Some remarks on interpolation for entire functions.

- $Hol(\mathbb{C})$ is a Fréchet space w.r.t. the seminorms $\|f\|_n = \max_{|z|\le n} |f(z)|$:

  $f_j \xrightarrow[u.c.]{} f \iff \|f_j - f\|_n \to 0 \quad \forall n.$

  We can define a metric

  $$d(f,g) = \sum_n \frac{1}{2^n} \frac{\|f-g\|_n}{1+\|f-g\|_n}$$

  with respect to which $Hol(\mathbb{C})$ is complete by Weierstrass Thm.

- $\mathbb{C}^{\mathbb{N}}$ is similarly Fréchet w.r.t.

  $\|\{a_\ell\}_{\ell=1}^\infty\|_\nu = \|a\|_\nu = \max_{1\le \ell\le \nu} |a_\ell|$:

  $a_j \xrightarrow[\text{pointwise}]{} a \iff \|a_j - a\|_\nu \to 0 \quad \forall \nu$

- The restriction map $\left(Z = \{z_n\} \subseteq \mathbb{C},\ |z_n| \to \infty\right)$

  $$Hol(\mathbb{C}) \xrightarrow{R_Z} \mathbb{C}^{\mathbb{N}}$$
  $$f \longmapsto \{f(z_n)\}_{n=1}^\infty = f|_Z$$

  is continuous:

  $$f_j \xrightarrow[u.c.]{} f \Rightarrow f_j|_Z \xrightarrow[\text{pointwise}]{} f|_Z$$

- The interpolation result shows that R_Z is onto as well. This implies that R_Z is <u>open</u> as well.

- Openness means that $R_Z(Hol(\mathbb{C})_\varepsilon) \supseteq (\mathbb{C}^{\mathbb{N}})_\delta$ $\forall \varepsilon>0\ \exists\delta>0$, where $X_\delta = \{x\in X : d(x,0)<\delta\}$, ~~and~~ X being a Fréchet space.

  i.e. $\forall\varepsilon>0\ \exists\delta>0$ s.t. if $\{a(\ell)\}_{\ell=1}^\infty = a$ satisfies $d(a,0)\le\delta$ then $\exists f\in Hol(\mathbb{C})$ with $d(f,0)\le\varepsilon$ s.t.

  $$f(z_\ell) = a_\ell.$$

- <u>Question</u>: does R_Z admit a linear, continuous, right inverse T_Z?

  $$\mathbb{C}^{\mathbb{N}} \xrightarrow{T_Z} Hol(\mathbb{C}) \xrightarrow{R_Z} \mathbb{C}^{\mathbb{N}}$$

  $$R_Z \circ T_Z = Id$$ <u>Spoiler</u>: no.

- Suppose it does. Then $\forall n\ \exists f_n \in Hol(\mathbb{C})$ s.t. $f_n(z_m) = \delta_{nm}$ $\left(f_n = T_Z\{\delta_{nm}\}_m\right) = (T_Z\delta_n)(z_m)$

  ~~but~~ No problem: this is just onto.

  Let $a = \{a_n\} \in \mathbb{C}^{\mathbb{N}}$ and $a^\nu(n) = \begin{cases} a_n & \text{if } n\le\nu \\ 0 & \text{if } n>\nu\end{cases}$

  Then, by linearity,

  $$T_Z a^\nu = T_Z\left(\sum_{n=1}^\nu a_n\delta_n\right) = \sum_{n=1}^\nu a_n T_Z\delta_n = \sum_{n=1}^\nu a_n f_n$$

  Since $a^\nu \xrightarrow[\nu\to\infty]{} a$ in $\mathbb{C}^{\mathbb{N}}$, by continuity we have that $\sum_{n=1}^\nu a_n f_n = T_Z a^\nu \xrightarrow[\nu\to\infty]{} T_Z a$ uniformly on compact sets.

This says that
$$\sum_{n=1}^{\infty} a_n f_n = R_Z a \quad \text{converges u.c.}$$
for all $\{a_n\}$ in $\mathbb{C}^{\mathbb{N}}$.

- We show that this is not possible. Each f_n has at most countably many zeros, hence $\exists b \in \mathbb{C}: f_n(b) \neq 0 \ \forall n$. Set $a_n = \frac{1}{f_n(b)}$: then $\sum_{n=1}^{\infty} a_n f_n(b) = \sum_{n=1}^{\infty} 1$ diverges.

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- A rereading of the proof of the interpolation theorem shows that, in fact, the Mittag-Leffler function used to construct the interpolating f depends on $\{a_n\}$ (in terms of growth).
- Sampling: In the space $Hol(\mathbb{C})$ no (discrete) sampling sequence exists, by Weierstrass factorization theorem.
- Question. Does R_Z admit a linear (unbounded) right inverse? I don't know. (Zorn's Lemma?).

- With the notation used in the M-L theorem, consider, for a fixed sequence $\{b_n\} = Z$ in $\mathbb{C}$, a fixed function $g \in Hol(\mathbb{C})$ s.t. $g(b_n) = 0 \ \forall n$ and a fixed increasing sequence $\{N_n\}$ of positive integers, let
$$\mathcal{F} = \left\{ a = \{a_n\} \in \mathbb{C}^{\mathbb{N}} : \lim_{n\to\infty} \frac{|a_n|^{1/N_n}}{|g'(b_n)|^{1/N_n} \cdot |b_n|} = 0 \right\}$$
$\mathcal{F}$ is a (nonclosed) linear subspace of $\mathbb{C}^{\mathbb{N}}$.

For sequences in $\mathcal{F}$ we have a linear interpolating formula:
$$Ta(z) = \sum_{n=1}^{\infty} a_n f_n(z)$$
converges uniformly on compacts and $Ta(b_n) = a_n$, where f_n is the solution to the interpolating problem $f_n(b_m) = \delta_{nm}$ constructed as $f_n = g h_n$, h_n having principal part
$$Q_n\left(\frac{1}{z-b_n}\right) = \frac{1}{g'(b_n)(z-b_n)}$$
and the right decay:
$h_n(z) = Q_n\left(\frac{1}{z-b_n}\right) - q_n(z)$ where q_n is the Taylor polynomial of $Q_n\left(\frac{1}{z-b_n}\right)$ having degree N_n. The union of such $\mathcal{F}$'s exhausts $\mathbb{C}^{\mathbb{N}}$ ☒