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Interpolation, sampling and zeros (2023).

Ph.D. course with N. Chalmoukis.

Sampling requires a measure of <u>rigidity</u> in $\mathcal{F}$. Interpolation requires a measure of <u>flexibility</u>. The problem of zeros is related to both sampling and interpolation: if Z is a zero set, it is not feasable for sampling and it solves a very specific problem of interpolation.

<u>Prologue</u>. Let $\mathcal{F}$ be a family (a vector space) of functions $X \xrightarrow{f} \mathbb{C}$ on some space X.

<u>Sampling problem</u>. For which $Z \subseteq X$ the functions in $\mathcal{F}$ can be reconstructed by their values on Z?

$$\left(Z \xrightarrow{f|_Z} \mathbb{C}\right) \xmapsto{T} \left(f : X \to \mathbb{C}\right)$$

Z is "large enough" to allow reconstruction.

<u>Interpolation problem</u>. For which $Z \subseteq X$ we find functions in $\mathcal{F}$ with prescribed values on Z?

The problem can be stated in several nonequivalent ways.

<u>Zero problem</u>. For which $Z \subseteq X$ there is $f \neq 0$ in $\mathcal{F}$ such that $f|_Z = 0$?

Polynomials in $\mathbb{C}[z]$

<u>Zeros.</u> If $P(z) \in \mathbb{C}_n[z]$ has degree n, $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, then it has n zeros $z_1, \dots, z_n$ counting multiplicities.
Viceversa, given $z_1, \dots, z_n \in \mathbb{C}$ the polynomial $P(z) = \prod_{j=1}^{n} (z - z_j)$ has that zero set.

~~Below~~ Moreover, if $\deg(P) \le n$ and its zero set $Z(P)$ has $\#(Z(P)) > n$, then $P \equiv 0$.

Below, $Z = \{z_1, \dots, z_n\}$ is a set of n points in $\mathbb{C}$ and $\mathbb{C}_{\le n-1}[z]$ is the space of polynomials P having $\deg(P) \le n-1$

<u>Interpolation</u>. Given $Z = \{z_1, \dots, z_n\}$ and $w_1, \dots, w_n \in \mathbb{C}$, find $P \in \mathbb{C}_{\le n-1}[z]$:

$$P(z_j) = w_j$$

for $j = 1, \dots, n$.

i.e.
$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} a_0 + a_1 z_1 + \dots + a_{n-1} z_1^{n-1} \\ \vdots \\ a_0 + a_1 z_n + \dots + a_{n-1} z_n^{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & z_1 & z_1^2 & \dots & z_1^{n-1} \\ 1 & z_2 & z_2^2 & \dots & z_2^{n-1} \\ & & \vdots & & \\ 1 & z_n & z_n^2 & \dots & z_n^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

For fixed Z the problem has solution $\forall\, w_1, \dots, w_n$ iff

$$0 \ne \begin{vmatrix} 1 & z_1 & z_1^2 & \dots & z_1^{n-1} \\ & & \vdots & & \\ 1 & z_n & z_n^2 & \dots & z_n^{n-1} \end{vmatrix} = \prod_{i<j} (z_i - z_j)$$

which in fact holds since $z_i \ne z_j$ if $i \ne j$.

The linear algebra suggests a way to write down the interpolating polynomial.
For $\ell = 1, 2, \dots, n$ write

$$f_\ell(z) = \frac{\prod_{i \ne \ell} (z - z_i)}{\prod_{i \ne \ell} (z_\ell - z_i)} = \frac{\det \begin{vmatrix} 1 & z_1 & \dots & z_1^{n-1} \\ 1 & z & \dots & z^{n-1} \\ 1 & z_n & \dots & z_n^{n-1} \end{vmatrix} \ell}{\det \begin{vmatrix} 1 & z_1 & \dots & z_1^{n-1} \\ & \vdots & & \\ 1 & z_n & \dots & z_n^{n-1} \end{vmatrix}}$$

So that $f_\ell(z_\ell) = 1$, $f_\ell(z_i) = 0$ if $i \ne \ell$, $\deg(f_\ell) = n-1$. Set $P(z) = \sum_\ell w_\ell f_\ell(z)$.

Exercise The solution to
$$\begin{cases} P(z_i) = w_i & i = 1 \dots n \\ \deg(P) \le n-1 \end{cases}$$
is unique. Parametrize all solutions to
$$\begin{cases} P(z_i) = w_i & i = 1 \dots n \\ \deg(P) \le n+m \end{cases}$$

Sampling: The obvious solution to the sampling problem ~~[illegible]~~ is
$$P(z) = \sum_{i=1}^{n} P(z_i) f_i(z) :$$
the R.H.S. is a polynomial of degree $\le n-1$, having the right value at $z = z_i$; $i = 1 \dots n$.
If q is another polynomial in $\mathbb{C}_{\le n-1}[z]$ having the same values at $z_1, \dots, z_n$ then $q - P \equiv 0$ since it has n distinct zeros.

Obs. Interpolation at $\{z_1, \dots, z_n\}$ is possible requiring $p \in \mathbb{C}_{\le N}[z]$ with $N \ge n-1$; while sampling holds requiring $p \in \mathbb{C}_{\le N}[z]$ with $N \le 1$.

Shannon sampling theorem.

$PW(a) \ni \mathbb{R} \xrightarrow{f} \mathbb{C}$ s.t. $\operatorname{supp}(\hat{f}) \subseteq [-\frac{a}{2}, \frac{a}{2}]$, $f \in L^2(\mathbb{R})$

Paley-Wiener space "band limited"

Handbook of Fourier basics:
$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i \omega x} dx ; \quad f(x) = \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{2\pi i \omega x} d\omega$$
$$\int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 d\omega = \int_{-\infty}^{+\infty} |f(x)|^2 dx \quad \text{converges since } \hat{f} \in L^1, f \in C.$$
If $\varphi \in L^2[-\frac{a}{2}, \frac{a}{2}]$ and $\hat{\varphi}(n) = \int_{-a/2}^{a/2} \varphi(\omega) e^{2\pi i n \omega/a} \cdot d\omega$
then $\varphi(\omega) = \frac{1}{a} \sum_{n=-\infty}^{+\infty} \hat{\varphi}(n) e^{-2\pi i n \omega/a}$ converges in L^2
and $\int_{-a/2}^{a/2} |\varphi(\omega)|^2 d\omega = \frac{1}{a} \sum_n |\hat{\varphi}(n)|^2$

Theorem (Shannon, ...) If $f \in PW(a)$, then
$$f(x) = \sum_{n=-\infty}^{+\infty} f\left(\frac{n}{a}\right) \operatorname{sinc}(\pi(ax - n)) \quad \text{converges } \forall x \in \mathbb{R}$$
where $\operatorname{sinc}(y) = \frac{\sin(y)}{y}$.

Proof. Set $a_n(f) := \int_{-a/2}^{a/2} \hat{f}(\omega) e^{2\pi i n \omega/a} d\omega = f\left(\frac{n}{a}\right)$.
Then, $\hat{f}(\omega) = \frac{1}{a} \sum_n a_n(f) e^{-2\pi i n \omega/a} = \frac{1}{a} \sum_n f\left(\frac{n}{a}\right) e^{-2\pi i \omega \frac{n}{a}}$
with convergence in $L^2[-\frac{a}{2}, \frac{a}{2}]$.
Hence, $f(x) = \int_{-a/2}^{a/2} \frac{1}{a} \sum_n f\left(\frac{n}{a}\right) e^{2\pi i \omega (x - n/a)} d\omega$
$$= \sum_n f\left(\frac{n}{a}\right) \frac{1}{a} \left[\frac{e^{2\pi i \omega (x - n/a)}}{2\pi i (x - n/a)} \right]_{-a/2}^{a/2}$$

$$= \sum_n f\left(\frac{n}{a}\right) \cdot \mathrm{sinc}(\pi(ax-n)) \;\blacksquare$$

The proof says much more.

<u>Isometry</u>: $\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-a/2}^{a/2} |\hat{f}(\omega)|^2 d\omega =$

$$= \frac{1}{a} \sum_{-\infty}^{+\infty} |a_n(f)|^2 = \frac{1}{a} \sum_{-\infty}^{+\infty} \left|f\left(\frac{n}{a}\right)\right|^2.$$

<u>Reproducing formulas</u>.

Set $\hat{K}_z(\omega) = \chi_{[-a/2, a/2]}(\omega) \cdot e^{-2\pi i \omega z}$, $\hat{K}_z \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$

Then,

$$K_z(x) = \int_{-\infty}^{+\infty} \hat{K}_z(\omega) e^{2\pi i \omega x} d\omega$$

$$= \int_{-a/2}^{a/2} e^{2\pi i \omega (x-z)} d\omega = a\,\mathrm{sinc}(\pi a(x-z)).$$

In particular, Shannon's formula becomes:

$$f(x) = \frac{1}{a} \sum_n f\left(\frac{n}{a}\right) K_{n/a}(x)$$

which is very similar to Lagrange formula.

We also have the reproducing formula:

$$f(x) = \int_{-a/2}^{a/2} \hat{f}(\omega) e^{2\pi i \omega x} d\omega = \langle \hat{f}, \hat{K}_x \rangle_{L^2}$$

$$= \langle f, K_x \rangle_{L^2}$$

We express this by saying that $\{K_x\}_{x \in \mathbb{R}}$ is the reproducing kernel for $PW(a)$ (on $\mathbb{R}$: we shall extend this to $\mathbb{C}$).

It is immediate that

$$\langle K_{\frac{n}{a}}, K_{\frac{m}{a}} \rangle_{L^2} = a \cdot \delta_{mn}$$

<u>Theorem</u> $\left\{\frac{1}{\sqrt{a}} K_{n/a}\right\}_{n \in \mathbb{Z}}$ is a o.n.b. for $PW(a)$ and the n^{th} coefficient of $f \in PW(a)$ is $\langle f, \frac{K_{n/a}}{\sqrt{a}} \rangle = \frac{1}{\sqrt{a}} f\left(\frac{n}{a}\right)$

$$f \underset{L^2}{=} \sum_n \frac{1}{\sqrt{a}} f\left(\frac{n}{a}\right) \frac{K_{n/a}}{\sqrt{a}} = \frac{1}{a} \sum_n f\left(\frac{n}{a}\right) K_{n/a},$$

which is Shannon's formula again.

<u>Corollary</u>. $\sum_n f\left(\frac{n}{a}\right) \mathrm{sinc}(\pi(ax-n)) = f(x)$ also converges in $L^2(\mathbb{R})$.

<u>Interpolation</u> comes for free from Shannon's formula and its L^2 interpretation.

Suppose $\frac{1}{a} \sum_{-\infty}^{+\infty} |a_n|^2 < +\infty$ and let $f(x) = \sum_n a_n \,\mathrm{sinc}(\pi(ax-n)) = \frac{1}{a} \sum_n a_n K_{n/a}(x)$.

A priori the series converges in L^2 to $f \in L^2$ s.t. $\mathrm{supp}(\hat{f}) \subseteq [-\frac{a}{2}, \frac{a}{2}]$. Hence, by ~~the~~ the discussion around Shannon's formula, we also have pointwise convergence and

$$f\left(\frac{n}{a}\right) = a_n.$$

A special instance of Shannon's formula is:

$$\mathrm{sinc}(\pi a(x-z)) = \frac{1}{a} K_z(x) = \sum_n \mathrm{sinc}(\pi(n-za)) \cdot \mathrm{sinc}(\pi(ax-n)) = \frac{1}{a^2} \sum_n \overline{K_{n/a}(z)}\, K_{n/a}(x)$$

which can also deduced ~~[illegible]~~ by general R.K.H.S. theory (with L^2 convergence of the series w.r.t. the x variable, while here we have both L^2 and uniform convergence).