots \leq |a_n| \leq \cdots$

so that $n \leq \nu(|a_n|) \leq |a_n|^{\frac{1}{\lambda+\varepsilon}}$ for n large

(use here that $\limsup_{\rho\to\infty} \dfrac{\nu(\rho)}{\rho^{\lambda+\varepsilon}} \leq 2^{\lambda+\varepsilon}$

hence that, by picking any $\varepsilon' > \varepsilon$, $\limsup_{\rho\to\infty} \dfrac{\nu(\rho)}{\rho^{\lambda+\varepsilon'}} = 0$).

Thus, $\sum_{n \geq N} \frac{1}{|a_n|^{h+1}} \leq \sum_{n \geq N} \frac{1}{n^{\frac{h+1}{\lambda+\varepsilon}}}$, which converges is ε is so small that $\lambda + \varepsilon < h+1$.

By Weierstrass factorization,

$P(z) = \prod_{n=1}^{\infty} E_h\left(\frac{z}{a_n}\right)$ converges (absolutely) and $f(z) = P(z) \cdot e^{g(z)}$, with $g(z)$ entire (and $\mathrm{genus}(P(z)) = h \leq \lambda(f)$).

We have to prove that $g(z) \in \mathbb{C}[z]$ and $\deg(g) \leq \lambda(f)$. We first write down Poisson–Jensen formula and we will let $|z| \leq \rho/2$ there.

$$\log|f(z)| = -\sum_{|a_i|\le\rho} \log\left|\frac{\rho^2 - \bar{a}_i z}{\rho(z-a_i)}\right| + \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{\rho e^{it}+z}{\rho e^{it}-z}\right)\log|f(\rho e^{it})|\,dt$$

The idea is taking enough derivatives to ~~have~~ make some summands to vanish:

$$\frac{f'(z)}{f(z)} = \frac{P'(z)e^{g(z)} + P(z)g'(z)e^{g(z)}}{P(z)e^{g(z)}} = \frac{P'(z)}{P(z)} + g'(z)$$

The h-derivative of $\frac{P'(z)}{P(z)}$ will be reabsorbed in the one of $\frac{f'(z)}{f(z)}$, so we will have that the h derivative of $g'(z)$ vanishes.

We start with a ~~[illegible]~~ calculation.

<u>Lemma</u>. If $v(z) = \mathrm{Re}\,\varphi(z)$ is the real part of $\varphi \in Hol(\Omega)$, then ($\partial = \partial_x - i\partial_y$)

$$\partial v = \varphi'.$$

<u>Pf</u>. ~~$0 = \Delta v = \bar\partial(\partial v)$, $\bar\partial = \partial_x + i\partial_y$, because v is harmonic; hence, $\partial v \in Hol(\Omega)$. Moreover,~~ We have:

$$\varphi' = \partial_x \varphi = \partial_x v + i\,\partial_x w \qquad (\varphi = v + iw)$$
$$= \partial_x v - i\partial_y v \quad \text{(Cauchy-Riemann)}$$
$$= \partial v \quad \blacksquare$$

If $v(z) = \log|f(z)|$, then

$$\partial v(z) = \frac{f'(z)}{f(z)}.$$

Apply this to (PJ):

$$\frac{f'(z)}{f(z)} = \sum_{|a_i|\le\rho}\frac{1}{z-a_i} + \sum_{|a_i|\le\rho}\frac{\bar{a}_i}{\rho^2 - \bar{a}_i z}$$
$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2\rho e^{it}}{(\rho e^{it}-z)^2}\log|f(\rho e^{it})|\,dt$$

$= A + B + C$, since $\frac{d}{dz}\left(\frac{a+z}{a-z}\right) = \frac{2a}{(a-z)^2}$

Take now h derivatives w.r.t. z:

$$D^{(h)}\left[\frac{f'(z)}{f(z)}\right] = -h!\sum_{|a_i|\le\rho}\frac{1}{(a_i - z)^{h+1}} + h!\sum_{|a_i|\le\rho}\frac{\bar{a}_i^{h+1}}{(\rho^2 - \bar{a}_i z)^{h+1}}$$
$$+ \frac{(h+1)!}{2\pi}\int_{-\pi}^{\pi}\frac{2e^{it}\rho}{(\rho e^{it}-z)^{h+2}}\cdot\log|f(\rho e^{it})|\,dt = A' + B' + C'$$

Since by the Theorem of Residues, for $|z|<\rho$:

$$0 = \frac{1}{2\pi i}\int_{|w|=\rho}\frac{dw}{(w-z)^{h+2}} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\rho e^{it}\,dt}{(\rho e^{it}-z)^{h+2}},$$

$$|C'| = \left|\frac{(h+1)!}{\pi}\int_{-\pi}^{\pi}\frac{\rho e^{it}}{(\rho e^{it}-z)^{h+2}}\{\log|f(\rho e^{it})| - \log M(\rho)\}\,dt\right|$$

$$\underset{|z|\le\rho/2}{\le} \frac{(h+1)!\,\rho}{\pi}\left(\frac{2}{\rho}\right)^{h+2}\int_{-\pi}^{\pi}\log\frac{M(\rho)}{|f(\rho e^{it})|}\,dt$$

$$\le \frac{(h+1)!\, 2^{h+2}}{\pi \, \rho^{h+1}} \left\{ \int_{-\pi}^{\pi} \log M(\rho)\, dt - 2\pi \cdot \log|f(0)| \right\}$$

Jensen

$$= \frac{(h+1)!\, 2^{h+2} \cdot \left(\log M(\rho) - \log|f(0)|\right)}{\rho^{h+1}}$$

$\underset{\rho \to \infty}{\longrightarrow} 0$ because $\lambda(f) < h+1$.

$$|B'| \le \sum_{|a_i| \le \rho} \frac{\rho^{h+1}}{(\rho^2 - \rho^2/2)^{h+1}} \le \nu(\rho)\, \rho^{-h-1}$$

$|z| \le \rho/2$

$$\le \rho^{\lambda(f)+\varepsilon-h-1} \underset{\rho \to \infty}{\longrightarrow} 0$$

if $\varepsilon > 0$ is small enough.

As $\rho \to \infty$, then, we are left with

$$D^{(h)} \frac{f'(z)}{f(z)} = -h! \sum_{n=1}^{\infty} \frac{1}{(a_n - z)^{h+1}} \qquad (A)$$

$$\| $$

$$D^{(h)} \frac{P'(z)}{P(z)} + D^{(h)} g'(z).$$

Recall that $P(z) = \prod_{n=1}^{\infty} E_h\left(\frac{z}{a_n}\right)$

so $$\frac{P'(z)}{P(z)} = \frac{d}{dz} \log(P(z)) = \sum_n \frac{E_h'(z/a_n)}{E_h(z/a_n)}$$

We wish to show that

$\frac{P'(z)}{P(z)} = \text{R.H.S.}\,(A)$, so proving that $g^{(h+1)}(z) = 0$, hence $\deg(g(z)) \le h$.

Now, $E_h(z) = (1-z)\, e^{z + z^2/2 + \cdots + z^h/h}$

hence

$$\frac{E_h'(z)}{E_h(z)} = \frac{-e^{z+\cdots+z^h/h} + (1-z)\, e^{z+\cdots+z^h/h} \cdot (1+\cdots+z^{h-1})}{(1-z)\, e^{z+\cdots+z^h/h}}$$

$$= 1 + z + \cdots + z^{h-1} - \frac{1}{1-z},$$

$(*) \Rightarrow D^{(h)}\left(\frac{E_h'(z)}{E_h(z)}\right) = -D^{(h)}(1-z)^{-1} = -h!\,(1-z)^{-h-1}.$

and so

$$\frac{d^h}{dz^h}\left(\frac{\frac{d}{dz} E(z/a)}{E(z/a)}\right) = \frac{-h!}{a^{h+1}} \cdot \frac{1}{(1-z/a)^{h+1}}$$

$$= -\frac{h!}{(a-z)^{h+1}} \text{ as wished } \blacksquare$$

$(*)$ Proof that if $g \in Hol(\mathbb{C})$ and $\operatorname{Re} g(z) \le |z|^m$ $\exists m > 0 \Rightarrow g$ is a polynomial.

Borel - Carathéodory Lemma

If $f \in Hol(\overline{D(0,R)})$ and $M(\operatorname{Re} f, R) = \sup_{|z| \le R} \operatorname{Re} f(z)$, then for $0 < r < R$

$$M(f,r) \le \frac{2r}{R-r} M(\operatorname{Re} f, R) + \frac{R+r}{R-r} |f(0)|$$

Pf. Let $A = M(\operatorname{Re} f, R)$, so that $f(\overline{D(0,R)}) \subseteq \{\operatorname{Re} w \le A\}$.

Assume $f(0) = 0$ first.

Then $z \mapsto \dfrac{R\, f(z)}{f(z)-2A} = g(z)$ maps $\overline{\Delta(0,R)}$ into itself, $g(0)=0$ and Schwarz' Lemma implies

$$|z| \geq |g(z)| = \frac{R\,|f(z)|}{|f(z)-2A|}$$

So that $R|f(z)| \leq |z|\,|f(z)| + 2|z|A$

and so $(|z|=r)$

$$(R-r)\,M(f,r) \leq 2r \cdot M(\operatorname{Re} f, R)$$

If $f(0) \neq 0$, then

$$|f(z)| - |f(0)| \leq |f(z)-f(0)| \underset{|z|=r}{\leq} \frac{2r\, M(\operatorname{Re}(f(z)-f(0)))}{R-r}$$

So $|f(z)| \leq \dfrac{2r}{R-r} M(\operatorname{Re} f, R) + |f(0)| \cdot \left(1 + \dfrac{2r}{R-r}\right)$ ■

Suppose now $M(\operatorname{Re} g, R) \leq R^m$ and set $r = R/2$, so

$$|g(z)| \underset{|z|=r}{\leq} \frac{2r}{2r-r} M(\operatorname{Re} g, 2r) + 3|g(0)| \leq 2 \cdot (2r)^m + 3|g(0)|$$

$$\Rightarrow M(g,r) \leq C(r^m+1)$$

That implies that $\dfrac{g(z)}{z^{m+1}} \xrightarrow[z\to\infty]{} 0$

so $w^{m+1} g(1/w)$ is holomorphic in a neighbourhood of $w=0$, and it easily follows that g is a polynomial ■

<u>An example of $f \in \mathrm{Hol}(\mathbb{C})$ with genus$(f)=0$ and order$(f)=1$.</u>

Consider $a_n = n \log^2 n$, $n \geq 1$

and set $f(z) = \prod \left(1 + \dfrac{z}{a_n}\right)$

so that genus$(f)=0$ since $\sum \dfrac{1}{a_n} < \infty$

and $M(f,R) = \prod_n \left(1 + \dfrac{R}{a_n}\right)$

Observe that $\log(1+x) \geq \delta \cdot x$ if $0 \leq x \leq 1$ $(\exists \delta > 0)$

so $\log M(f,R) \geq \delta \cdot \sum\limits_{R/a_n \leq 1} \dfrac{R}{a_n}$.

Consider $R = \bar{n} \cdot \log^2 \bar{n}$ with $\bar{n} \nearrow \infty$

$$\sum_{R/a_n \leq 1} \frac{1}{a_n} = \sum_{\bar{n}}^{\infty} \frac{1}{n \log^2 n} \sim \int_{\bar{n}}^{\infty} \frac{dt}{t \cdot \log^2 t} = \frac{1}{\log \bar{n}}$$

so that $R \cdot \sum\limits_{R/a_n \leq 1} 1/a_n = \bar{n} \cdot \log \bar{n}$

$$\frac{\log \log M(R)}{\log R} \geq \frac{\log(\delta) + \log(\bar{n} \cdot \log \bar{n})}{\log(\bar{n} \log^2 \bar{n})}$$

$$= \frac{\log(\bar{n}) + o(\log(\bar{n}))}{\log(\bar{n}) + o(\log(\bar{n}))} \xrightarrow[\bar{n}\to\infty]{} 1$$

hence order$(f) \geq 1$ ■