We saw that an entire function can have arbitrary zeroes, and it can assume arbitrary values on an arbitrary sequence.

We would like to study similar problems for ~~[illegible]~~ holomorphic functions in hyperbolic space i.e. functions in $\mathbb{B}$.

§ Infinite Blaschke products.

Recall that a finite Blaschke product of degree n is a function of the form $e^{i\theta}\prod_{k=1}^{n}\frac{z-a_k}{1-\bar{a}_k z}$. We would like to study the convergence of finite Blaschke products as the degree n tends to infinity.

Theorem. Let $\{a_n\} \subseteq \mathbb{D}$. There are two possibilities

(i) $\sum_1 (1-|a_n|) < +\infty$. Then the infinite product $(*)$ converges absolutely and ~~ane~~ locally uniformly in $\mathbb{D}$. to a function $B(z) = z^r \prod_{n\geq 1} \frac{|a_n|}{a_n}\frac{z-a_n}{1-\bar{a}_n z}$. $(*)$ $r = \#\{a_n = 0\}$.

Furthermore ~~B~~ $z_0 \in \mathbb{D}$ is a zero of B of order m $\iff$ there exist exactly m of the a_n's equal to z_0.

(ii) $\sum_1 (1-|a_n|) = \infty$ $\qquad \prod_{n=1}^{N} \cancel{e^{i\theta}} \left|\frac{z-a_n}{1-\bar{a}_n z}\right| \longrightarrow 0$ uniformly on compacts in $\mathbb{D}$.

~~for every $\theta\in\mathbb{R}$.~~

p.2.

Recall the basic theorem for convergence of infinite products of holomorphic functions.

Thm Let $f_n \in Hol(\Omega)$ then $f = \prod (f_n + 1)$ converges uniformly on compact sets and absolutely in Ω if $\sum_1 f_n$ converges absolutely and locally uniformly. If a is a zero of f then $Z(f,a) = \sum_{n \geq 1}^{\infty} Z(f_n, a)$

Proof of (i) Without loss of generality $r = 0$.

$$\prod_{n=1}^{\infty} \underbrace{-\frac{|a_n|}{a_n} \frac{z - a_n}{1 - z\bar{a}_n} - 1}_{f_n} + 1 \qquad \text{let } |z| \leq \rho < 1.$$

$$|f_n(z)| = \left| \frac{-|a_n| z + |a_n| a_n - a_n + z|a_n|^2}{|a_n| |1 - z\bar{a}_n|} \right|$$

$$\leq \frac{\left| z|a_n|(|a_n| - 1) + a_n(|a_n| - 1) \right|}{|a_n| |1 - z\bar{a}_n|}$$

$$\leq \frac{|z + a_n| (1 - |a_n|)}{|a_n| |1 - z\bar{a}_n|} \leq \frac{2}{\min_{n \geq 1} |a_n| (1 - \rho)} \cdot (1 - |a_n|)$$

By Weierstrass M-test. $\sum_1 f_n$ converges absolutely and uniformly in $\mathbb{D}(0, \rho)$. for all $\rho < 1$. which concludes the first part.

For the second part.

p2.

$$1 - \left|\frac{a_n - z}{1-\bar{a}_n z}\right|^2 = \frac{(1-|a_n|^2)(1-|z|^2)}{|1-a_n z|^2} \geq \frac{(1-|a_n|^2)(1-\rho^2)}{4}$$

$$\prod_{n=1}^{N} \left|\frac{a_n - z}{1-\bar{a}_n z}\right|^2 = \exp\left\{\sum_{n=1}^{N} \frac{1}{2} \ln \left|\frac{a_n - z}{1-\bar{a}_n z}\right|^2\right\}$$

$$= \exp\left\{\frac{1}{2}\sum_{n=1}^{N} \ln\left(1 - \frac{(1-|a_n|^2)(1-|z|^2)}{|1-z\bar{a}_n|^2}\right)\right\}$$

$$\leq \exp\left\{-\frac{1}{2}\sum_{n=1}^{N} \frac{(1-|a_n|^2)(1-|z|^2)}{|1-\bar{a}_n z|^2}\right\} \leq \exp\left\{-\frac{1-\rho^2}{4}\sum_{n=1}^{N}(1-|a_n|^2)\right\}$$

$\ln(1-x) \leq -x$. ■

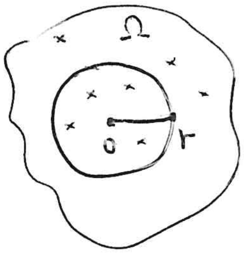
Clearly when $\sum (1-|a_n|^2) < \infty$ the associated Blaschke product B is a holomorphic self map of $\mathbb{D}$. (Because $|B(z)| < 1$).

Or in other words if $\sum (1-|a_n|) < \infty$ $\exists f \in \mathbb{B}$ s.t. $f(a_n) = 0$.

Is this condition necessary?

To answer this question we shall need Jensen's formula.

<u>Proposition.</u> Suppose that f is holomorphic in $\overline{D(0,r)}$. $f(0) \neq 0$.



$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta - \sum_{k=1}^{n} \log\frac{r}{|a_k|}$$

f does not vanish on $|z| = r$ and $a_1, a_2, -, a_n$ are the zeros of f in $|z| < r$ canted with multiplicity.

proof. Case I. f does not have any zeros in $\overline{D(0,r)}$.

Then, there exists a $g \in Hol\left(\overline{D(0,r)}\right)$ such that

$$e^g = f. \implies e^u = e^{\mathrm{Re}\, g} = |e^g| = |f|.$$

$g = u + iv$

$$\implies u = \log|f|$$

$$g(0) = \frac{1}{2\pi i} \int_{|z|=r} \frac{g(t)}{t}\, dt = \frac{1}{2\pi i} \int_0^{2\pi} \frac{g(re^{i\theta})\, ire^{i\theta}\, d\theta}{re^{i\theta}}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta})\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})\, d\theta + i\, \frac{1}{2\pi} \int_0^{2\pi} v(re^{i\theta})\, d\theta$$

$$\implies \log|f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta.$$

Case II let $a_1, a_2, \ldots, a_n$ the zeros of f in $|z| < r$. (According to multiplicity).

Consider the Blaschke product $B(z) = \prod_{k=1}^{n} \frac{rz - a_k}{r - z\bar{a}_k}$. Then $\overline{D(0,r)}$ and f/β is holomorphic in $D(0,r)$ and has no zeros in $\overline{D(0,r)}$. Therefore, by case I.

$$\frac{1}{2\pi} \int_0^{2\pi} \log\left|\frac{f(re^{i\theta})}{\beta(re^{i\theta})}\right| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \ln|f(re^{i\theta})|\, d\theta = \ln\left[|f(0)| \big/ |\beta(0)|\right]$$

$$= \ln|f(0)| - \ln \prod \frac{|a_k|}{r}$$

$$\implies \ln|f(0)| + \sum_{k=1}^{n} \ln\left(\frac{r}{|a_k|}\right) = \frac{1}{2\pi} \int_0^{2\pi} \ln|f(re^{i\theta})|\, d\theta.$$

p.5 <u>Exercise</u>.

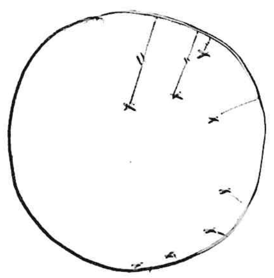
Show that ~~the~~ the hypothesis "f does not vanish on $|z|=r$" is superfluous.

<u>Proposition</u>. Suppose that $f \in \mathrm{Hol}(\mathbb{D})$ and

$$\underbrace{\sup_{0<r<1} \int_0^{2\pi} \ln^+|f(re^{i\theta})| \frac{d\theta}{2\pi}}_{=\|f\|_N} < +\infty.$$

Then, if $\alpha_1, \alpha_2, \dots$ are the zeros of f c.w.m (counted with multiplicity)

$$\Longrightarrow \sum_{n=1}^{\infty} (1-|a_n|) < \infty$$



<u>Proof</u>. Without loss of generality $f(0) \neq 0$.

Then,

$$\cancel{\ln|f(0)|} + \cancel{\sum_{K=1}}$$

$$\ln|f(0)| + \sum_{K=1}^{\infty} \ln^+(r/|a_K|) = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|\,d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \ln^+|f(re^{i\theta})|\,d\theta \leq \|f\|_N < \infty$$

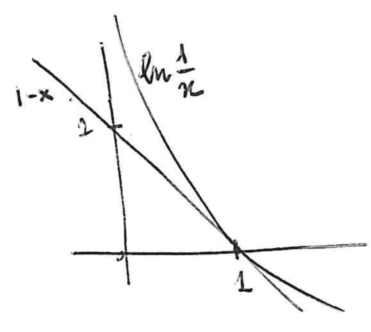
$$\Longrightarrow \ln|f(0)| + \sum_{K=1}^{\infty} \ln^+(1/|a_K|) = \ln|f(0)| + \lim_{r \nearrow 1^-} \sum_{K=1} \ln^+(r/|a_K|)$$

"monotone convergence theorem."

$$\leq \|f\|_N < \infty.$$

<u>Hence</u> $\displaystyle\sum_{K=1}^{\infty} \ln(1/|a_K|) < \infty$

$$1 - x \leq \ln\frac{1}{x}$$



~~BLO~~ p6.

Corollary. Let $\{a_n\} \subseteq \mathbb{D}$. There exists $f \in B$ such that $f(a_n) = 0$ $\forall n$. iff $\sum_n (1-|a_n|) < \infty$.

This latter condition is called Blaschke condition

Notice that in Pick's interpolation problem we were given a single set of data $w_1, \ldots, w_n$ and points, and we asked if there exists $f \in B$ such that $f(z_i) = w_i \quad i = 1, \ldots, n$.

<u>Exercise</u>. If $z_1, \ldots \quad w_1, \ldots \in \mathbb{D} \quad \exists f : \mathbb{D} \to \mathbb{D}$ s.t. $f(z_i) = w_i$ if-f every finite interpolation problem has a solution. (Hint: B is a normal family).

Carleson - Newmann problem: what if I fix a sequence $z_1, \ldots, z_n, \ldots \in \mathbb{D}$. and I ask to interpolate <u>every</u> $\{w_n\}_{n=1}^{\infty}$ ~~of such that $|w_n| < 1$~~ of "sufficiently small norm".

we need the open mapping theorem for $\Leftarrow$.

* in this equivalence

$\iff \forall \{w_n\} \in \ell^\infty(\mathbb{N}) \quad \exists f \in H^\infty(\mathbb{D}) = \{ f : \mathbb{D} \to \mathbb{C} \ ; \ \sup_{|z|<1} |f(z)| = \|f\|_\infty < \infty \}$ such that $f(z_i) = w_i. \quad i = 1, 2, \ldots,$

<u>Theorem</u> [Carleson '58] $\{z_n\}_{n=1}^{\infty}$ is $H^\infty(\mathbb{D})$-interpolating if-f.

$$\boxed{\inf_k \prod_{n: n \neq k} \left| \frac{z_n - z_k}{1 - z_n \bar{z}_k} \right| > \delta > 0.}$$

"Carleson condition"

Preparation for the proof: for $k=1$ $\displaystyle\prod_{n \geq 2} \left| \frac{z_n - z_1}{1 - z_n \bar{z}_1} \right|^2$ converges to a non zero number. $\Rightarrow \displaystyle\sum_{n \geq 2} \left[1 - \left| \frac{z_n - z_1}{1 - z_1 \bar{z}_n} \right|^2 \right] < \infty$

$$\left(\frac{(1-|z_1|^2)}{4} \right) \cdot \sum_{n \geq 2} (1 - |z_n|) \leq \sum_{n \geq 2} \frac{(1-|z_1|^2)(1-|z_n|^2)}{|1 - z_1 \bar{z}_n|^2} < \infty \Rightarrow \{z_n\}_{n=1}^{\infty} \text{ is}$$

a Blaschke sequence. Let $B(z) = \displaystyle\prod_{n=1}^{\infty} \left(-\frac{|z_n|}{z_n} \right) \cdot \frac{z - z_n}{1 - \bar{z}_n z}$

and $B_n(z) = \displaystyle\prod_{k \neq n} \left(-\frac{|z_k|}{z_k} \right) \cdot \frac{z - z_k}{1 - z \bar{z}_k}$

Then Carleson's condition translates to

$$\inf_{n} |B_n(z_n)| > \delta$$

Suppose we dream for a second... define $\tilde{B}_n(z) = \dfrac{B_n(z)}{B_n(z_n)}$

and given $w_1, \ldots, w_n, \ldots \in \ell^\infty(\mathbb{N})$ define for $N \in \mathbb{N}$ $F_N(z) = \sum_{n=1}^{N} w_n \dfrac{B_n(z)}{B_n(z_n)}$. Then

for $k = 1, 2, \ldots, N$ $\quad F_N(k) = \sum_{n=1}^{N} w_n \dfrac{B_n(z_k)}{B_n(z_n)}$

$$= \sum w_n \frac{\delta_{nk} \cancel{B_n(z_n)}}{\cancel{B_n(z_n)}} = w_k.$$

but the estimate $|F_N(z)| \leq \sum_{n=1}^{N} |w_n| \dfrac{|B_n(z)|}{|B_n(z_n)|}$

$$\leq \|\{w_n\}\|_{\ell^\infty} \frac{1}{\delta} \sum_{n=1}^{N} |B_n(z)| \leq \frac{\|\{w_n\}\|_{\ell^\infty} \cdot N}{\delta} \xrightarrow[N\to\infty]{} \infty,$$

is not enough to show that $F_N \to F$ in $H^\infty(\mathbb{D})$

So the difficulty is to modify B_n so that

$$\sup_{|z|<1} \sum_n |\Phi_n(z)| < C(\delta)$$

[1981 S.A. Vinogradov, E.A. Gorin, S.V. Khrushchev].

Proof. Let $\varepsilon = \left(2 \log \dfrac{e}{\delta^2}\right)^{-1}$, $\quad \alpha_n(z) = \sum_{k \geq n} \dfrac{1+\bar{z}_k z}{1-\bar{z}_k z}(1-|z_k|^2)$

$\boxed{|z_n| \text{ increasing.}}$

[where we have ordered z_k so that $|z_k|$ is increasing.]

Let $\Phi_n(z) = \left(\dfrac{1-|z_n|^2}{1-\bar{z}_n z}\right)^2 \dfrac{B_n(z)}{B_n(z_n)} \cdot \exp\left\{\varepsilon\left(\alpha_n(z_n) - \alpha_n(z)\right)\right\}$.

Theorem $S(z) = \sum_{n=1}^{\infty} |\Phi_n(z)| \leq \dfrac{2e}{\delta} \log \dfrac{e}{\delta^2}$

If $k \geq n$, $|z_k| \geq |z_n|$, and hence

$$\operatorname{Re} \alpha_n(z_n) = \sum_{k \geq n} \frac{(1-|z_k|^2|z_n|^2)(1-|z_k|^2)}{|1-z_k\bar{z}_n|^2} \quad (*)$$

$$\operatorname{Re}\left\{\frac{1+\bar{z}_k z_n}{1-\bar{z}_k z_n}\right\} = \frac{1}{2}\left\{\frac{1+\bar{z}_k z_n}{1-\bar{z}_k z_n} + \frac{1+z_k\bar{z}_n}{1-z_k\bar{z}_n}\right\}$$

$$= \frac{1}{2}\left\{\frac{(1+\bar{z}_k z_n)(1-z_k\bar{z}_n) + (1+z_k\bar{z}_n)\cdot(1-\bar{z}_k z_n)}{|1-\bar{z}_k z_n|^2}\right\}$$

$$= \frac{1}{2}\,\frac{1 - \cancel{z_k\bar{z}_n} + \cancel{\bar{z}_k z_n} - |z_k z_n|^2 + 1 - \cancel{\bar{z}_k z_n} + \cancel{z_k\bar{z}_n} - |z_k z_n|^2}{|1-\bar{z}_k z_n|^2}$$

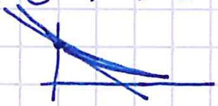
$$= \frac{1-|z_k|^2|z_n|^2}{|1-\bar{z}_k z_n|^2}$$

$$(*) \leq \sum_{k \geq n} \frac{(1-|z_n|^4)(1-|z_k|^2)}{|1-z_k\bar{z}_n|^2} \leq 2\sum_{k=n}^{\infty} \frac{(1-|z_n|^2)(1-|z_k|^2)}{|1-z_n\bar{z}_k|^2}$$

$$\leq 2\left[\sum_{k \neq n}^{\infty} \underbrace{1 - \left|\frac{z_n-z_k}{1-z_n\bar{z}_k}\right|^2}\right] + 2 \leq 2\sum_{k=1}^{\infty} -\ln\left(\left|\frac{z_n-z_k}{1-z_n\bar{z}_k}\right|^2\right) + 2$$

$t \leq -\log(1-t) \iff e^{-t} \geq 1-t$

$t \in (0,1)$



$$= 2\ln\left(\left[\prod_{k \neq n} \left|\frac{z_n-z_k}{1-z_n\bar{z}_k}\right|^2\right]^{-1}\right) + 2 \leq 2\ln\frac{1}{\delta^2} + 2$$

$$= 2\ln\left(\frac{e}{\delta^2}\right) =: \varepsilon.$$

If $|z|<1$, let $\gamma_n = \sum_{k\geq n} \frac{(1-|z_k|^2|z|^2)(1-|z_k|^2)}{|1-\bar{z}_k\cdot z|^2}$.

$$S(z) = \sum_{n\geq 1} \underbrace{\left(\frac{1-|z_n|^2}{|1-\bar{z}_n z|}\right)^2}_{\substack{\text{III} \\ \gamma_{n+1}-\gamma_n}} \frac{\overset{\leq 1}{\boxed{|B_n(z)|}}}{\underset{\geq \delta}{\boxed{|B_n(z_n)|}}} \cdot \exp\left\{\varepsilon\left(\underbrace{\mathrm{Re}\,\alpha_n(z_n)}_{\leq 1/\varepsilon} - \mathrm{Re}\,\alpha_n(z)\right)\right\}$$

[because $(1-|z_n|^2) \leq 1-|z|^2|z_n|^2$.]

$$\leq \frac{e}{\delta\cdot\varepsilon} \sum_{n\geq 1} \varepsilon\cdot(\gamma_n-\gamma_{n+1})\cdot e^{-\varepsilon\gamma_n}$$

$$\leq \frac{e}{\varepsilon\cdot\delta}\sum_{n\geq 1}\left[e^{\varepsilon(\gamma_n-\gamma_{n+1})}-1\right]\cdot e^{-\gamma_n\cdot\varepsilon}$$

$\left(\begin{array}{l} t \leq e^t - 1 \text{ for } t\in\mathbb{R} \\ \text{where } t = \varepsilon(\gamma_n-\gamma_{n+1}) \end{array}\right)$.

$$= \frac{e}{\delta\cdot\varepsilon}\sum_{n\geq 1}\left[e^{-\varepsilon\gamma_{n+1}} - e^{-\gamma_n\cdot\varepsilon}\right] = \frac{e}{\delta\cdot\varepsilon}\left\{\left(1-\overset{\leq 1}{e^{-\gamma_1\cdot\varepsilon}}\right)\right\} \blacksquare$$

↑ this converges absolutely and it is a telescopic sum.

$\boxed{\gamma_n \to 0}$ for fixed z

<u>Necessity</u> of Carleson's condition

~~Consider the points $\{z_k\}$ ... for some fixed $N\in\mathbb{N}$~~

~~Consider the restriction operator $R: H^\infty(\mathbb{D}) \to \ell^\infty(\mathbb{N})$~~
~~By hypothesis this is surjective. It is always linear and bounded.~~

~~Fix some $N \in \mathbb{N}$.~~

Let $R: H^\infty(\mathbb{D}) \longrightarrow \ell^\infty(\mathbb{N})$ the restriction operator. By hypothesis is surjective hence by the open mapping thm. let $e_n = \{\delta_{nk}\}_{k=1}^\infty$ and

$\exists\ g_n \in H^\infty(\mathbb{D})$, $\quad \|g\|_\infty \leq M \cdot \|e_n\|_{\ell^\infty} = M$

$$R g_n = e_n \implies \boxed{\begin{array}{l} g_n(z_k) = \delta_{nk} \\ \|g_n\|_\infty \leq M \end{array}}$$

Fix some $N \in \mathbb{N}$. let $B_n(z) = \prod_{k \neq n}^{N} \frac{z - z_k}{1 - \bar{z}_k z}$

g_n / B_n is in H^∞ and $\sup_{|z|<1} \left| \frac{g_n(z)}{B_n(z)} \right| = \limsup_{|z| \to 1^-} \left| \frac{g_n(z)}{B_n(z)} \right|$

$= \|g\|_\infty \implies \| g_n / B_n \| = \| g_n \|$.

Then, $\ {}_{1 \leq k \leq N}\ \ |B_n(z)| = \dfrac{|B_n(z_n)|}{|g_n(z_n)|}$

In particular for every $N \in \mathbb{N}$, $1 \leq n \leq N$

$$\sup \left\{ |f(z_n)| \ ; \|f\|_{H^\infty} \leq 1 \ ; \ f(z_k) = 0, \ {}_{\substack{k \neq n \\ 1 \leq k \leq N}} \right\} := C(N, n) > \delta > 0.$$

uniformly in N, k.

Suppose that $g \in H_1^\infty(\mathbb{D})$ such that $g(z_k) = 0$ $\ k \neq n, 1 \leq k \leq N$.

then; $\quad \| g / B_n \|_\infty \xlongequal{\text{MAXIMUM PRINCIPLE}} \|g\|_\infty \leq 1$.

$$\implies |g(z)| \leq |B_n(z)| \implies \delta < |g(z_n)| \leq |B_n(z_n)|$$

$$\implies \inf_{n \in \mathbb{N}} |B_n(z_n)| > \delta > 0.$$

Linear interpolation operator.

let Φ_n as constructed in Carleson's theorem

the operator $T : \ell^\infty(\mathbb{N}) \to H^\infty(\mathbb{D})$.

$$T(\{w_n\}) \doteq \sum \phi_n \cdot w_n \quad \text{is}$$

linear, bounded $\|T\|_{\ell^\infty \to H^\infty} \leq \frac{2e}{\delta} \log \frac{e}{\delta^2}$.

and $\boxed{RT = id_{\ell^\infty(\mathbb{N})}}$

~~Proposition~~

Def. let X a Banach space a closed subspace $Y \leq X$ is called complemented in X if $\exists \underset{\text{bounded}}{P : X \to X}$ such that $P^2 = P$ and $\operatorname{Im} P = Y$

$\Longleftrightarrow$ $X = Y + Z$, $Y \cap Z = (0)$ and $P : X \to X$ $x \mapsto y$ or z is continuous

- If $\mathcal{H}$ is a Hilbert space every closed subspace in $\mathcal{H}$ is complemented. $M \oplus M^\perp = \mathcal{H}$.

- c_0 is not complemented in $\ell^\infty(\mathbb{N})$.

Proposition. Let $Z = \{z_n\}_{n \geq 1}$ an $H^\infty(\mathbb{D})$-interpolating sequence. Then $M_Z = \{ f \in H^\infty(\mathbb{D}) \,;\, f(z_i) = 0 \}$ is complemented in $H^\infty(\mathbb{D})$ and $H^\infty(\mathbb{D}) / M_Z \simeq \ell^\infty$ as "Banach algebras"

let $X = X_1 \oplus_B X_2$ then by definition $P_1 : X \to X$ proj on X_1 is bounded and $P^2 = P$, $\operatorname{Im} P = X_1$. $\operatorname{Ker} P_1 = X_2$.

If P is such ~~that~~ $X_1 = \operatorname{Im} P$ $X_2 = \operatorname{Ker} P$. and $x \in X$, $\underbrace{x - Px}_{\operatorname{Ker}P} + \underbrace{Px}_{\operatorname{Im}P}$

$x \in \operatorname{Ker} P \cap \operatorname{Im} P$ $\quad \underbrace{x = Px}_{P^2 = P} \underbrace{= 0}_{x \in \operatorname{Ker} P}$.

proof. Let $P : H^\infty(\mathbb{D}) \longrightarrow H^\infty(\mathbb{D})$

$$Pf(z) = f(z) - \sum_{n\geq 1} \phi_n f(z_n)$$

$Pf \in M_Z$ and if $f \in M_Z$ $Pf = f$ because $f(z_n) = 0$.

$\Longrightarrow P^2 = P$ and $\operatorname{Im} P = M_Z$.

$R : H^\infty(\mathbb{D}) \to \ell^\infty$ $\operatorname{Ker} R = M_Z$ $\Rightarrow$

$H^\infty(\mathbb{D}) / M_Z \simeq \ell^\infty$

So for every $Z = \{z_n\}_{n\geq 1}$ $H^\infty(\mathbb{D}) = M_Z \oplus X_Z$

$X_Z \simeq \ell^\infty$.

Def. Let X a Banach space. $\Omega \subseteq \mathbb{C}$ open
$f : \Omega \to X$ is analytic if $(f, x^*) : \Omega \to \mathbb{C}$.
$z \longmapsto (f(z), x^*)$ is analytic for every $x^* \in X^*$.

Corollary. If $\{z_n\}$ is an $H^\infty(\mathbb{D})$ - interpolating sequence then for every $\{x_n\} \in X$ $\sup_n \|x_n\| < \infty$
$\exists$ $f : \mathbb{D} \to X$ bounded analytic such that
$f(z_n) = x_n$.

Consider, $f(z) = \sum_1 \phi_n(z) x_n$

$$\sum |\phi_n(z)| \, \|x_n\|_X \leq \sup \|x_n\| \sum_1 |\phi_n(z)| \leq \sup \|x_n\| \frac{2}{\delta} \log \frac{e}{\delta^2}.$$

$\Longrightarrow$ (X is Banach) $\sum \phi_n(z) x_n$ converges in <u>norm</u> for every $z \in \mathbb{D}$. if $x^* \in X$.

$\left(\sum_1^\infty \phi_n(z) x_n , x^* \right) = \sum \phi_n(z) (x_n, x^*)$ but this a

series of holomorphic functions converging a.o.c.

hence holomorphic. ∎