Electrostatics

1 Basic concepts

If q, Q are charges in position $\mathbf{x}(q), \mathbf{x}(Q)$, an electric force $\mathbf{F}$ is exherted on q:

$$\boldsymbol{F} = KqQ\frac{\mathbf{x}(q) - \mathbf{x}(Q)}{|\mathbf{x}(q) - \mathbf{x}(Q)|^3},$$

with $K = \frac{1}{4\pi\epsilon_0}$. Thnis is *Coulomb's law.*

The *electric field* generated by Q at $\mathbf{x}$ is

$$\boldsymbol{E}(x) = KQ\frac{\mathbf{x}(q) - \mathbf{x}(Q)}{|\mathbf{x}(q) - \mathbf{x}(Q)|^3}.$$

Passing to a distribution μ of charges (typically, μ is a compactly supported signed measure),

$$\boldsymbol{E}^\mu(x) = K\int_{\mathbb{R}^3}\frac{\mathbf{x}(q) - \mathbf{x}(Q)}{|\mathbf{x}(q) - \mathbf{x}(Q)|^3}d\mu(\mathbf{x}).$$

The electric field generated by a charge, being radial, is conservative: $-\nabla(|\mathbf{x}|^{-1}) = \frac{\mathbf{x}}{|\mathbf{x}|^3}$. Hence, the electric field generated by μ is the gradient of a *potential*,

$$\boldsymbol{E}^\mu = -\nabla V^\mu, V^\mu(\mathbf{x}) = K\int_{\mathbb{R}^3}\frac{d\mu(\mathbf{x})}{|\mathbf{x}(q) - \mathbf{x}(Q)|} = V^{\delta_0} * \mu(\mathbf{x}).$$

For each $\mathbf{x}$ consider a smooth curve $\gamma_{\mathbf{x}}$ joining ∞ to $\mathbf{x}$. Then,

$$V^\mu(\mathbf{x}) = -\int_{\gamma_{\mathbf{x}}}\boldsymbol{E}^\mu(\mathbf{y})\cdot d\mathbf{y}$$

is the work required to move $q = +1$ from ∞ to $\mathbf{x}$ against the field $\boldsymbol{E}^\mu$. The *energy* $\frac{1}{2}\mathcal{E}$ stored in a configuration of charges $Q_1, \ldots, Q_N$ in positions $\mathbf{x}_1, \ldots, \mathbf{x}_N$ is that needed to arrange the configuration moving the charges in place from a distant location. If $\mu = \sum_{j=1}^N Q_j\delta_{\mathbf{x}_j}$, then

$$\begin{aligned}\frac{1}{2}\mathcal{E}(\mu) &= -Q_2\int_{\gamma_{\mathbf{x}_2}}\boldsymbol{E}^{Q_1}(\mathbf{y})\cdot d\mathbf{y} - Q_3\int_{\gamma_{\mathbf{x}_3}}\boldsymbol{E}^{Q_1,Q_2}(\mathbf{y})\cdot d\mathbf{y} - \cdots - Q_N\int_{\gamma_{\mathbf{x}_N}}\boldsymbol{E}^{Q_1,\ldots,Q_{N-1}}(\mathbf{y})\cdot d\mathbf{y} \\ &\quad Q_2V^{Q_1}(\mathbf{x}_2) + Q_3(V^{Q_1}(\mathbf{x}_3) + V^{Q_2}(\mathbf{x}_3)) + \cdots + Q_N(V^{Q_1}(\mathbf{x}_N) + \ldots + V^{Q_{N-1}}(\mathbf{x}_N)) \\ &= \frac{K}{2}\sum_{i\neq j=1}^N\frac{Q_iQ_j}{|\mathbf{x}_i - \mathbf{x}_j|}.\end{aligned}$$

Passing to a general (atomless) distribution μ:

$$\mathcal{E}(\mu) = K\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{d\mu(\mathbf{x})d\mu(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}.$$

2 Some applications of calculus

Writing the Laplace operator $\Delta = \partial_{11} + \partial_{22} + \partial_{33} = \partial_{rr} + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^2}$ in polar coordinates $\mathbf{x} = r\boldsymbol{\omega}$, with $|\boldsymbol{\omega}| = 1$ belonging to the 2-sphere $\mathbb{S}^2$, it is easy to verify the crucial fact that

$$\Delta(|\mathbf{x}|^{-1}) = 0 \text{ for } \mathbf{x} \neq 0,$$

hence that

$$\Delta V^\mu = -\frac{\mu}{\epsilon_0} \text{ (Laplace equation)}$$

in some (distributional) sense, and $\Delta V^\mu(\infty) = 0$, if the support of μ is compact.

We roughly follow here Gauss' reasoning, with no pretension of rigour. Let $A \subset \mathbb{R}^3$ be a region with smooth boundary ∂A. The (Lagrange)-Gauss-(Ostrogradsky) formula says that for a smooth vector field U defined on the closure $\bar{A}$ of A one has:

$$\int_{\partial A}U\cdot n\,d\sigma = \int_A\nabla\cdot U\,d\,\mathrm{Vol}$$

where n is the outer unit normal, $d\sigma$ is surface measure, $\nabla\cdot U = \partial_1 U_1 + \partial_2 U_2 + \partial_3 U_3$ is the divergence of U and $d\,\mathrm{Vol}$ is volume measure. Applying it to $U(\mathbf{x}) = \nabla(|\mathbf{x}|^{-1}) = -\frac{\mathbf{x}}{|\mathbf{x}|^3}$ we find that

$$\int_{\partial A} \nabla(|\mathbf{x}|^{-1})\cdot n\, d\sigma = \int_A \Delta(|\mathbf{x}|^{-1}) dx_1 dx_2 dx_3 = 0 = -4\pi\delta_0(A) \text{ if } 0 \notin A.$$

If $0\in A$, let $B_\epsilon = \{\mathbf{x}\colon |\mathbf{x}|\le\epsilon\}$ be contained in A, and apply Gauss' formula to $A\setminus B_\epsilon$:

$$\begin{aligned} 0 &= \int_{A\setminus B_\epsilon} \Delta(|\mathbf{x}|^{-1}) dx_1 dx_2 dx_3 \\ &= \int_{\partial A} \nabla(|\mathbf{x}|^{-1})\cdot n\, d\sigma + \int_{\partial B_\epsilon} \nabla(|\mathbf{x}|^{-1})\cdot n\, d\sigma \\ &= \int_{\partial A} \nabla(|\mathbf{x}|^{-1})\cdot n\, d\sigma - \int_{\epsilon=|\mathbf{x}|} \frac{\partial}{\partial r}\left(\frac{1}{r}\right) d\sigma \\ &= \int_{\partial A} \nabla(|\mathbf{x}|^{-1})\cdot n\, d\sigma - \left(\frac{-1}{\epsilon^2}\right) 4\pi\epsilon^2, \end{aligned}$$

so that

$$\int_{\partial A} \nabla(|\mathbf{x}|^{-1})\cdot n\, d\sigma = -4\pi = -4\pi\delta_0(A).$$

We might "distributionally" close here the discourse by writing (as $\epsilon\to 0$)

$$\forall A\colon \int_A \Delta(|\mathbf{x}|^{-1}) dx_1 dx_2 dx_3 = -4\pi\delta_0(A) \Rightarrow \Delta(|\mathbf{x}|^{-1}) = -4\pi\delta_0(\mathbf{x}),$$

hence that

$$\Delta(V^\mu) = \Delta(V^{\delta_0} * \mu) = -4\pi K\delta_0 * \mu = -4\pi K\mu = -\frac{\mu}{\epsilon_0}.$$

3 Getting rid of vectors and derivatives

The energy integral can be written in several different ways:

$$\begin{aligned} \mathcal{E}(\mu) &= K \iint \frac{d\mu(x)d\mu(y)}{|x-y|} \\ &= \int V^\mu(x) d\mu(x) \\ &= -\epsilon_0 \int V^\mu(y)\Delta V^\mu(y) d\,\mathrm{Vol}(y) \\ &= \epsilon_0 \int |\nabla V^\mu|^2 d\,\mathrm{Vol} = \epsilon_0 \int |E^\mu|^2 d\,\mathrm{Vol} \\ &= \epsilon_0 \int |(-\Delta)^{1/2} V^\mu|^2 d\,\mathrm{Vol}. \end{aligned}$$

The fractional Laplacian $(-\Delta)^{1/2}$ is defined "spectrally" by means of Fourier transforms:

$$\begin{aligned} \hat{f}(\omega) &= \int f(x) e^{-2\pi i x\cdot\omega} dx, \\ f(x) &= \int \hat{f}(\omega) e^{2\pi i x\cdot\omega} d\omega \\ (\partial_j f)\hat{}(\omega) &= 2\pi i\omega_j \hat{f}(\omega) \\ ((-\Delta) f)\hat{}(\omega) &= 4\pi^2|\omega|^2 \hat{f}(\omega) \\ ((-\Delta)^a f)\hat{}(\omega) &= (4\pi^2|\omega|^2)^a \hat{f}(\omega) \end{aligned}$$

In the case $a = -1/2$ we have

$$I_1\, f(x) := (-\Delta)^{-1/2} f(x) = \frac{1}{2\pi^2}\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|^2} dy,$$

where I_1 is a Riesz potential, to be discussed in more detail below.

For a measure,

$$I_1\,\mu(x) := \frac{1}{2\pi^2}\int_{\mathbb{R}^3}\frac{d\mu(y)}{|x-y|^2}.$$

Most of what we have seen so far can be written in terms of U alone:

$$\begin{aligned} V^{\mu}(x) &= \frac{1}{\epsilon_0}(-\Delta)^{-1}\mu(x) = \frac{1}{\epsilon_0}I_1\,I_1\mu(x),\\ \mathcal{E}(\mu) &= \epsilon_0\int|(-\Delta)^{-1/2}\mu|^2 d\,\mathrm{Vol}\\ &= \frac{1}{\epsilon_0}\int|I_1\mu|^2 d\,\mathrm{Vol}. \end{aligned}$$

4 Capacity of conductors

A conductor C in $\mathbb{R}^3$ is a set where charges are free to move (in most cases is a compact set, not necessarily connected, although we fictionally assume charges are free to "jump" between one component and the other). Let $M>0$ be an amount of charge on C. The charges of a positive distribution μ with $\|\mu\| = M$ will start moving in C under Coulomb's forces, until they reach an equilibrium configuration $\mu^{^C}$, which can be proven to exist and to be unique. For such distribution we have that the potential

$$E^{\mu^C} = 0 \text{ on } \mathrm{supp}(\mu^C),$$

otherwise some charges will still move under the action of the electric force. It is easy to see that, even if C is not connected, V^{μ^C} must be constant on $\mathrm{supp}(\mu^C)$. Actually, V^{μ^c} must be constant on the whole of C.

The *capacity* of C is the maximum amount M of charge for which $V^{\mu^C} = 1$ on $\mathrm{supp}(\mu^C)$. In fact, Gauss realized that the equilibrium distribution simultaneously possess a number of properties, some of them of an extremal nature:

i. $\mu(C) = \mu^C(C)$;

ii. $\mathcal{E}(\mu^C) \leqslant \mathcal{E}(\mu)$;

iii. $V^{\mu^C} = V$ is constant on C;

iv. $V^{\mu^C} \leqslant 1$ on $\mathbb{R}^3$.