Tilings and Potential Theory on trees

We consider here for simplicity a `rooted, finite, sub-binary tree` T. The edge set $E(T)$ has a distinguished `root-edge` ω, one of which endpoints, the `pre-root-vertex` $b(\omega)$, is endpont of ω alone. Each vertex x is endpoint of no more than three edges, and it is a `leaf` if $x \neq b(\omega)$ is endpoint of just one edge. The set of the leaves is the boundary ∂T of T. Having a pre-root-vertex and a root edge induces a partial order on vertices and edges. We might consider T as a subtree of a sufficiently deep dyadic tree $T_{2,N}$.

Alternatively, we define the dyadic tree $T_{2,N}$, consider a subset $F \subseteq V(T_{2,N})$, and consider the tree $T = T(F)$ generated by F (or we might not: it will appear naturally). We might of curse assume that F is a boundary set.

$$\mathrm{Cap}(F) = \inf\{\|\varphi\|^2_{\ell^2(E(T))} \colon \varphi \colon E(T) \to \mathbb{R}, I\varphi \geqslant 1 \text{ on } E, \varphi \geqslant 0\}.$$

We proceed step by step.

i. The requirement $\varphi \geqslant 0$ might be dropped.

ii. There exists an extremal φ, by Weierstrass Theorem.

iii. $I\varphi = 1$ on F (otherwise I can make $\|\varphi\|^2_{\ell^2(E(T))}$ smaller while keeping $I\varphi \geqslant 1$ on E).

iv. $\mathrm{supp}(\varphi) \subseteq T(E)$.

v. $\varphi(\alpha) = \varphi(\alpha_+) + \varphi(\alpha_-)$. Fix the attention on the vertex $e(\alpha) = b(\alpha_-) = b(\alpha_+)$, leaving φ unaltered outside $\{\alpha.\alpha_\pm\}$: $\varphi(\alpha) + \varphi(\alpha_-) = A_-$ and $\varphi(\alpha) + \varphi(\alpha_+) = A_+$ with A_-, A_+ fixed, i.e. with $t = \varphi(\alpha)$, we want to minimize $f(t) = t^2 + (A_+ - t)^2 + (A_- - t)^2$, obtaining

$$\varphi(\alpha) = \frac{A_- + A_+}{3}, \varphi(\alpha_-) = \frac{2A_- - A_+}{3}, \varphi(\alpha_+) = \frac{2A_+ - A_-}{3},$$

which satisfies the relation.

vi. Thus, φ can be thought of as a measure on F: $\varphi(\alpha) = \mu(F \cap S(\alpha))$. At this point we have the tiling.

vii. The extermal φ is unique. If φ_1 and φ_2 are extremals, so that in particular $\|\varphi_1\|^2_{\ell^2} = \|\varphi_2\|^2_{\ell^2}$ and $I\varphi_1 = I\varphi_2 = 1$ on F, then $\frac{\varphi_1 + \varphi_2}{2}$ is better, unless $\varphi_1 = \varphi_2$.

viii. $\mathcal{E}(\mu) = \sum_\alpha I^*\mu(\alpha)^2 = \mathrm{Cap}(F)$ (look at the tiling) and $\varphi = I^*\mu$.

ix. $I\varphi = 1$ on F has a unique solution satisfying $\varphi(\alpha) = \varphi(\alpha_+) + \varphi(\alpha_-)$. It suffices to show that $\Phi = I\varphi$ is harmonic and vanishes on the boundary.

x. $\mathrm{Cap}(F) = \sup\left\{\frac{\nu(F)^2}{\mathcal{E}(\nu)} \colon \mathrm{supp}(\nu) \subseteq F\right\}$. Here we need min/max.

xi. Recursive relation.