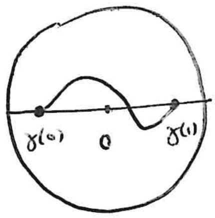
§1 Hyperbolic plane - Poincaré model.

The hyperbolic plane is the unit disc $\mathbb{D} = \{ |z| < 1 \}$ equipped with the Riemannian metric $ds_h = \frac{2|dz|}{1-|z|^2}$.

$$\ell(\gamma) = \int_\gamma \frac{2|dz|}{1-|z|^2} = \int_0^1 \frac{2|\dot\gamma(t)|\,dt}{1-|\gamma(t)|^2} \geqslant \int_0^1 \frac{2|\dot\Gamma(t)|\,dt}{1-|\Gamma(t)|^2} \qquad \Gamma(t) = \operatorname{Re}\gamma(t)$$



Hence diameters are geodesics.

Note; $ds^2 = \frac{4dx^2}{1-x^2-y^2} + \frac{4\,dy^2}{1-x^2-y^2}$ Hence in Gauss first fundamental form notation $E = G = \frac{4}{1-x^2-y^2}$ $F \equiv 0$.

$$K = \frac{-1}{2\sqrt{EG}}\left(\frac{\partial}{\partial x}\frac{G_x}{\sqrt{EG}} + \frac{\partial}{\partial y}\frac{E_y}{\sqrt{EG}}\right)$$ is the Gauss curvature for orthogonal coordinates, i.e $F \equiv 0$

Let $\psi(z,w) = \inf_\gamma \{ \ell(\gamma) ; \gamma(0) = z, \gamma(1) = w \}$

$$\psi(0,z) = \psi(0,|z|) = \int_0^{|z|} \frac{2\,dt}{1-t^2} = \log \frac{1+|z|}{1-|z|}.$$

The easiest way to proceed is with hindsight;

let $\operatorname{Aut}(\mathbb{D}) = \left\{ e^{i\theta} \frac{z-a}{1-\bar a z} ; \theta \in \mathbb{R}, a \in \mathbb{D} \right\}$, let $\tau_a(z) = \frac{z-a}{1-\bar a z}$

$$1 - \tau_a(z)\overline{\tau_a(w)} = 1 - \frac{a-z}{1-\bar a z}\,\frac{\bar w - \bar a}{1-a\bar w} = \frac{1-\bar a z - a\bar w + |a|^2 z\bar w - [a\bar w - |a|^2 - z\bar w + \bar a z]}{(1-\bar a z)(1-a\bar w)}$$

$$= \frac{(1-z\bar w)(1-|a|^2)}{(1-\bar a z)(1-a\bar w)}.$$

In particular,

$$1-|\tau_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar a z|^2}.$$

Hence $\tau_a(\mathbb{D}) \subseteq \mathbb{D}$ $\quad \tau_a(\partial\mathbb{D}) \subseteq \partial\mathbb{D}$

Furthermore, $\boxed{\tau_a^{-1} = \tau_{-a}}$, Hence, $\tau_a(\mathbb{D}) = \mathbb{D}$, $\tau_a(\partial\mathbb{D}) = \partial\mathbb{D}$.

$$\tau_a'(z) = \frac{1-\bar a z - (a-z)(-\bar a)}{(1-\bar a z)^2} = \frac{1-|a|^2}{(1-\bar a z)^2}$$

$$\Rightarrow |\tau_a'(z)| = \frac{(1-|a|^2)}{|1-\bar a z|^2} = \frac{1-|\tau_a(z)|^2}{1-|z|^2} \Rightarrow$$

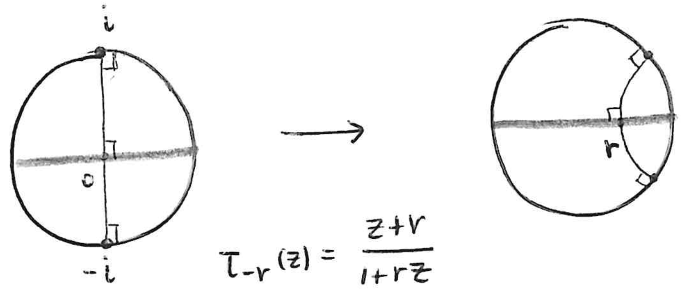
$$\frac{|\tau_a'(z)|\,|dz|}{1-|\tau_a(z)|^2} = \frac{1}{1-|z|^2}\cdot|dz|$$

, i.e. τ_a are isometries

In particular, $\psi(z,w) = \psi(\tau_z(z), \tau_z(w)) = \psi(0, \tau_z(w))$

$$= \log\frac{1+|\tau_z(w)|}{1-|\tau_z(w)|}.$$

The quantity $\rho(z,w) \doteq |\tau_z(w)| = \left|\frac{z-w}{1-z\bar w}\right|$ is also a metric called pseudo hyperbolic, but not an arc length one.



$$T_{-r}(z) = \frac{z+r}{1+rz}$$

by using an automorphism of the form $e^{i\theta} T_r$, $\theta \in \mathbb{R}$, $-1<r<1$, I can send the diameter $\{it\,;\ -1 \leq t \leq 1\}$ to any arc segment perpendicular to the boundary.

$\boxed{\text{So these are all the geodesics in this model}}$

Note; $\mathrm{Isom}(\mathbb{D}, d_h s)$ is a Lie group with two connected components. The one containing the identity is $\mathrm{Aut}(\mathbb{D})$, the other one is

$$\left\{ e^{i\theta} \frac{\bar{z}-a}{1-\bar{z}a}\ ;\ a \in \mathbb{D},\ \theta \in \mathbb{R} \right\}.$$

($\mathrm{Aut}\,\mathbb{D}$ is the orientation preserving.)

§2 Schwarz-Pick lemma

Let $B = \{ f : \mathbb{D} \to \mathbb{D}\ ,\ f \text{ holomorphic} \}$.

Lemma 0 Suppose that $f(0)=0$ then,

$$\left.\begin{aligned} &|f(z)| \leq |z| \ , \quad z \neq 0 \\ &|f'(0)| \leq 1 \end{aligned}\right\} \quad (1)$$

If equality holds in (1) for some point z_0 then

$f(z) = e^{i\theta} z$, $|z|<1$.

proof. Consider $g(z) = \frac{f(z)}{z}$, $\limsup_{|z|\to 1} |g(z)| \le 1$

$\xRightarrow[\text{max principle}]{}$ $|g(z)| \le 1$, $|z|<1$ $\Rightarrow$ $|f(z)| \le |z|$

$g \in \mathcal{O}(\mathbb{D})$ $\qquad |g(0)| = |f'(0)| \le 1.$

furthermore, $|g(z_0)| = 1 \xRightarrow[\text{principle}]{\text{Strong maximum}} g(z_0) = \text{const} = e^{i\theta}$ ∎

Schwarz-Pick lemma.

Let $f \in B$ then for $z, z_0 \in \mathbb{D}$,

(i) $$\left| \frac{f(z) - f(z_0)}{1 - f(z)\overline{f(z_0)}} \right| \le \left| \frac{z - z_0}{1 - z\bar{z}_0} \right|.$$

(ii) $$\frac{|f'(z_0)|}{1 - |f(z_0)|^2} \le \frac{1}{1 - |z_0|^2}.$$

If equality holds for some $z_0 \ne z$ in (i) or for some z_0 in (ii)

$\Rightarrow$ $f \in Aut(\mathbb{D})$.

Proof.
$f(0) = a$

$$\begin{tikzcd} 0 \arrow[r, "\tau_{-z_0}"] \arrow[rrr, bend right, "\tau_{f(z_0)} \circ f \circ \tau_{-z_0}"'] & z_0 \arrow[r, "f"] & f(z_0) \arrow[r, "\tau_{f(z_0)}"] & 0 \end{tikzcd}$$

$|\tau_{f(z_0)} \circ f \circ \tau_{-z_0}(w)| \le w \quad (*)$

$w \in \mathbb{D}$, $\quad w = \tau_{z_0}(z)$

Lemma 0

$\Rightarrow |\tau_{f(z_0)}(f(z))| \le |\tau_{z_0}(z)|$

(ii) is either by (ii) in Lemma 0, or by deviding by $|z - z_0|$ in (1) and taking the limit.

Equality at some point in (i) or (ii) gives equality in $(*)$

$\Rightarrow \tau_{f(z_0)} \circ f \circ \tau_{-z_0}(w) = w e^{i\theta}$

$\Rightarrow \tau_{f(z_0)} \circ f(z) = \tau_{z_0}(z) e^{i\theta} \Rightarrow f(z) = \tau_{f(z_0)}\left(\tau_{z_0}(z) e^{i\theta}\right) \in Aut(\mathbb{D})$

§ Finite Blaschke products.

In Euclidean space $\mathbb{C}$, a polynomial is a product of orientation preserving isometries of $\mathbb{C}$ and a dilation factor. $p(z) = \delta \cdot e^{i\theta} \prod_{j=1}^{n}(z-z_j)$.
$\delta > 0$, $\theta \in \mathbb{R}$

In $\mathbb{D}$ the analogue is finite Blaschke products i.e. products of elements in $\mathrm{Aut}(\mathbb{D})$.

Def. A Blaschke product of degree n is a function

$$\beta(z) = e^{i\theta} \prod_{i=1}^{n} \frac{z-a_i}{1-z\bar{a}_i} \quad , \ a_i \in \mathbb{D}.$$

It has the properties

(i) β is continuous in $\overline{\mathbb{D}}$

(ii) $|\beta(z)| = 1$ if $|z| = 1$

(iii) β has n zeros in $\mathbb{D}$. (counting multiplicities).

If f satisfies (i)–(iii). the f is a Blaschke product of degree n because if $a_1, -, a_n$ are the zeros of f. Let $\beta(z) = \prod \frac{z-a_i}{1-\bar{a}_i z}$ f/β and β/f are of modulus 1 on the boundary and holomorphic inside, hence $f/\beta = e^{i\theta}$,

For this reason, pre or post composing by $\varphi \in \mathrm{Aut}(\mathbb{D})$ preserves Blaschke products of degree n.

§ Finite interpolation is hyperbolic space.

[Pick '1916]

Theorem. There exists $f \in B$ satisfying $f(z_j) = w_j$ if and only if the quadratic form

$$Q_n(t_1, \ldots, t_n) = \sum_{j,k=1}^{n} \frac{1 - w_j \overline{w_k}}{1 - z_j \overline{z_k}} t_j \overline{t_k}.$$

is nonnegative, $Q_n \geq 0$. When $Q_n \geq 0$ there exists a Blaschke product of degree $\leq n$ which solves the interpolation problem.

proof. For $n=1$ the theorem holds true because $\tau_{w_1}^{-1} \circ \tau_{z_1}$ is a Blaschke product of degree 1 interpolating at 1 point.

Suppose that it holds for $n-1$.

CLAIM 1. We can assume that $z_n = w_n = 0$. Otherwise,

let $z_j' = \tau_{z_n}(z_j)$, $w_j' = \tau_{w_n}(w_j)$ $1 \leq j \leq n$.

then $f \in B$ solves the n-point interpolation problem

iff $g = \tau_{w_n} \circ f \circ \tau_{z_n}^{-1}$ satisfies $g(z_j') = w_j'$, $1 \leq j \leq n$.

and $g \in \mathcal{B}_n \Leftrightarrow f \in \mathcal{B}_n$

Furthermore if Q_n' is the quadratic form associated to $\{z_1', z_2', \ldots, z_{n-1}', 0\}$ $\{w_1', \ldots, w_{n-1}', 0\}$

$$\frac{1 - w_j' \overline{w_k'}}{1 - z_j' \overline{z_k'}} = \frac{1 - \tau_{w_n}(w_j) \overline{\tau_{w_n}(w_k)}}{1 - \tau_{z_n}(z_j) \overline{\tau_{z_n}(z_k)}}$$

$$= \left[\frac{(1-|w_n|^2)\boxed{(1-w_j\bar{w}_k)}}{(1-\bar{w}_n w_j)(1-w_n\bar{w}_k)}\right]\left[\frac{(1-\bar{z}_n z_j)(1-z_n\bar{z}_k)}{(1-|z_n|^2)\boxed{(1-z_j\bar{z}_k)}}\right]$$

$$\cancel{= \left[\frac{(1-|w_n|^2)(1-\bar{z}_n z_j)}{(1-\bar{w}_n w_j)}\right]\left[\frac{1-w_j\bar{w}_k}{1-z_j\bar{z}_k}\right]\left[\frac{1-z_n\bar{z}_k}{(1-|z_n|^2)(1-w_n\bar{w}_k)}\right]}$$

$$= \underbrace{\left[\frac{1-|w_n|^2}{1-|z_n|^2}\right]}_{c_n} \cdot \left[\frac{1-w_j\bar{w}_k}{1-z_j\bar{z}_k}\right] \cdot \left[\underbrace{\left(\frac{1-\bar{z}_n z_j}{1-\bar{w}_n w_j}\right)}_{\alpha_j} \cdot \underbrace{\left(\frac{1-z_n\bar{z}_k}{1-w_n\bar{w}_k}\right)}_{\bar{\alpha}_k}\right]$$

Hence $\quad Q'_n(t_1,\dots,t_n) = \sum c_n \cdot \dfrac{1-w_j\bar{w}_k}{1-z_j\bar{z}_k}(t_j\alpha_j)\overline{(t_k\alpha_k)}$

$$= c_n Q_n(\alpha_1 t_1,\dots,\alpha_n t_n). \quad \text{Hence}$$

$Q_n \geq 0 \iff Q_n \geq 0.$

Hence we assume that $z_n = w_n = 0$, There exists $f \in B_n$ $f(0)=0$, $f(z_j)=w_j$, $j=1,2,\dots,n-1$ $\iff$ $\exists\, g \in B_{n-1};\ g(z_j) = w_j/z_j$ $j=1,2,\dots,n-1$.

by induction this is true $\iff$

$$\tilde{Q}_{n-1}(s_1,\dots,s_{n-1}) = \sum_{j,k=1}^{n-1} \frac{1-w_j/z_j\cdot(\bar{w}_k/\bar{z}_k)}{1-z_j\bar{z}_k} s_j\bar{s}_k \geq 0.$$

We need to show therefore that $\tilde{Q}_{n-1} \geq 0 \iff Q_n \geq 0$
if $z_n = w_n = 0$.

$$Q_n(t_1,\dots,t_n) = |t_n|^2 + 2\,\mathrm{Re}\sum_{j=1}^{n-1} t_j\,\overline{t_n} + \sum_{j,k=1}^{n-1} \frac{1-w_j\overline{w_k}}{1-z_j\overline{z_k}}\, t_j\,\overline{t_k}$$

$$= \Big|t_n + \sum_{j=1}^{n} t_j\Big|^2 + \sum_{j,k=1}^{n-1} \Big(\underbrace{\frac{1-w_j\overline{w_k}}{1-z_j\overline{z_k}} - 1}_{= \frac{1-\frac{w_j}{z_j}\cdot\frac{\overline{w_k}}{\overline{z_k}}}{1-z_j\overline{z_k}}\, z_j z_k} \Big)\, t_j\,\overline{t_k}\,.$$

$$= \Big|\sum_{j=1}^{n} t_j\Big|^2 + \widetilde{Q}_{n-1}(z_1t_1,\dots,z_{n-1}t_{n-1})$$

$\widetilde{Q}_{n-1} \geq 0 \Rightarrow Q_n \geq 0$

Setting $t_n = -\sum_{j=1}^{n-1} t_j$, $Q_n \geq 0 \Rightarrow \widetilde{Q}_{n-1} \geq 0$. $\blacksquare$