What is a Paley-Wiener-type Thm.?

It is a holomorphic Fourier-type homomorphism (representation).

~~Plancherel~~ Fourier inverse-type theorems:

growth condition on f $\iff$ growth condition on $\hat{f}$

$$\int_0^{2\pi} |f(e^{it})|^2 \frac{dt}{2\pi} = \|f\|_{L^2}^2 < \infty; \quad \hat{f}(n) = \int_0^{2\pi} f(e^{it}) e^{-int} dt$$

growth on f

$$\Rightarrow f(e^{it}) \underset{L^2}{=} \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{int} \text{ and } \sum |\hat{f}(n)|^2 < \infty$$

growth on $\hat{f}$

(spectral analysis) is the deeper side of

$$\Big\| \sum_{n \in \mathbb{Z}} a_n e^{int} \Big\|_{L^2}^2 = \frac{1}{2\pi} \int_0^{2\pi} \Big|\sum a_n e^{int}\Big|^2 dt$$

$$= \sum_n |a_n|^2 \quad \text{(spectral synthesis).}$$

(Shallow) $\{a_n\} \mapsto \sum_{n \in \mathbb{Z}} a_n e^{int}$ maps $\ell^2(\mathbb{Z})$ into $L^2[0, 2\pi]$ homomorphically

(Deep) each $f \in L^2[0, 2\pi]$ has such representation.

There are ~~plenty of~~ non Hilbert versions of the phenomenon.

Paley-Wiener (baby) version.

(Shallow) $\{a_n\}_{n=0}^{\infty} \mapsto \sum_{n=0}^{\infty} a_n z^n = f(z)$

maps $\ell^2(\mathbb{N})$ into $H^2(\mathbb{D})$ homomorphically,

growth on $\hat{f}$

$$\|f\|_{H^2}^2 = \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt < \infty$$

growth on f

$f \in \mathrm{Hol}(\mathbb{D})$.

(Deeper) If $f \in H^2(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ~~converges~~ satisfies $\|f\|_{H^2} = \|\{a_n\}\|_{\ell^2}$.

The novelty here is that the growth condition on f involves the values of $f(z)$ on $\mathbb{D} \ni z$, not on $\partial\mathbb{D}$. In fact, the way one can talk about $f(e^{it})$ is a discipline by itself:

$f(e^{it}) = \lim_{r \to 1} f(re^{it})$? (Abel-type Thm.)

$f(e^{it}) = \lim_{N \to \infty} \frac{1}{2\pi} \sum_{0 \le n \le N} \hat{f}(n) e^{int}$? (Carleson-type Thm.)

Paley-Wiener Theorem

Recall $PW(a) = \{\mathbb{R} \xrightarrow{f} \mathbb{C} : f \in L^2, \operatorname{supp}(\hat{f}) \subseteq [-\frac{a}{2}, \frac{a}{2}]\}$ is the Paley-Wiener space, having reproducing kernel

$$K_y(x) = a \cdot \operatorname{sinc}(\pi a(x-y)) = \frac{\sin(\pi a(x-y))}{\pi(x-y)}.$$

Proposition. Each $f \in PW(a)$ has a holomorphic extension to $\mathbb{C}$,

$$f(z) = \int_{-a/2}^{a/2} \hat{f}(\omega) e^{2\pi i \omega z} \cdot d\omega \quad (*)$$

Moreover, order$(f) \leq 1$. In fact,

$$|f(z)| \leq \sqrt{a} \cdot \|f\|_{L^2} \cdot e^{\pi a |z|} \quad (**)$$

Pf. Estimate $(**)$ follows from $(*)$ and Cauchy-Schwarz. Holomorphicity is a consequence of Morera's theorem. If $[\alpha, \beta] \xrightarrow{\gamma} \mathbb{C}$ is a closed curve in $\mathbb{C}$, then

$$F(\omega, t) = \hat{f}(\omega) e^{2\pi i \omega \gamma(t)} \dot{\gamma}(t),$$

$$[-\tfrac{a}{2}, \tfrac{a}{2}] \times [\alpha, \beta] \xrightarrow{F} \mathbb{C},$$

satisfies the hypothesis of Fubini's Theorem, hence

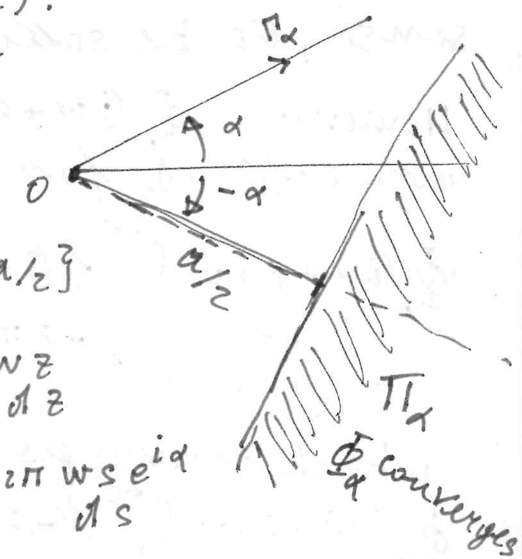
$$\int_\gamma f(z)\, dz = \int_{-a/2}^{a/2} \hat{f}(\omega) \int_\gamma e^{2\pi i \omega z} dz \cdot d\omega$$
$$= 0 \quad \blacksquare$$

Paley and Wiener proved the converse result.

Theorem. If $f \in \operatorname{Hol}(\mathbb{C})$, $f|_{\mathbb{R}} \in L^2(\mathbb{R})$, and $|f(z)| \leq C \cdot e^{\pi a |z|}$, then $f \in PW(a)$.

Proof. For $\alpha \in \mathbb{R}$ set

$\Gamma_\alpha(s) = s e^{i\alpha} \quad (s \geq 0)$



and

$\Pi_\alpha = \{w : \operatorname{Re}(w \cdot e^{i\alpha}) > a/2\}$

and

$$\Phi_\alpha(w) = \int_{\Gamma_\alpha} f(z) e^{-2\pi i w z} dz$$
$$= e^{i\alpha} \int_0^\infty f(se^{i\alpha s}) e^{-2\pi w s e^{i\alpha}} ds$$

which converges because

$$\left| f(se^{i\alpha s}) e^{-2\pi w s e^{i\alpha}} \right| \leq C_1 \cdot e^{\pi a s} \cdot e^{-2\pi s \operatorname{Re}(we^{i\alpha})}$$
$$= C_1 \cdot e^{-\pi s (\operatorname{Re}(we^{i\alpha}) - \frac{a}{2})}$$

which is integrable ~~on~~ on $[0, \infty)_s$.

Also observe that, formally:

$$\hat{f}(\omega) = \int_0^\infty f(s)\, e^{-2\pi i \omega s}\, ds + \int_0^{+\infty} f(-s)\, e^{2\pi i \omega s}\, ds$$

$$= \Phi_0(i\omega) - \Phi_\pi(i\omega),$$

although $\Phi_0(i\omega)$, $\Phi_\pi(i\omega)$ are not a priori well defined, if not in some measure theoretic (L^2) sense, to be specified later.

However, $\Phi_0(i\omega+\varepsilon)$ and $\Phi_\pi(i\omega-\varepsilon)$ are defined for $\varepsilon > 0$:

$$\Phi_0(i\omega+\varepsilon) = \int_0^\infty f(s)\, e^{-2\pi(i\omega+\varepsilon)s}\, ds$$

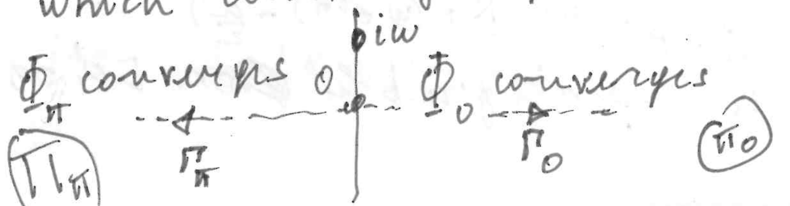
$$= \int_0^\infty f(s)\, e^{-2\pi i \omega s} \cdot e^{-2\pi\varepsilon s}\, ds,$$

which converges by Cauchy-Schwarz;

$$\Phi_\pi(i\omega-\varepsilon) = \int_0^\infty f(-s)\, e^{-2\pi(i\omega-\varepsilon)\underset{(-1)}{(-s)}}\, ds$$

$$= -\int_0^\infty f(s)\, e^{+2\pi i \omega s}\, e^{-2\pi\varepsilon s}\, ds,$$

which converges for the same reason.



<u>Lemma 1.</u> Φ_α is holomorphic in Π_α.

Pf. We apply Morera's Theorem. Let $w = \gamma(t)$ be a smooth closed curve in Π_α: $|\gamma(t)| \leq C$ and $\mathrm{Re}(\gamma(t)e^{i\alpha}) \geq \frac{\alpha}{2} + \varepsilon$ if $\alpha \neq 0, \pi$; $\mathrm{Re}(\gamma(t)) \geq \varepsilon$ in Π_0; and $\mathrm{Re}(\gamma(t)) \leq -\varepsilon$ in Π_π $(\varepsilon > 0)$.

We want to change order of integration in

$$\int_\gamma \Phi_\alpha(w)\, dw = \int_a^b dt \int_0^\infty ds\, f(s e^{i\alpha})\, e^{i\alpha} \cdot \dot{\gamma}(t) \cdot e^{-2\pi \gamma(t) e^{i\alpha} s} = \int_a^b dt \cdot \int_0^\infty ds \cdot F(s,t)$$

and we have the estimates:

$|F(s,t)| \leq C \cdot e^{-\pi s \varepsilon}$ $(\alpha \neq 0, \pi)$

$|F(s,t)| \leq C \cdot |f(s)|\, e^{-2\pi\varepsilon s}$ ($\alpha = 0$, similar for $\alpha = \pi$)

so that in both cases $F \in L^1([0,+\infty) \times [a,b])$ and

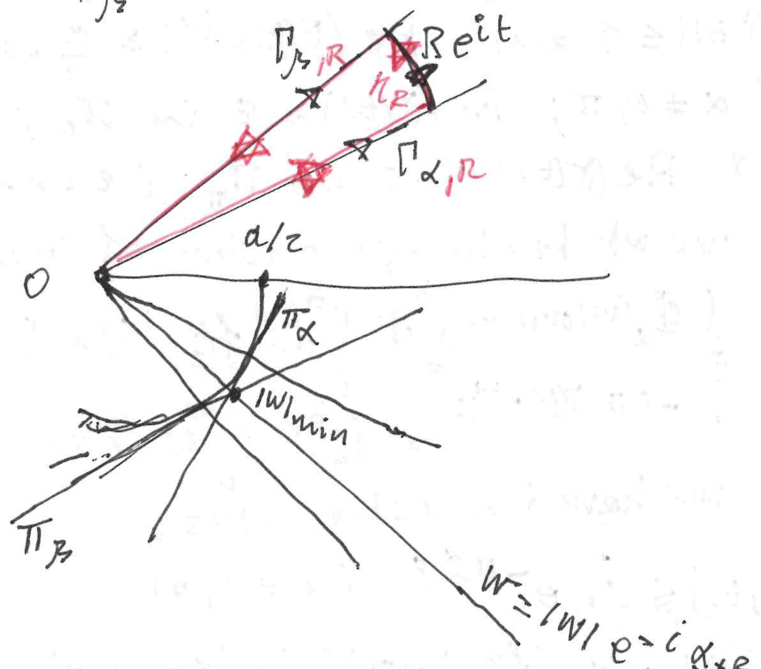
$$\int_\gamma \Phi_\alpha(w)\, dw = \int_\gamma \int_{\Gamma_\alpha} f(z)\, e^{-2\pi w z}\, dz\, dw$$

$$= \int_{\Gamma_\alpha} f(z) \int_\gamma e^{-2\pi w z}\, dw \cdot dz = 0$$

hence Φ_α is holomorphic in Π_α ∎

<u>Lemma 2.</u> If $0 < \beta - \alpha < \pi$, then

$\Phi_\alpha = \Phi_\beta$ on $\Pi_\alpha \cap \Pi_\beta$.



<u>Pf.</u>

It suffices to show that $\Phi_\alpha(w) = \Phi_\beta(w)$ for $w = |w| e^{-i\frac{\alpha+\beta}{2}}$ and $|w|$ large enough (we'll get a precise value for $|w|_{min}$, which is the one in the picture).

For $R>0$ consider the curve γ_R in the picture:

$$0 = \int_{\gamma_R} f(z)\, e^{-2\pi w z}\, dz = \int_{\Gamma_{\alpha,R}} - \int_{\Gamma_{\beta,R}} + \int_{\eta_R}$$

$$\left|\int_{\eta_R} f(z)\, e^{-2\pi z w}\, dz\right| = \left|\int_\alpha^\beta f(Re^{it})\, e^{-2\pi Re^{it}|w|e^{-i\frac{\alpha+\beta}{2}}}\, Rie^{it}\, dt\right|$$

$$\le \int_\alpha^\beta c \cdot e^{\pi a R} \cdot R \cdot e^{-2\pi R|w| \cdot \cos\left(t - \frac{\alpha+\beta}{2}\right)}\, dt$$

$$\le c \cdot (\beta-\alpha) \cdot R \cdot e^{-2R\pi\left(-\frac{a}{2} + |w| \cdot \cos\left(\frac{\beta-\alpha}{2}\right)\right)}$$

$\xrightarrow[R\to\infty]{} 0$ if $|w| > |w|_{min} = \frac{a}{2} \Big/ \cos\left(\frac{\beta-\alpha}{2}\right)$

Letting $R \to \infty$, $\Phi_\alpha(w) = \int_{\Gamma_\alpha} = \int_{\Gamma_\beta} = \Phi_\beta(w)$. ∎

To finish the proof, for $a>0$ consider $f_\varepsilon(x) = f(x)\, e^{-\varepsilon|x|2\pi}$, so that $\|f - f_a\|_{L^2} \xrightarrow{\varepsilon \to 0} 0$

We wish to show that $\hat{f}_a(t) \xrightarrow[\varepsilon\to 0]{} 0 \quad \forall |t| > a/2$, $\|\hat{f} - \hat{f}_a\|_{L^2}$

which implies that $\hat{f}(t) = 0$ for a.e. $|t| > \frac{a}{2}$.

By <u>Lemma 2</u>, $\hat{f}_\varepsilon(t) =$

$$= \int_{-\infty}^{+\infty} f_\varepsilon(x)\, e^{-2\pi i t x}\, dx = \int_0^\infty f(x)\, e^{-2\pi(\varepsilon + it)x}\, dx$$

$$- \int_0^\infty f(-x)\, e^{2\pi(-\varepsilon + it)x}\, dx =$$

$$= \Phi_0(it + \varepsilon) - \Phi_\pi(it - \varepsilon) = \Phi_{\pi/2}(it+\varepsilon) - \Phi_{\pi/2}(it - \varepsilon)$$

$\xrightarrow[\varepsilon\to 0]{} 0$ ∎

A few more (classical) facts on PW.

(1) L. Schwartz 1952. TFAE

(i) $f \in \mathrm{Hol}(\mathbb{C})$, $|f(z)| \leq C\cdot(1+|z|)^N \cdot e^{\pi|\mathrm{Im}(z)|a}$ $\exists C, N>0$

(ii) there is a distribution $T \in \mathcal{S}'(\mathbb{R})$ with $\mathrm{supp}(T) \subseteq [-\frac{a}{2}, \frac{a}{2}]$ s.t.

$$f(z) = \hat{T}(z)$$

(2) $PW(a)$ is a R.K.H.S. on $\mathbb{C}$, not just on $\mathbb{R}$:

$$f(z) = \int_{-a/2}^{a/2} \hat{f}(\omega) e^{2\pi i z\omega} d\omega = \langle f, K_z \rangle_{PW}$$

where $K_z(\zeta) = \int_{-a/2}^{a/2} \hat{K}_z(\omega) e^{2\pi i \zeta\omega} d\omega$

$$= \int_{-a/2}^{a/2} e^{-2\pi i \bar{z}\omega} e^{2\pi i \zeta\omega} d\omega$$

$$= \left[\frac{e^{2\pi i(\zeta-\bar{z})\omega}}{2\pi i(\zeta-\bar{z})}\right]_{-a/2}^{a/2}$$

$$= \frac{\sin(\pi(\zeta-\bar{z})a)}{\pi(\zeta-\bar{z})} = a\cdot \mathrm{sinc}(\pi(\zeta-\bar{z}));$$

$$\boxed{K_z(\zeta) = a\cdot \mathrm{sinc}(\pi(\zeta-\bar{z}))}$$

In particular, $|f(z)| \leq \|f\|_{PW} \cdot K_z(z)^{1/2}$.

Compute for $z = x+iy$:

$$K_z(z) = \frac{\sin(2i\pi y a)}{2\pi i y} = \frac{e^{2\pi y a} - e^{-2\pi y a}}{4\pi y}$$

$$= \frac{\sinh(2\pi y a)}{2\pi y} = a\cdot\mathrm{sinhc}(2\pi y a);$$

$$|f(x+iy)| = |f(z)| \leq \sqrt{a}\cdot \mathrm{sinhc}(2\pi y a)^{1/2} \cdot \|f\|_{PW}$$

$$\leq C\, e^{\pi|y|a} \cdot \|f\|_{PW}$$

Also, from $f(z) = f(x+iy) = \int_{-a/2}^{a/2} \hat{f}(\omega) e^{-2\pi y\omega} \cdot e^{2\pi i\omega x} d\omega$,

$$\int_{-\infty}^{+\infty} |f(x+iy)|^2 dx = \int_{-a/2}^{a/2} |\hat{f}(\omega)|^2 e^{-4\pi y\omega} d\omega$$

$$\leq e^{2\pi a|y|} \cdot \|f\|_{PW}^2,$$

which (with $0<p<\infty$ instead of 2) is called <u>Plancherel - Polya inequality</u> (1936).

(3) The proof of Paley-Wiener works if we just assume that $\forall \varepsilon>0 \Rightarrow |f(z)| \leq C(\varepsilon)\cdot e^{(\pi+\varepsilon)a|z|}$! We then have $|f(z)| \leq C\cdot e^{\pi|\mathrm{Im}\, z|a}$ from (2).

(4) ~~[illegible]~~

<u>No multipliers.</u> $\mathcal{M}(PW) = \mathbb{C}$ (constants).

If $m \in \mathcal{M}(PW)$, then m is bounded by the general theory. Also, $m \in Hol(\mathbb{C})$: $m(z)K_w(z)$ is holomorphic and $K_w(z_0) \neq 0$ if $w \neq z_0 + i\bar{a}^{-1}\mathbb{Z}$, hence $m(z)$ is holomorphic in a neighb. of each $z_0 \in \mathbb{C} \Rightarrow m$ is constant.

<u>Corollary.</u> $PW(a)$ is not a Pick space.

(5) ~~[illegible]~~