lecture 4-4-2023

Hilbert spaces of analytic functions
– reproducing kernel Hilbert spaces.

Putting in a unified frame the Shannon and Carleson theorem. In general it gives a nice abstract frame work for quantitative interpolation problems

If we want to interpolate we want expressions like $f(z_n) = w_n$ to have meaning so our functions f have to be defined point wise.

This motivates the following definition.

Def. let $\mathcal{H} \subseteq \mathbb{C}^X$, $X \neq \emptyset$ set a Hilbert space and we require that $L_x : f \mapsto f(x)$ is in H^*.

Hence by Riesz representation theorem $\exists\, K_x \in H$ such that $\langle f, K_x \rangle = f(x)$. assume that $K_x \neq 0 \ \forall x$

[~~Recall the Fourier transform~~ :

$$\langle K_x, K_y \rangle = K_x(y) = \overline{\langle K_y, K_x \rangle} = \overline{K_y(x)}$$

$$K_x(x) = \langle K_x, K_x \rangle = \|K_x\|_H^2 > 0.$$

Paley-Wiener space $\{f : \mathbb{R} \to \mathbb{C} \ ; \ \operatorname{Supp} \hat{f} \subseteq [-\pi, \pi]\}$.

$\|f\|^2_{PW_\pi} = \|f\|^2_{L^2(\mathbb{R})}$ $\quad f \in L^2(\mathbb{R})$

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i \xi x}\, dx$$

~~Shannon's theorem tell's us that~~

$$\cancel{f(\alpha) = (f^\vee)^\wedge(\alpha) =}\ \cancel{f(x) =} \int_{-\pi}^{\pi} f^\vee(x) e^{-2\pi i x} dx \Rightarrow$$

$$f(x_0) = (f^\vee)^\wedge(a) = \int_{-\infty}^{+\infty} f^\vee(x)\, e^{-2\pi i x x_0} dx = \int_{-\infty}^{+\infty} \hat{f}(-x)\, e^{-2\pi i x x_0} dx$$

$$= \int_{-\infty}^{+\infty} \hat{f}(x)\, \chi_{(-\frac{a}{2}, \frac{a}{2})}(x)\, e^{2\pi i x x_0} dx = \int_{-\infty}^{+\infty} \hat{f}(x)\, \overline{\hat{K}_{x_0}(+x)}\, dx$$

$$= \int_{-\infty}^{+\infty} f(x) K_{x_0}(x)\, dx.$$

$|f(x_0)| \leq \|f\|_{PW_a} \|K_{x_0}\|_{L^2(\mathbb{R})}$ $\qquad K_{x_0}(x) = a \operatorname{sinc}(\pi a (x - x_0))$

Hardy space; $H^2(\mathbb{D})$; $\left\{ \underbrace{\sum_{n \geq 0} a_n z^n}_{f(z)} \ ; \ \underbrace{\sum |a_n|^2}_{\|f\|^2_{H^2(\mathbb{D})}} < \infty \right\}$

$|z| < r$ $\quad \sum |a_n| |z|^n \leq \left\{ \sum |a_n|^2 \right\}^{1/2} \left\{ \sum r^{2n} \right\}^{1/2} = \|f\|^2_{H^2} / (1 - r^2)^{1/2}$

This proves that (i) f are well defined in $\mathbb{D}$
(ii) $\{L_z\}_{z \in \mathbb{D}}$ are BLO

$K_z(w) = \sum c_n w^n$ $\qquad \langle f, K_z \rangle_{H^2} = \sum a_n \cdot \bar{c}_n = \sum a_n z^n$

$\Rightarrow c_n = \bar{z}^n$. and $\quad K_z(w) = \dfrac{1}{1 - \bar{z} w}$. (Szegö Kernel).

<u>Multipliers</u>. $\mathcal{M}(\mathcal{H}) = \{ f : X \to \mathbb{C} ; \quad fH \subseteq H \}$.

(i) If $\varphi \in \mathcal{M}(\mathcal{H})$ $\quad M_\varphi : \mathcal{H} \to \mathcal{H}$, $f \mapsto \varphi f$ is a bounded linear operator

(ii) $\quad M_\varphi^*(K_x) = \overline{\varphi(x)}\, K_x \quad \forall x \in X$

(iii) $\quad \sup_{x \in X} |\varphi(x)| \leq \|M_\varphi\|_{\mathcal{B}(H)}$

<u>Proof</u>. Let $f_n \to f$ in H and $M_\varphi f_n \to g$ in H

$f_n \xrightarrow{H} f \Rightarrow f_n(x) \to f(x) \quad \forall x$

$\varphi(x) f_n(x) \longrightarrow g(x) \quad \Rightarrow g(x) = f(x)\varphi(x). \quad \forall x.$

$\Rightarrow M_\varphi$ bounded
closed graph theorem

$(f, M_\varphi^* K_x) = (M_\varphi f, k_x) = \varphi(x) f(x) = (f, \overline{\varphi(x)} k_x) \quad \forall f \in H$

$\Rightarrow M_\varphi^* K_x = \overline{\varphi(x)}\, Kx.$

$\overline{\phi(x)} \in \sigma(M_\varphi^*) \subseteq \overline{D}(0, \|M_\varphi^*\|) = \overline{D}(0, \|M_\varphi\|)$

- Nicola will prove a theorem about PW_a which will have as a consequence that $\mathcal{M}(PW_a) = \{0\}$.

- For $H^2(\mathbb{D})$ are much richer

<u>Prop</u>. $\displaystyle\sum_{n=0}^{\infty} |a_n|^2 = \sup_{0<r<1} \sum_{n=0}^{\infty} |a_n r^n|^2 =$

$\displaystyle\sup_{0<r<1} \int_0^{2\pi} \Big| \sum_{n\geq 0} a_n r^n e^{i\theta n} \Big|^2 \frac{d\theta}{2\pi} = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi}.$

<u>Corollary</u>. $\mathcal{M}(H^2(\mathbb{D})) = H^\infty(\mathbb{D})$.

$\varphi \in H^\infty(\mathbb{D}) \Rightarrow \displaystyle \|\varphi f\|^2_{H^2(\mathbb{D})} = \sup_{0<r<1} \int_0^{2\pi} |\varphi(re^{i\theta}) f(re^{i\theta})|^2 \frac{d\theta}{2\pi}$

$\leq \|\varphi\|^2_{H^\infty} \cdot \|f\|^2_{H^2}.$

If $\varphi \in \mathcal{M}(H^2(\mathbb{D}))$, $1 \in H^2 \Rightarrow \varphi \cdot 1 = \varphi \in H^2 \Rightarrow \varphi \in \mathcal{O}(\mathbb{D})$

and by (iii) from before $\sup_{|z|<1} |\varphi(z)| \leq \|M_\varphi\| < \infty.$

$\|M_\varphi\|_{\mathcal{M}(H^2)} = \|\varphi\|_{H^\infty}.$ ▨

Abstract pick problem

~~Concrete~~ Pick problem. $z_1, \dots, z_n \in \mathbb{D}$, $w_1, \dots, w_n \in \mathbb{D}$
$\Longleftrightarrow$ $\exists\, \varphi \in H^\infty(\mathbb{D})$ $\|\varphi\|_{H^\infty} \le 1$ s.t. $\varphi(z_i) = w_i$

Let H a rkHs. Let $\mathcal{M}_1(\mathcal{H}) = \{\varphi \in \mathcal{M}(\mathcal{H}) ; \|\varphi\|_{\mathcal{M}(\mathcal{H})} \le 1\}$

$z_1, \dots, z_n \in X$, $w_1, \dots, w_n \in \mathbb{C}$ $\exists\, \phi \in \mathcal{M}_1(\mathcal{H})$

$\phi(z_i) = w_i$.

Remember that Pick's theorem tells us that the problem is solvable $\Longleftrightarrow$

$$\left[\frac{1 - \bar{w}_i w_j}{1 - \bar{z}_i z_j}\right]_{i,j=1}^n \ge 0 \text{ PSD}. \quad \left[(1 - w_i \bar{w}_j) K_{z_j}(z_i)\right] \ge 0$$

We will see that the necessary condition is a general fact, it holds for ALL rk Hs.

Suppose that $\exists$ such φ. consider the subspace of $\mathcal{H}$ $= \operatorname{span}\{K_{z_1}, \dots, K_{z_n}\} = V$

if $f \in V$ $\|M_\varphi^* f\| \le \|f\|$, $f = \sum_{i=1}^n t_i K_{z_i}$

$$\|M_\varphi f\|^2 = \langle M_\varphi M_\varphi^* f, f\rangle = \sum_{i,j=1}^n \langle M_\varphi M_\varphi^* K_{z_i}, K_{z_j}\rangle$$

$$= \sum_{i,j=1}^n \langle M_\varphi \overline{\varphi(z_i)} K_{z_i}, K_{z_j}\rangle$$

$$= \sum_{i,j=1}^n \overline{\varphi(z_i)} \varphi(z_j) K_{z_i}(z_j)$$

$$= \sum_{i,j=1}^n \bar{w}_i w_j K_{z_i}(z_j) t_i \bar{t}_j$$

$$\|f\|^2 = \sum_{i,j} t_i \bar{t}_j K_{z_i}(z_j) \Longrightarrow$$

$$\sum_{i,j=1}^n (1 - \bar{w}_i w_j) K_{z_i}(z_j) t_i \bar{t}_j \ge 0$$

$$\Longleftrightarrow \left[(1 - w_i \bar{w}_j) K_{z_j}(z_i)\right]_{i,j=1}^n \text{ PSD}.$$

The sufficiency of the condition is NOT a Hilbert space property and it very much depends on the particular kernel

In fact for spaces, like PWa, with trivial multipliers the problem is never solvable

Def. If ~~the posit~~ $\left[(1 - w_i \bar{w}_j) K_{z_j}(z_i)\right]_{i,j=1}^n \ge 0 \Longrightarrow$

$\exists\, \varphi \in \mathcal{M}_1(\mathcal{H})$; $\varphi(z_i) = w_i$, then $\mathcal{H}$

is called a Pick space.

Corollary. $H^2(\mathbb{D})$ is a pick space.

I do not want to leave you with the impression that being a Pick space is a rare event.

For example in with M. Abate we proved that the space

$$\{ f \in AC_{loc}(\mathbb{R}) ; \int_{-\infty}^{+\infty} |f'(x)|^2 dx < \infty \} \text{ is a rkhs}$$

with the Pick property and there are many more such spaces.

<u>Zero sets for Pick spaces</u> H is a normalized pick space $K_{z_0} = 1 \in H$.

<u>Proposition.</u> Let $\sum_{i=1}^{\infty} K_{z_i}(z_i)^{-1} < \infty$. Then $\exists \varphi \in M(H)$ $\varphi \not\equiv 0$ and $\varphi(z_i) = 0$.

$M(H)$ is a Banach algebra. $\{\varphi_n\}$

Cauchy $M_{\varphi_n} \xrightarrow{\text{~~operator norm~~}} T$ SOT. $\varphi_n f \underset{n}{\rightarrow} Tf$

$\varphi_n(x) f(x) \longmapsto Tf(x) = \varphi(x) f(x)$. Hence φ_n converges $\forall x$ (pick $f_x(x) = 1$)

$$\|\varphi\psi\|_{M(H)} = \sup_{\|f\|_H \leq 1} \|\varphi\psi f\|_H \leq \sup_{\|f\| \leq 1} \|\varphi\|_{M(H)} \|\psi f\|_H$$

$$= \|\varphi\|_{M(H)} \|\psi\|_{M(H)}.$$

~~$M(H) \subset B(H)$ closed.~~

<u>Proof.</u> Consider $z_0 \neq z_i$ $i = 1, 2, 3, \ldots$ and $\delta_i > 0$ to be fixed.

Consider the two-point pick problem

$z_0 \longrightarrow \delta_i$

$z_i \longrightarrow 0$

$$\begin{bmatrix} (1-\delta_i^2) K_{z_0}(z_0) & K_{z_0}(z_i) \\ K_{z_i}(z_0) & K_{z_i}(z_i) \end{bmatrix} \geq 0$$

$\iff \delta_i^2 \leq 1$ and $0 \leq (1-\delta_i^2) K_{z_0}(z_0) K_{z_i}(z_i) - |K_{z_i}(z_0)|^2$

$\iff (1-\delta_i^2) K_{z_i}(z_i) - 1 \geq 0 \iff$

$1 - \delta_i^2 \geq \frac{1}{K_{z_i}(z_i)} \iff \delta_i \leq \sqrt{1 - \frac{1}{K_{z_i}(z_i)}}$

Set $\boxed{\delta_i = 1 - \frac{1}{K_{z_i}(z_i)}}$

So there exist multipliers $\varphi_i(z_i) = 0$

$\varphi_i(z_0) = \delta_i$

then consider the functions $\Phi_n = \prod_{i=1}^{n} \varphi_i$.

$\|\Phi_n\|_{\mathcal{H}} \leq \|\Phi_n\|_{M(H)} \leq \prod_{i=1}^{n} \|\varphi_i\| \leq 1.$

$$\Phi_n(z_0) = \prod_{i=1}^{n} \varphi_i(z_0) = \prod_{i=1}^{n} \left\{1 - \frac{1}{K_{z_i}(z_i)}\right\}$$

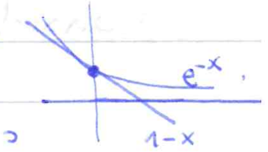
$$\geq \cancel{\prod_{i=1}^{n} \frac{1}{K_{z_i}}} = \exp\left\{\sum_{i=1}^{n} \ln\left(1 - \frac{1}{K_{z_i}(z_i)}\right)\right\}$$

$$\geq \exp\left\{-\sum_{i=1}^{n} \frac{1}{K_{z_i}(z_i)}\right\} \geq \exp\left\{-\sum_{i=1}^{\infty} \frac{1}{K_{z_i}(z_i)}\right\} > 0.$$

$\Phi_n(z_i) = 0 \quad 1 \leq i \leq n.$

$\ln(1 - x) \geq -x$

$\iff 1 - x \geq e^{-x}$



$\Phi_{n_k} \xrightarrow{w^*} \Phi \implies \Phi_{n_k}(z) \longrightarrow \Phi(z) \quad \forall z.$

$\implies \Phi(z_0) > 0 \quad \Phi(z_i) = 0 \quad i = 1, 2, 3, 4 \ldots$

Remarks. It can be proven that $\Phi \in M(\mathcal{H})$

but we need to know that $M(H)$ has a weak* topology.

and that $\phi \longmapsto \phi(z)$ are continuous in this w^* topology.

If $M \leq X^*$. $(X/M_\perp)^* \underset{\text{ISOMETRIC}}{\simeq} M$. $M_\perp = \{x \in X;\ (x, m) = 0\ \forall m \in M\}$

if $x^* \in M$, $\widehat{x^*}: X/M_\perp \to \mathbb{C}$

$\widehat{x^*}(x + M_\perp) = x^*(x)$

if $x_0^*: X/M_\perp \to \mathbb{C}$ $\check{x}_0^* \in M$, $\check{x}_0^*: X \to \mathbb{C}, (x, \check{x}_0^*) = (x + M_\perp, x_0^*)$

lecture 5/4/2023.

Rescalling of kernels

H is a rkHs with kernel $K_x : X \to \mathbb{C}$ and $\delta : X \to \mathbb{C}$

$$K^{\delta}_x(y) = \delta(x)\, K_x(y)\, \overline{\delta}(y)$$

$M_\delta : H \to H_\delta$ $\quad \|\delta f\|_{H_\delta} = \|f\|_H$

$\delta(x) f(x) = \langle \delta f, K^{\delta}_x \rangle_{H_\delta} = \langle f, K^{\delta}_x \frac{1}{\delta} \rangle_H$

$= \langle f, K_x \cdot \overline{\delta(x)} \rangle$ $\Rightarrow$

$$K^{\delta}_x(y) = \delta(y)\, K_x(y)\, \overline{\delta(x)}.$$

Hence, assuming that $\delta(x) \neq 0$ $\forall x \in X$. $\exists \varphi \in H$; $\phi(z_i) = 0$ $i = 1,2,\ldots$ $\phi(z_0) \neq 0$ $\iff$ $\exists \varphi \in H_\delta$ with the same property.

pick $x_0 \in X$ $\quad \delta(x) = \dfrac{\sqrt{K_{x_0}(x_0)}}{K_{x_0}(x)}$ $\quad \dfrac{K_{x_0}(x)}{K_{x_0}(x)\, K_{x_0}(x_0)}$

$$K^{\delta}_{x_0}(x) = \delta(x_0)\, K_{x_0}(x)\, \delta(x)$$

$$= \frac{\sqrt{K_{x_0}(x_0)}}{K_{x_0}(x_0)}\, K_{x_0}(x)\, \frac{\sqrt{K_{x_0}(x_0)}}{K_{x_0}(x)} = 1.$$

$$\sup_{\|f\| \le 1} \|\varphi \delta f\|_{H_\delta} = \sup_{\|f\| \le 1} \|\varphi f\|_H = \|\varphi\|_{m(H)}.$$

$\|$

$\|\varphi\|_{m(H_\delta)}$

$$\boxed{m(H_\delta) \underset{\text{ISOM.}}{=} m(H)}$$

"Interpolation for the Hilbert space."

Shannon sampling theorem"

In PW_a, let $K^a_x(y) = a\, \mathrm{sinc}(\pi a(y-x))$. $\left\{\frac{K^a_n}{\|K^a_n\|}\right\}_{n\in\mathbb{Z}}$ is an ONB of PW_a

$$f(\cdot) = \sum_{n\in\mathbb{Z}} \left(f, \frac{K^a_n}{\|K^a_n\|}\right)_{PW_a} \cdot \frac{K^a_n}{\|K^a_n\|} = \sum_{n\in\mathbb{Z}} f(n)\, \mathrm{sinc}[\pi a(\cdot - n)]$$

The infinite sum converges in norm and pointwise

This is part of a general idea

$$\left\{\begin{matrix}\text{interpolation \& sampling} \\ \text{in } \mathcal{H}\end{matrix}\right\} \rightsquigarrow \left\{\begin{matrix}\text{base properties} \\ \text{of } \{K_{z_n}/\|K_{z_n}\|\}\end{matrix}\right\}.$$

Definition (i) $\{g_n\}_{n\in\mathbb{N}}$ is a Riesz system in H if $A \sum |a_n|^2 \le \|\sum_{n=1}^{\infty} a_n g_n\|^2 \le B \sum |a_n|^2$, $\forall \{a_n\} \in C_{00}$.

"almost" pythagorian thm

(ii) $\{g_n\}$ is a frame is.

$$A' \|x\|^2 \le \sum_{n=1}^{\infty} |(x, g_n)|^2 \le B' \|x\|^2$$

"almost" parseval identity

Definition (iii) Riesz Basis if both hold.

In a rkHs. (i) $\{z_n\} \subseteq X$. $\{z_n\}$ is sampling $\iff$ $\{k_{z_n}/\|k_{z_n}\|\}_{n=1}^{\infty}$ is a Frame.

(ii) $\{z_n\} \subseteq X$ is interpolating $\iff$ $\{k_{z_n}/\|k_{z_n}\|\}$ is a Riesz system.

~~Frame~~

Therefore sampling sequence means that

$$\boxed{\text{for all } f\in\mathcal{H},\ A\|f\|^2 \le \sum_{n\ge 0} \frac{|f(z_n)|^2}{\|K_{z_n}\|^2} \le B\|f\|_{\mathcal{H}}^2}$$

$f(z_n)=g(z_n) \implies f=g.$

$f(z_n)=g(z_n)+\varepsilon_n \implies \implies \|f-g\|_H^2 \lesssim \sum |\varepsilon_n|^2/\|k_{z_n}\|^2.$

Remark $H^2(\mathbb{D})$ does not have sampling sequences.

let $f_N(z)=z^N$. $\quad A = A\|f_N\|^2 \le \sum_{n=0}^{\infty} (1-|z_n|^2)|z_n|^{2N}$

by monotone convergence

$$0=\lim_{N\to\infty}\sum_{n=0}^{\infty}(1-|z_n|^2)|z_n|^{2N} \ge A \implies \boxed{A=0}$$

PW_a does! ~~all~~ $\mathbb{Z}$ for example.

Interpolating sequences.

$\iff R:\mathcal{H}\to\ell^2$

$f\mapsto \left\{\frac{f(z_n)}{\|K_{z_n}\|}\right\}$

is bounded and surjective.

$$\langle R^*\{a_n\}, f\rangle_H = \left\langle \{a_n\}_n, \left\{\frac{f(z_n)}{\|k_{z_n}\|}\right\}_n\right\rangle_{\ell^2} = \sum_{n\ge 0}^{\infty} a_n \overline{\frac{f(z_n)}{\|k_{z_n}\|}}$$

$$\iff R^*\{a_n\} = \sum_{n\ge 0} a_n \frac{K_{z_n}}{\|K_{z_n}\|}.$$

~~Lemma $T: H\to K$ is bounded below $\iff$ $T^*: K\to H$ is surjective~~

Proof $\mathrm{Ker}\,T = \mathcal{R}(T^*)^\perp$ ~~$\|Tx\|^2 \ge \varepsilon\|x\|^2$~~ ~~$x_n\in K$, $T^*x_n\to y$.~~

~~$\iff (T^*Tx,x)\ge\varepsilon\|x\|^2$~~

~~$\varepsilon\|x_n-x_m\|^2 \le \|T(x_n-x_m)\|^2$~~

~~T has closed range $\implies$ $M=\mathrm{Range}(T)$ $K=M\oplus M^\perp$~~

$\mathrm{Ker}\,T = [\mathrm{Range}\,T^*]^\perp$

Lemma $T: H\to K$ is bounded ISOMORPHISM iff $T^*: K\to H$ is an ISOMORPHISM.

Proof By symmetry suffices to prove "$\implies$"

$\mathrm{Ker}\,T = (\mathrm{Range}\,T^*)^\perp \Rightarrow T^*$ has dense range and

if $T^*x_n \to y$, let $k\in K$ and $Th=k$.

$T: H \to K \leq K_1$

$T^*: K_1 \to K$

$(x_n, k) = (x_n, Th) = (T^* x_n, h) \to (y, h) = (y, T^{-1}k) = ((T^{-1})^* y, k)$

$\Rightarrow x_n \xrightarrow{\text{weakly}} x \Rightarrow T^* x_n \xrightarrow{\text{weakly}} T^* x = y. \Rightarrow y \in \text{Range}(T^*)$

$\text{Ker } T^* = [\text{Range}(T)]^{\perp} = (0). \Rightarrow T^*$ is invertible.

$T: H \to K = R(T) \oplus R(T)^{\perp}$

$T^*: R(T) \oplus \underset{\text{Ker}(T^*)}{\underset{\shortparallel}{R(T)^{\perp}}} \to H.$

Hence $\{z_n\}$ is interpolating $\iff$

R is bounded and surjective

$$\iff \begin{cases} \sum_{n=0}^{\infty} \frac{|f(z_n)|^2}{\|k_{z_n}\|^2} < C \cdot \|f\|^2 \\ \forall \{a_n\} \in \ell^2;\ \exists f \in H;\ f(z_n) = a_n \|k_{z_n}\|. \end{cases}$$

Lemma $T: H \to K$ is bounded $\overset{\text{above}}{\text{below}} \iff T^*: K \to H$ is surjective

T is bounded above and below $\iff$ $\text{Ker}\,T = (0)$ and $R(T)$ is closed. let $\Pi_1: K \to R(T)$ orthogonal projection

$T^*: R(T) \oplus R(T)^{\perp} \to H$

$\shortparallel$

$R(T) \oplus \text{Ker}\,T^* \to H \Rightarrow R(T^*) = T^*(R(T)) =$

$\Pi_1^* = id: R(T) \to K.$

$\Pi_1 T$ is isomorphism $\iff$

$T^* \Pi_1^*$ is an isomorphism $\Rightarrow T^*$ surjective

$T^*: \text{Ker}(T^*) \oplus (\text{Ker}\,T^*)^{\perp} \to H \qquad \Pi_2: K \to \underset{R(T)}{\underset{\shortparallel}{(\text{Ker}\,T^*)^{\perp}}}$ proj

$T^* \Pi_2$ isom $\iff \Pi_2^* T$ isom $\Rightarrow T$ isomorphism onto its image.

Our next goal is to show that the interpolation problem is equivalent to an interpolation problem for the multiplier algebra in spaces with the Pick property.

~~NP~~

~~later~~

Carleson interpolation theorem and interpolation in the multiplier algebra.

$$S: \mathcal{M}(H) \to \ell^{\infty}(\mathbb{N})$$
$$\varphi \mapsto \{\varphi(z_n)\}_{n \geq 1}.$$

we say that $\{z_n\}$ is $\mathcal{M}(H)$-interpolating $\iff$ S is surjective.

So that $H^{\infty}(\mathbb{D})$-interpolating sequences are exactly the ones described by Carleson's theorem

<u>Theorem.</u> The following are equivalent for a Pick space $\mathcal{H}$.

(I) $\{z_n\}$ is $\mathcal{M}(\mathcal{H})$-interpolating

(II) $\{z_n\}$ is interpolating for $\mathcal{H}$.

<u>Corollary.</u> <u>TFAE.</u> for $\{z_i\} \subseteq \mathbb{D}$

(α) $\forall \{a_n\}_{n=1}^{\infty} \in \ell^2 \ \exists f \in H^2(\mathbb{D});\ f(z_n) = a_n (1-|z_n|^2)^{1/2}$ and $\sum_{n \geq 1} |f(z_n)|^2 (1-|z_n|^2) \leq C \|f\|_{H^2}^2$

(β) $\inf_n \prod_{m \neq n} \left| \frac{z_n - z_m}{1 - \bar{z}_n z_m} \right| \geq \delta > 0.$

Proof

"$\Longrightarrow$" let $g_n = K_{z_n}/\|K_{z_n}\|$, let $w_j = e^{2\pi i t \cdot j}$

$\exists\, \varphi \in \mathcal{M}(\mathcal{H})$ such that $\varphi_t(z_j) = w_j$ $\|\varphi_t\| \le M_{\mathcal{M}(H)}$.

Then, $M^*_{\varphi_t} : H \to H$ is a BLO.

$\|M^*_{\varphi_t} f\|^2 \le M^2 \|f\|^2$. let $f = \sum a_n g_n$, $a_n \in C_{00}(\mathbb{N})$

$$\sum_{n,m=1}^{\infty} a_n \bar{a}_m e^{2\pi i (n-m) t} \langle g_n, g_m \rangle \le M^2 \sum_{n,m=1}^{\infty} a_n \bar{a}_m \langle g_n, g_m \rangle$$

$$\xRightarrow{\int_0^1 \cdot\, dt} \sum_{n=1}^{\infty} |a_n|^2 \le M^2 \Big\| \sum_{n=1}^{\infty} a_n g_n \Big\|_{\mathcal{H}}^2$$

$$f = \sum_{n=1}^{\infty} a_n e^{+2\pi i n t} g_n \qquad M^*_{\varphi_t} f = \sum_{n=1}^{\infty} a_n g_n$$

symmetrically. $\Big\| \sum_{n=1}^{\infty} a_n g_n \Big\|^2 \le M^2 \sum_{n=1}^{\infty} |a_n|^2$.

~~f(t) so~~

$(\Longleftarrow)$ $$A \sum_n |a_n|^2 \le \Big\| \sum_n a_n g_n \Big\|^2 \le B \sum_n |a_n|^2$$

$A, B > 0$. let $\{w_n\}$ such that $\sup |w_n| \le \frac{A}{B}$.

consider $w_1, \ldots, w_N$.
$\uparrow \qquad \uparrow$
$z_1 \qquad z_N$

$$\sum_{n,m=1}^{N} (1 - w_n \bar{w}_m)(g_n, g_m) = \sum_{n,m}^{N} (g_n, g_m) - \sum_{n,m=1}^{N} w_n \bar{w}_m (g_n, g_m)$$

$$= \Big\| \sum_1^N g_n \Big\|^2 - \Big\| \sum_1^N w_n g_n \Big\|^2$$

$$\ge A \cdot N - \sup|w_n| \cdot N \cdot B = N\big(A - \sup|w_n| \cdot B\big) \ge 0$$

$\Longrightarrow \exists\, \varphi_N \in \mathcal{M}_1(H)$ $\varphi_N(z_n) = w_n$.

$\phi_{N_j} \xrightarrow{w^*} \phi$.

Lemma. $T: H \to K$ bounded above and below
$\iff$ $R(T)$ is closed and $\Pi : K \to R(T)$ projection
ΠT is an isomorphism