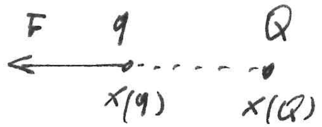
Electrostatics.



Coulomb's Law

$$F = K \cdot q \cdot Q \, \frac{x(q) - x(Q)}{|x(q) - x(Q)|^3}$$

$$K = \frac{1}{4\pi\varepsilon_0}$$

$$E(x) = KQ \, \frac{x - x(Q)}{|x - x(Q)|^3}$$: electric field...

$$E^{\mu}(x) = K \cdot \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \, d\mu(y)$$ .. generated by a charge distribution μ

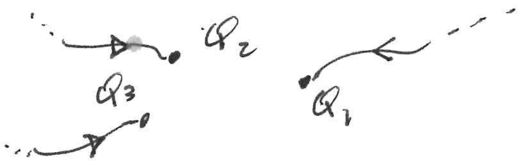
E radial => E exact:

$$\frac{x}{|x|^3} = -\nabla\left(\frac{1}{|x|}\right) \Rightarrow E^{\mu} = -\nabla V^{\mu}$$

where $$V^{\mu}(x) = K \cdot \int_{\mathbb{R}^3} \frac{d\mu(y)}{|x - y|} = V^{\delta_0} * \mu(x)$$

$$V^{\mu}(x) = -\int_{\gamma_x} E^{\mu}(y) \cdot dy$$

What is the energy stored in a charge distribution?



Energy: $\frac{1}{2}\mathcal{E}$ = work to be done against electric forces.

$$\frac{1}{2}\mathcal{E} = -KQ_2 \int_{\gamma_{x_2}} E^{Q_1}(y) \cdot dy - KQ_3 \int_{\gamma_{x_3}} E^{Q_1, Q_2}(y) \cdot dy$$

$$- \dots - K Q_n \int_{\gamma_{x_n}} E^{Q_1 \dots Q_{n-1}}(y) \cdot dy$$

$$= K \cdot \sum_{i<j} Q^j V^{Q_i}(x_j) = \frac{K}{2} \sum_{i \neq j} \frac{Q_i Q_j}{|x_i - x_j|}$$

Thus in general:

$$\mathcal{E} = K \iint_{\mathbb{R}^3 \mathbb{R}^3} \frac{d\mu(x)\, d\mu(y)}{|x - y|}$$

We can apply some tools from calculus:

$$\Delta = \partial_{11} + \partial_{22} + \partial_{33} + = \partial_{rr} + \frac{2}{r}\partial_r + \frac{\Delta_{S^2}}{r^2}$$

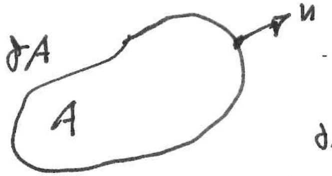
is Laplace' operator in polar coordinates $x = r\omega$; $r > 0$, $\omega \in S^2$.

Theorem (Gauss, Green, ...).

$\Delta\left(\frac{1}{|x|}\right) = 0$ for $x \neq 0$, hence:

$$\Delta V^{\mu} = -\frac{\mu}{\varepsilon_0} \quad \text{(Laplace equation)}$$

and $V^{\mu}(\infty) = 0$ if μ has compact support.



$$\int_{\partial A} v \cdot n \, d\sigma = \int_A \nabla \cdot v \, dVol$$

if $v : \bar{A} \to \mathbb{R}^3$ is C^1,

where $\nabla \cdot v = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3$

If $0 \notin \bar{A}$: $\displaystyle\int_{\partial A} \nabla\left(\frac{1}{|x|}\right) \circ n \, d\sigma =$

$$= \int_A \Delta\left(\frac{1}{|x|}\right) dx_1 \, dx_2 \, dx_2 = 0 = -4\pi \delta_0(A)$$

If $0 \in A$ and $B_\varepsilon = \{x : |x| \leq \varepsilon\}$:

$$0 = \int_{A \setminus B_\varepsilon} \Delta\left(\frac{1}{|x|}\right) dx_1 \, dx_2 \, dx_3$$

$$= \int_{\partial A} \nabla\left(\frac{1}{|x|}\right) \circ n \, d\sigma + \int_{\partial B_\varepsilon} \nabla\left(\frac{1}{|x|}\right) \circ n \, d\sigma$$

$$= \int_{\partial A} \nabla\left(\frac{1}{|x|}\right) \circ n \, d\sigma - \int_{|x| = \varepsilon} \frac{\partial}{\partial r}\left(\frac{1}{r}\right) d\sigma$$

$$= \int_{\partial A} \nabla\left(\frac{1}{|x|}\right) \circ n \, d\sigma + \frac{1}{\varepsilon^2} 4\pi \varepsilon^2$$

so that $\displaystyle\int_{\partial A} \nabla\left(\frac{1}{|x|}\right) \cdot n \, d\sigma = -4\pi$

$$= -4\pi \delta_0(A).$$

This might be taken as evidence that

$$\int_A \underbrace{\Delta\left(\frac{1}{|x|}\right) dx_1 \, dx_2 \, dx_3}_{\text{in some generalized sense}} = -4\pi \delta_0(A)$$

hence that $\Delta\left(\frac{1}{|x|}\right) = -4\pi \delta_0$,

so: $\displaystyle\Delta V^\mu = \Delta\left(V^{\delta_0} * \mu\right) = -\frac{4\pi}{4\pi\varepsilon_0} \delta_0 * \mu$

$$= -\frac{\mu}{\varepsilon_0}.$$

<u>Obs.</u> We avoided the issue $0 \in \partial A$.
One might use $\varphi \in C_c(\mathbb{R}^3)$ instead of $A \subseteq \mathbb{R}^3$ and show that

$$0 = \int_{\mathbb{R}^3 \setminus B_\varepsilon} \Delta\left(\frac{1}{|x|}\right) \varphi(x) \, dx$$

$$= -\int_{\mathbb{R}^3 \setminus B_\varepsilon} \nabla\left(\frac{1}{|x|}\right) \circ \nabla\varphi(x) \, dx + \int_{\partial B_\varepsilon} \varphi \nabla\left(\frac{1}{|x|}\right) \circ n \, d\sigma$$

$$\xrightarrow[\varepsilon \to 0]{} -\int_{\mathbb{R}^3} \nabla\left(\frac{1}{|x|}\right) \circ \nabla\varphi(x) \, dx + 4\pi \cdot \varphi(0)$$

i.e. $-4\pi \varphi(0) = -4\pi \delta_0(\varphi) =$

$$= -\int_{\mathbb{R}^3} \nabla\left(\frac{1}{|x|}\right) \cdot \nabla\varphi(x) \, dx$$

in some general sense

$$= \int_{\mathbb{R}^3} \Delta\left(\frac{1}{|x|}\right) \varphi(x) \, dx$$

We want to write as much as possible without vectors and derivatives. The energy integral is:

$$\mathcal{E}(\mu) = K \iint \frac{d\mu(x)\, d\mu(y)}{|x-y|}$$

$$= \int V^{\mu}(x)\, d\mu(x)$$

$$= -\varepsilon_0 \int V^{\mu}(x)\, \Delta V^{\mu}(x)\, dx$$

$$= \varepsilon_0 \int |\nabla V^{\mu}(x)|^2\, dx$$

$$= \varepsilon_0 \int |E^{\mu}(x)|^2\, dx$$

$$= \varepsilon_0 \int |(-\Delta)^{1/2} V^{\mu}(x)|^2\, dx$$

where $(-\Delta)^{1/2}$ is defined "spectrally":

$$\hat{f}(\omega) = \int f(x)\, e^{-2\pi i x \omega}\, dx$$

$$f(x) = \int \hat{f}(\omega)\, e^{2\pi i x \omega}\, d\omega$$

$$(\partial_j f)^{\wedge}(\omega) = 2\pi i \omega_j \hat{f}(\omega)$$

$$[(-\Delta) f]^{\wedge}(\omega) = 4\pi^2 |\omega|^2 \hat{f}(\omega)$$

$$[(-\Delta)^{1/2} f]^{\wedge}(\omega) = 2\pi |\omega| \cdot \hat{f}(\omega)$$

The best expression of the energy follows from the 3rd (or 4th) and Plancherel.

One has:

$$I_1 f(x) = (-\Delta)^{-1/2} f(x) = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|^2}\, dy$$

and more generally:

$$I_1 \mu(x) = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \frac{d\mu(y)}{|x-y|^2}.$$

Back to our objects:

$$V^{\mu}(x) = \frac{1}{\varepsilon_0} (-\Delta)^{-1} \mu(x) = \frac{1}{\varepsilon_0} I_1 I_1 \mu(x)$$

$$\mathcal{E}(\mu) = \varepsilon_0 \int |(-\Delta)^{+1/2} V^{\mu}|^2\, dx$$

$$= \varepsilon_0^{-1} \int |(-\Delta)^{-1/2} \mu|^2\, dx$$

$$= \frac{1}{\varepsilon_0} \int |I_1 \mu|^2\, dx.$$

Relevant objects (set $\varepsilon_0 = 1$):

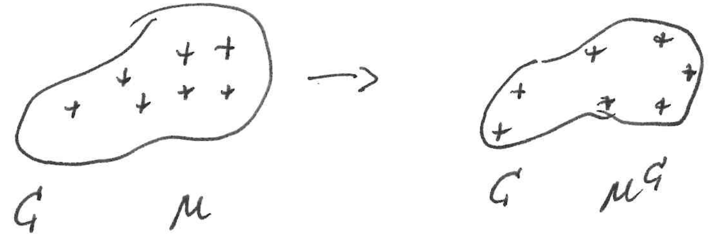
μ

$I_1 \mu$ $\qquad \mu = (-\Delta)^{1/2} I_1 \mu$

$V^{\mu} = I_1 I_1 \mu$ $\qquad = (-\Delta) I_1 I_1 \mu$

$\mathcal{E}(\mu)$

Gauss' view of conductors.



$\|\mu\| = \mu(G)$: amount of charge.

It moves in G until it reaches equilibrium configuration μ^G:

$$\mu^G(G) = \mu(G)$$

$E^{\mu^G} = 0$ on G otherwise charges start moving

(actually: $E^{\mu^G} = 0$ on $\mathrm{supp}(\mu^G)$).

Hence, $V^{\mu^G} = \text{const}$ on G (on $\mathrm{supp}(\mu^G)$).

$\mathrm{Cap}(G) = \max\{\|\mu\| : V^{\mu^G} = 1 \text{ on } G\}$

$V^{\mu^G} \leq 1$ on $\mathbb{R}^3$

$\mathcal{E}(\mu^G) \leq \mathcal{E}(\mu)$.

All this was made very precise by FROSTMAN in the '30s.

Axiomatic Nonlinear Potential Theory.1

Data. $1 < p < \infty$; $\frac{1}{p} + \frac{1}{p'} = 1$

(X, ρ): locally compact metric space

(M, m): a measure space

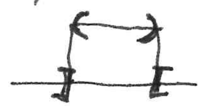
$K = K(x, \alpha) : X \times M \to [0, +\infty]$

satisfies: $K(x, \cdot)$ is measurable $\forall x \in X$

$K(\cdot, \alpha)$ is lower semicontinuous $\forall \alpha \in M$

where f is l.s.c. at x_0 if

$$\underline{\lim}_{x \to x_0} f(x) \geq f(x_0)$$



$f \geq 0$ on $M \longmapsto Kf(x) = \int_M K(x, \alpha) f(\alpha)\, d\alpha$

$\mu \geq 0$ on X Borel $\longmapsto \check{K}\mu(\alpha) = \int_X K(x, \alpha)\, d\mu(x)$

Energy: $\mathcal{E}(\mu) = \int_M (\check{K}\mu)^{p'} d\mu$

Potential: $V^{\mu}(x) = K(\check{K}\mu)^{p'-1}(x)$

Capacity: $E \subseteq X \longmapsto \mathrm{Cap}(E) \in [0, +\infty]$

$\mathrm{Cap}(E) = \inf\{\|f\|^p_{L^p} : Kf \geq 1 \text{ on } E,\ f \geq 0\}$

Basics:

(a) $\mathrm{Cap}(\emptyset) = 0$

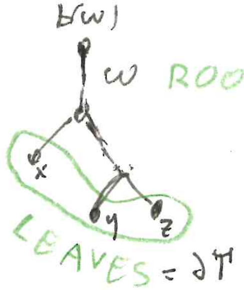
(b) $A \subseteq B \Rightarrow \mathrm{Cap}(A) \leq \mathrm{Cap}(B)$

(c) $\mathrm{Cap}\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mathrm{Cap}(E_n)$

Potential Theory on Finite Trees and tiling a rectangle by squares.

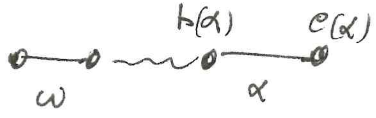
TILINGS

T: a sub-dyadic finite tree with a root-edge.



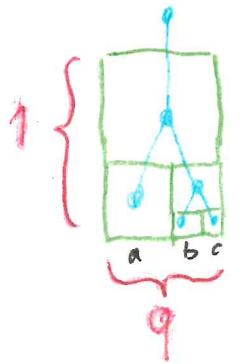
$T = (V(T), E(T))$

After choosing a root, $V(T)$, $E(T)$ become partially ordered



$e(\alpha) \le \alpha \le b(\alpha)$

To (T, ω) we associate a tiling of a rectangle by squares:



$q := \mathrm{Cap}(\partial T)$

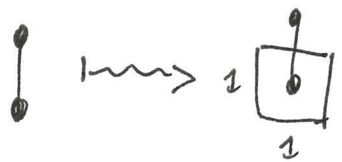
$\mu(x) = a$

$\mu(y) = b$

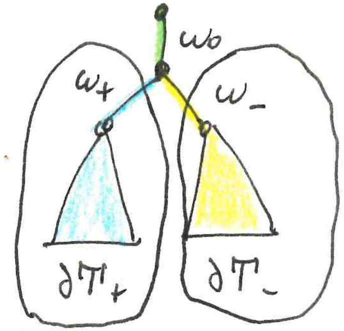
$\mu(z) = c$

μ: equilibrium measure of ∂T

The existence of such tilings can be proved recursively.

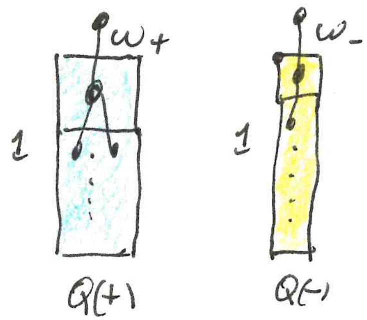


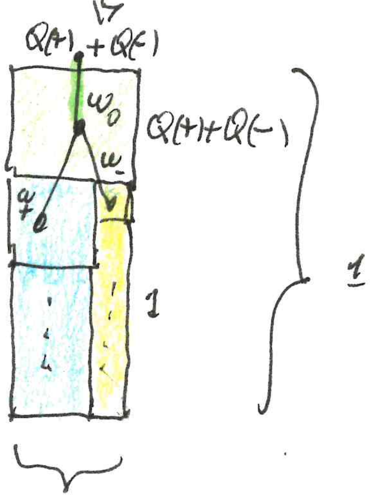
Step 0.



(T_+, ω_+) (T_-, ω_-)

Recursive step.





We juxtapose the tilings for T_+, T_- and produce one for T in the obvious way.

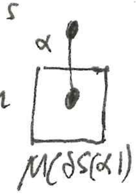
Rescaling sides we obtain a recursive relation for capacities:

$$\boxed{Q(0) = \frac{Q(+) + Q(-)}{1 + Q(+) + Q(-)}}$$

From the picture, for the equilibrium measure μ we obtain:

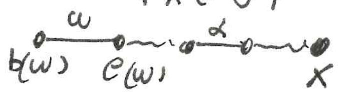
$$\boxed{\mathrm{Cap}(\partial T) = \sum_{\alpha \in E(T)} \mu(\partial S(\alpha))^2 = \mu(\partial T)}$$

where $\boxed{\mu(\partial S(\alpha)) = \sum_{x \in \partial S(\alpha)} \mu(x)}$ is the length of the side of the square corresponding to α



Set $\varphi(\alpha) = \mu(\partial S(\alpha))$; $E(T') \xrightarrow{\varphi} \mathbb{R}^+$

Then (i) $\sum_{\alpha \in [b(\omega), x]} \varphi(\alpha) = 1 \quad \forall x \in \partial T'$



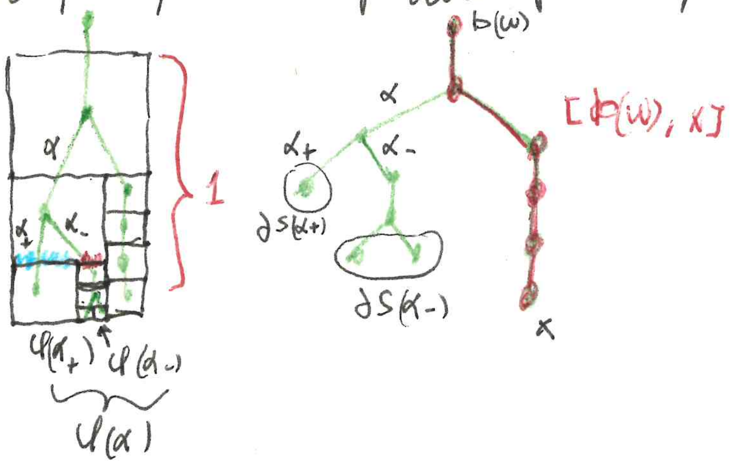
(ii) $\sum_{\alpha \in \partial T'} \varphi(\alpha)^2 = \mathrm{Cap}(\partial T')$

(iii) $\varphi(\alpha) = \varphi(\alpha_+) + \varphi(\alpha_-)$

(we set $\varphi(\alpha_-) = 0$ if the edge α_- is absent).

φ is the equilibrium function for $\partial T'$.

All properties follow from pictures.



Remark. If $\varphi \geq 0$, then (i),(ii) characterize it to be the equilibrium function for $\partial T'$, and (ii) follows.

Potential Theory on (T, ω). $b(\omega) = \sigma$

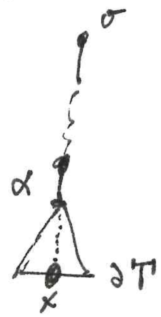
$p = 2$; $k : \partial T' \times E(T') \longrightarrow \mathbb{R}^+$

$$k(x, \alpha) = \chi(\alpha \in [\sigma, x]) = \chi(x \in \partial S(\alpha))$$

$\varphi : E(T') \to \mathbb{R}^+ \longmapsto I\varphi(x) = k\varphi(x) = \sum_{\alpha \in [\sigma, x]} \varphi(\alpha)$

Hardy operator

$\mu \geq 0$ on $\partial T' \longmapsto I^*\mu(\alpha) = \check{k}\mu(\alpha) = \mu(\partial S(\alpha))$



$$V^\mu(x) = I I^* \mu(x) = \sum_{\substack{\alpha \in [\sigma, x] \\ y \in \partial S(\alpha)}} \mu(y)$$

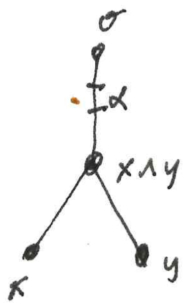
$$= \sum_y \#([\sigma, x] \cap [\sigma, y])\, \mu(y)$$

$$= \sum_y d(x \wedge y)\mu(y)$$

with $x \wedge y \in V(T)$:

$[\sigma, x] \cap [\sigma, y] = [\sigma, x \wedge y]$

and $d(x) = \#[\sigma, x]$



$$\mathcal{E}(\mu) = \sum_\alpha I^*\mu(\partial S(\alpha))^2 =$$

$$= \sum_\alpha \sum_{\alpha \in [\sigma, x]} \mu(x) \sum_{\alpha \in [\sigma, y]} \mu(y) =$$

$$= \sum_{x, y} \mu(x)\mu(y)\, d(x \wedge y)$$

$$= \sum_x V^\mu(x)\, \mu(x)$$

$Cap(\partial T) = \inf\{ \|\varphi\|^2_{\ell^2(E(T))} : \varphi \geq 0,\ I\varphi \geq 1 \text{ on } \partial T\}$

Remarks on $Cap(\partial T)$.

(i) $\inf = \max$ by T. of Weierstrass in the finite tree case

(ii) $\varphi \geq 0$ can be dropped: $\varphi \mapsto \max(\varphi, 0)$ produces better candidates

(iii) We can assume $I\varphi = 1$ on ∂T.

Suppose $I\varphi(x) > 1$.

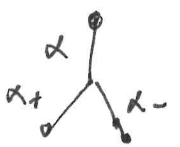
If $\varphi(\alpha_0) > 0$, by reducing it we obtain a better candidate.

Otherwise $\varphi(\alpha_0) = 0$ and $I\varphi(e(\alpha_1)) > 1$. If $\varphi(\alpha_1) > 0$, we reduce it; otherwise ...

Iterate. Since $I\varphi(x) > 1$, $\varphi(\alpha_j) > 0$ $\exists j$.

(iv) Suppose $I\varphi(x) = F(x)$; $\partial T \xrightarrow{F} \mathbb{R}$; is assigned and that φ has minimal $\|\varphi\|^2_{\ell^2(E(T))}$ with these values.

Then $\varphi(\alpha) = \varphi(\alpha_+) + \varphi(\alpha_-)\ \forall \alpha \in E(T)$.



Let $t \in \mathbb{R}$ and consider a variation ψ of φ with:

$\psi(\alpha) = \varphi(\alpha) + t$

$\psi(\alpha_\pm) = \varphi(\alpha_\pm) - t$

$\psi(\beta) = \varphi(\beta)$ for $\beta \neq \alpha, \alpha_\pm$

By minimality, $t = 0$ is a minimum of

$e(t) = (\varphi(\alpha)+t)^2 + (\varphi(\alpha_+)-t)^2 + (\varphi(\alpha_-)-t)^2$;

then

$0 = \frac{1}{2} e'(0) = \varphi(\alpha) - \varphi(\alpha_+) - \varphi(\alpha_-)$.

(v) For $x = e(\alpha) \in \partial T$ set $\mu(x) = \varphi(x)$, a measure.

Then $\varphi(\alpha) = \mu(\partial S(\alpha)) = I^*\mu(\alpha)$.

(vi) μ is the equilibrium measure of the tiling associated with T and

$Cap(\partial T) = Cap(\partial T)$

We already observed that $I\varphi = 1$ on ∂T and (iv) characterize the tiling.

(vii) The extremal φ, μ are unique. Geometrically, this follows from the uniqueness of the tiling (which we obtain by the recursive formulas). We could also use a convexity argument.

· If φ_1, φ_2 are extremal, then $\varphi = \frac{\varphi_1 + \varphi_2}{2}$ satisfies $I\varphi = 1$ on ∂T and

$\|\varphi\|^2_{\ell^2} = \sum_\alpha \left(\frac{\varphi_1(\alpha) + \varphi_2(\alpha)}{2}\right)^2 \leq \sum_\alpha \frac{\varphi_1(\alpha)^2 + \varphi_2(\alpha)^2}{2}$

$= Cap(\partial T)$, with $(=) \Leftrightarrow \varphi_1 = \varphi_2 = \varphi$.

Obs. We might consider T_2, the full dyadic tree, and $E \subseteq T_2'$, with $[o,x] \not\subseteq [o,y]$ $\forall x \neq y$ in E.

$Cap(E) = \inf\{\|\varphi\|^2_{\ell^2} : \varphi \geq 0 \text{ on } E(T_2),\ I\varphi \geq 1 \text{ on } E\} =$

$= Cap(\partial T(E))$ where $E(T(E)) = \bigcup_{x \in E} [o, x]$.

Axiomatic Nonlinear Potential Theory 2.

Proposition (a) $x \mapsto kf(x)$ is l.s.c. on X

(b) $\mu \mapsto \check{k}\mu(\alpha)$ is l.s.c. in the w^*-topology on $M_+(X)$ [positive measures on X]

(c) $\mu \mapsto \mathcal{E}(\mu, f) := \int_X kf \cdot d\mu = \int_M f(\alpha)\, \check{k}\mu(\alpha)\, d\alpha$ is l.s.c. in the w^*-topology.

Note on the w^*-topology on $M_+(X)$.

$M_+(X) := \{\mu \geq 0,$ Borel, $\|\mu\| := \mu(X) < +\infty\}$

$M(X) := \{\mu :$ signed, Borel measure on X, bounded, i.e. $\|\mu\| := |\mu|(X) < +\infty\}$ where $|\mu|$ is the variation of μ:

$|\mu|(E) = \sup\{\sum_{j=1}^{\infty} |\mu(E_j)| : \coprod_{j=1}^{\infty} E_j = X;$ E, E_j Borel measurable$\}$

$C_0(X) := \{\varphi \in C(X) : \lim_{x\to\infty} \varphi(x) = 0\}$

and the condition means:

$\forall \varepsilon > 0\ \exists K_\varepsilon \subseteq X$ compact: $|\varphi(x)| < \varepsilon\ \forall x \in X \setminus K_\varepsilon$

Riesz representation theorem.

$C_0(X)^* \equiv M(X)$ under the duality pairing: $\langle \varphi, \mu \rangle = \int_X \varphi\, d\mu$.

Banach-Alaoglu Theorem.

If V is a Banach space and $B_1(V^*)$ is the unit ball in the dual space V^* of V, then $B_1(V^*)$ is w^*-compact:

$\forall \{l_n\}_{n=1}^{\infty}$ in V^* with $\|l_n\| \leq 1$

$\exists \{l_{n_k}\}_{k=1}^{\infty}$ s.t. $l_{n_k} \xrightarrow[k\to\infty]{} l \in V^*$, $\|l\| \leq 1$;

i.e. $\forall v \in V \Rightarrow \lim_{k\to\infty} l_{n_k}(v) = l(v)$.

Moreover, $\|l\| \leq \liminf_{k\to\infty} \|l_{n_k}\|$.

Exercise. Verify that on $\mathbb{R}$ $\delta_n \xrightarrow[w^*]{} 0$.

Pf of (a) Let $x_0 \in X$ and $\{x_n\}$ in X:

$\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} kf(x_n) = \liminf_{y\to x_0} kf(y)$.

Then, $kf(x_0) = \int_M k(x_0, \alpha) f(\alpha)\, d\alpha$

$\leq \int_M \liminf_{n\to\infty} k(x_n, \alpha) f(\alpha)\, d\alpha$

$\leq \liminf_{n\to\infty} \int_M k(x_n, \alpha) f(\alpha)\, d\alpha$

$= \liminf_{n\to\infty} kf(x_n) = \liminf_{y\to x_0} kf(y)$.

(b) Fact: $h \geq 0$ is l.s.c. on $X \Leftrightarrow \exists h_n \nearrow h$, $h_n \in C_c(X)$.

Let $h_n(x) \uparrow k(x,\alpha)$, $h_n \in C_c(X)$.

Then $\check{k}\mu_m(\alpha) = \int_X k(x,\alpha)\, d\mu_m(x)$

$$= \lim_{n\to\infty} \int_X h_n(x)\, d\mu_m(x) \geq_{\forall n} \int_X h_n(x)\, d\mu_m(x)$$

and $\lim_{m\to\infty} \int_X h_n(x)\, d\mu_m(x) =$

$$= \int_X h_n(x)\, d\mu(x)$$

$$\Rightarrow \varliminf_{m\to\infty} \check{k}\mu_m(\alpha) \geq \lim_{m\to\infty} \int_X h_n(x)\, d\mu_m(x)$$

$$= \int_X h_n(x)\, d\mu(x) \quad \forall n$$

$$\Rightarrow \varliminf_{m\to\infty} \check{k}\mu_m(\alpha) \geq \lim_{n\to\infty} \int_X h_n(x)\, d\mu(x)$$
$$= \check{k}\mu(\alpha).$$

(c) $\varliminf_{n\to\infty} \mathcal{E}(\mu_n, f) = \varliminf_{n\to\infty} \int_M f(\alpha)\, \check{k}\mu_n(\alpha)\, d\alpha$

$$\geq \int_M f(\alpha) \varliminf_{n\to\infty} \check{k}\mu_n(\alpha)\, d\alpha$$

$$\geq \int_M f(\alpha)\, \check{k}\mu(\alpha)\, d\alpha = \mathcal{E}(f,\mu)$$

if $\mu_n \xrightarrow[w^*]{} \mu$. ∎

Capacity is <u>outer regular</u>.

<u>Thm.</u> $\forall E \subseteq X \Rightarrow \mathrm{Cap}(E) = \inf\{\mathrm{Cap}(V) : V \supseteq E$ is open$\}$.

<u>Pf.</u> WLOG $\mathrm{Cap}(E) < +\infty$. For $\varepsilon > 0$ let $f \in L_+(M)$ (measurable, positive): $kf \geq 1$ on E and $\int_M f^p\, dm \leq \mathrm{Cap}(E) + \varepsilon$

kf is l.s.c. $\Rightarrow V = \{kf > 1-\varepsilon\} \supseteq E$ is open and $\mathrm{Cap}(V) \leq \int_M \left(\frac{f}{1-\varepsilon}\right)^p dm \leq \frac{\mathrm{Cap}(E)+\varepsilon}{(1-\varepsilon)^p} \downarrow \mathrm{Cap}(E)$ ∎

Inner regularity is a delicate business.

<u>Obs.</u> The <u>weak capacitary inequality</u> is tautological:

$$\lambda^p \cdot \mathrm{Cap}(\{kf \geq \lambda\}) \leq \int_M f^p\, dm$$

The holy grail in the theory is a strong version of it (<u>Strong Capacitary Inequality</u>):

$$\int_0^{+\infty} \mathrm{Cap}(\{kf > \lambda\})\, d\lambda^p \leq C \cdot \int_M f^p\, dm$$

Sometimes it holds, sometimes it doesn't.

<u>Proposition.</u> $\mathrm{Cap}(E) = 0 \Leftrightarrow \exists f \in L^p_+ : kf = +\infty$ on E.

<u>Pf.</u> ($\Leftarrow$) $\mathrm{Cap}(E) \leq \frac{\int_M f^p\, dm}{N^p}$ $\forall N > 0$.

($\Rightarrow$) $\forall n\ \exists f_n : kf_n \geq 1$ on E and $\int_M f_n^p\, dm \leq \frac{1}{2^n}$

so $\sum \frac{1}{n} f_n = +\infty$ on E and $\int_M \left(\sum \frac{1}{n} f_n\right)^p \leq \left(\sum \frac{1}{n^{p'}}\right)^{p/p'} \cdot \sum \int_M f_n^p < +\infty$ ∎

The notion of negligeability in potential theory is in terms of capacity: a property $P(x)$ holds quasi everywhere (q.e.) on X if $\mathrm{Cap}(\{x : P(x) \text{ fails}\}) = 0$.

If $f = f_+ - f_-$ with $f_\pm \geq 0$, we can write $Kf = Kf_+ - Kf_-$ if $Kf_+ < \infty$ or $Kf_- < \infty$.

<u>Proposition</u> (Egorov-type Theorem).
Let $\{f_n\}$ be Cauchy in $L^p(M)$, $f_n \xrightarrow[L^p]{} f$.
Then $\exists \{f_{n_j}\}$ such that
$Kf_{n_j}(x) \xrightarrow[j\to\infty]{} Kf(x)$ $\forall x \in X \setminus V_\varepsilon$ where V_ε is open and $\mathrm{Cap}(V_\varepsilon) \leq \varepsilon$.

<u>Pf.</u> $g \in L^p \Rightarrow g_+, g_- \in L^p$, hence
$\mathrm{Cap}(\{Kg_+ = +\infty \text{ or } Kg_- = +\infty\}) = 0 =: \mathrm{Cap}(F)$
Apply this to $g = f_n, f$ and use subadditivity: Kf_n, Kf are defined and real ~~outside~~ q.e.
Find a subsequence $\{f_{n_j}\}$ s.t.
$$\int_M |f_{n_j} - f|^p \, dm < \frac{1}{4^{jp}} \quad \text{and set}$$
$E_j = \{x : K(|f_{n_j} - f|)(x) > 2^{-j}\}$, $G_m = \bigcup_{j=m}^{\infty} E_j$
so that $\mathrm{Cap}(E_j) \leq 2^{jp} \int_M |f_{n_j} - f|^p dm \leq 2^{-jp}$
and $\mathrm{Cap}(G_m) \leq \sum_{j=m}^{\infty} 2^{-jp} \xrightarrow[m\to\infty]{} 0$, hence
$\mathrm{Cap}\left(\bigcap_{m=1}^{\infty} G_m\right) = 0$.
If $x \notin F \cup G_m$ then $|Kf_{n_j}(x) - Kf(x)| \leq$
$\leq K(|f_{n_j} - f|)(x) \leq 2^{-j}$ $\forall j \geq m$
$\Rightarrow Kf_{n_j} \xrightarrow[j\to\infty]{} Kf$ uniformly on
$$X \setminus (F \cup G_m) \supseteq X \setminus (W \cup G_m)$$
with $W \supseteq F$ open, $\mathrm{Cap}(W) \leq \varepsilon$.

<u>Proposition</u>. $1 < p < \infty$; $E \subseteq \mathbb{R}^n$.
$$\overline{\Omega_E} = \overline{\{f \geq 0 : Kf \geq 1 \text{ on } E;\ f \in L^p\}}$$
$$\stackrel{*}{=} \{f \geq 0 : Kf \geq 1 \text{ q.e. on } E;\ f \in L^p\} = B$$
<u>Obs.</u> This is important because extremal functions will lie in $\overline{\Omega_E}$.
<u>Pf.</u> We show $B \supseteq \Omega_E$ is closed in L^p.
Let $f_n \in L^p_+$, $Kf_n \geq 1$ on $X \setminus E_n$, $\mathrm{Cap}(E_n) = 0$, $f_n \to f \in L^p_+$ in L^p. Find a subsequence $Kf_{n_j} \to Kf$ uniformly off a set of small capacity, hence q.e. on X.
Thus $Kf \geq 1$ q.e. so $f \in B$.

In the other direction if $f \in L^p_+$ and $kf \geq 1$ on $X \setminus E$ with $\operatorname{cap}(E) = 0$, we can find $h_n \in L^p_+ : \|h_n\|^p_{L^p} \to 0$ s.t. $k h_n = +\infty$ on E.

But $f + h_n \in \Omega_E \xrightarrow[n\to\infty]{} f$ in L^p. ■

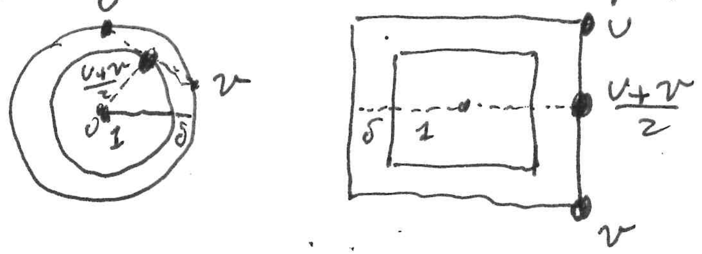
Theorem (existence of extremals). Suppose $\operatorname{cap}(E) < \infty$. Then $\exists!\, f^E \in L^p_+$: $k f^E \geq 1$ q.e. on E and

$$\operatorname{cap}(E) = \int_M (f^E)^p \, dm$$

Pf. The proof relies on a couple of deep results in Banach space theory, which are almost elementary in the Hilbertian case $p = 2$.

Def. A Banach space is uniformly convex if $\forall \varepsilon > 0\ \exists \delta > 0 : \forall u, v \in V$
$\|u\| < 1 + \delta$, $\|v\| < 1 + \delta$, $\left\|\frac{u+v}{2}\right\| \geq 1$
$\Longrightarrow \|u - v\| < \varepsilon$.

In a sense, the boundary of the ball in V is not "too flat".



Exercise. Set $\|(x,y)\|_p = \begin{cases} (|x|^p + |y|^p)^{1/p} & \text{if } 1 \leq p < +\infty \\ \max(|x|, |y|) & \text{if } p = +\infty \end{cases}$ in $\mathbb{R}^2$.
Show that $(\mathbb{R}^2; \|\cdot\|_p)$ is uniformly convex for $p = 2$ (in fact it is for $1 < p < \infty$) and it's not for $p = 1, +\infty$. In fact:

Theorem. L^p is uniformly convex if $1 < p < \infty$.

Theorem. Let $\Omega \subset V$ be closed and convex in a uniformly convex Banach space. Then, there exists a unique $v \in \Omega$:
$\|v\| = \inf\{\|\omega\| : \omega \in \Omega\} =: d$

In fact, if $\Omega \ni \omega_n$ s.t. $\|\omega_n\| \xrightarrow[n\to\infty]{} d$, then $\{\omega_n\}$ is Cauchy in V.

Let's see how this works for Hilbert spaces.

(*) $\|\omega_n + \omega_m\|^2 + \|\omega_n - \omega_m\|^2 = 2(\|\omega_m\|^2 + \|\omega_n\|^2)$

$\Downarrow$ since $(\omega_m + \omega_n)/2 \in \Omega$

$0 \leq \frac{1}{4}\|\omega_n - \omega_m\|^2 = \frac{\|\omega_m\|^2 + \|\omega_n\|^2}{2} - \left\|\frac{\omega_n + \omega_m}{2}\right\|^2$

$\leq \frac{\|\omega_m\|^2 + \|\omega_n\|^2}{2} - d^2 \xrightarrow[m,n\to+\infty]{} 0$

thus an extremal exists and two don't by the same relation (*).

Pf. of the existence of extremals. $\overline{\Omega_E}$ is closed in $L^p(m)$ and it's obviously convex ■

Capacitability.

Choquet Theorem. Let $C : 2^X \to [0, +\infty]$ s.t.

(a) $C(\emptyset) = 0$

(b) $E_1 \subseteq E_2 \Rightarrow C(E_1) \le C(E_2)$

(c) If compact $K_n \downarrow K \Rightarrow C(K) = \lim_n C(K_n)$

(d) If $E_n \uparrow E$ then $C(E_n) \uparrow C(E)$

Then all Suslin (hence, Borel) sets E are capacitable: $C(E) = \sup\{C(K) : K \subseteq E \text{ is compact}\}$

Cap verifies (a), (b). It verifies (c).

If U open, $U \supseteq K$ then $K_n \subseteq U$ $\exists n$.

~~(We can find open $V \subseteq \bar{V}$, compact, with $K \subseteq V \subseteq \bar{V} \subseteq U$.~~ ~~$\{U \setminus K_n\}_{n=1}^{\infty}$ is an open cover of~~

$U \cup \left(\bigcup_{n=1}^{\infty} (X \setminus K_n)\right)$ is an open cover of K_1, hence $K_m \subseteq K_1 \subseteq U \cup (X \setminus K_1) \cup \dots \cup (X \setminus K_n)$ thus $K_n \subseteq U$).

Thus $Cap(K) \le Cap(K_n) \le Cap(U)$ and outer regularity does the job.

Proposition Cap satisfies (d) and if $Cap(E) < \infty \Rightarrow f^{E_n} \to f^E$ in L^p.

We need a Lemma. If V is uniformly convex, $\lim_n \|x_n\| = 1$ and $\lim_{m,n} \left\|\frac{x_n + x_m}{2}\right\| \ge 1 \Rightarrow x_n \xrightarrow[n\to\infty]{} x_0$ in norm $\exists x_0$.

Pf. Fix $\varepsilon > 0$. For $\delta > 0 \Rightarrow \|x_n\| < 1 + \delta$ $\forall n \ge n_0$ and $\left\|\frac{x_n + x_m}{2}\right\| \ge 1 - \delta$ for $m, n > n_0$.

$\Rightarrow \left\{\frac{x_n}{1-\delta}\right\}_{n \ge n_0}$ satisfies the hp. in the def. of uniform convexity $\Rightarrow$

$\|x_n - x_m\| \le \varepsilon$ if $\delta > 0$ is small enough.

Pf. of Proposition $\forall m \ge n \Rightarrow kf^{E_n} \ge 1$ q.e. on E_m

$\Rightarrow Cap(E_m) \le \int_M \left(\frac{f^{E_n} + f^{E_m}}{2}\right)^p dm$

$\Downarrow$

$\sup_\ell Cap(E_\ell) < \infty$

On the other hand $\int_M f^{E_m} dm, \int_M f^{E_n} dm \le \sup_\ell Cap(E_\ell)$

Adjusting ε's and δ's: $f^{E_n} \xrightarrow[n\to\infty]{} f$ in L^p

and $f \in \overline{\Omega(E_n)}$ $\forall n \Rightarrow kf \ge 1$ q.e. on E_n $\forall n$

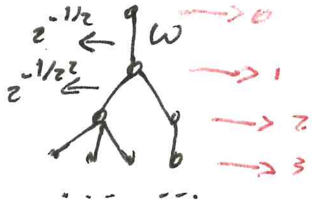
$\Rightarrow kf \ge 1$ q.e. on E, so

$Cap(E) \le \int_M f^p dm = \lim_n \int_M (f^{E_n})^p dm = \lim_n Cap(E_n)$

If $\sup_\ell Cap(E_\ell) = +\infty$ then $Cap(E) = +\infty$ ∎

Trees and their boundaries.

(T, ω) $\quad T = (V(T), E(T))$



$E(T) \ni \alpha \mapsto 2^{-d(e(\alpha), b(\omega))} = 2^{-|\alpha|-1}$

Define a metric ρ on $V(T)$ by $\rho(x,y) = \sum_{\alpha \in [x,y]} 2^{-|\alpha|-1}$

Observe that $\operatorname{diam}_\rho(V(T)) \leq 1$.

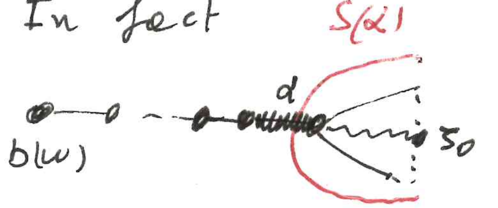
Consider $\overline{T}$: the metric completion of $V(T)$ w.r.t. ρ: $V(T) \subseteq \overline{T}$.

$\partial_\omega T := (\overline{T} \setminus V(T)) \cup \{x \in V(T) \text{ s.t. } x \neq b(\omega) \text{ is a leaf}\}$

is the <u>boundary</u> of (T, ω). Write $\partial_\omega T = \partial T$.

- $\overline{T} \setminus V(T)$ is made up of nonisolated points, while $V(T)$ is the set of isolated points.
- In most applications either $\overline{T} \setminus V(T) = \emptyset$ (finite trees) or $\partial T = \overline{T} \setminus V(T)$ (no leaves $\neq b(\omega)$)
- A basis of open neighborhood is given by $\{\{x\} : x \in V(T)\} \cup \{S(\alpha) = \{y : [y, b(\omega)] \ni \alpha\} : \alpha \in E(T)\}$

- In fact



$S(\alpha) = \{\xi \in \overline{T}:$

$\rho(\xi, \xi_0) \leq 2 \cdot 2^{-|\alpha|-2} \cdot (1 + \frac{1}{2} + \cdots) = 2^{-|\alpha|}$ for some (hence all) $\xi_0 \in \partial S(\alpha)$.

i.e. $S(\alpha) = \{\xi : \rho(\xi, \xi_0) \leq 2^{-|\alpha|}\} = \{\xi : \rho(\xi, \xi_0) < 2^{-|\alpha|+1/2}\}$

is closed and open (d-open).

- Formulas for ρ are easily computed. If $\xi, \zeta \in \partial T = \overline{T} \setminus V(T)$ then $\rho(\xi, \zeta) = 2^{-d(\xi \wedge \zeta)+1}$

where the <u>confluent</u> $\xi \wedge \zeta \in V(T)$ is defined by $[b(\omega), \xi] \cap [b(\omega), \zeta] = [b(\omega), \xi \wedge \zeta]$

- Assume $\partial T = \overline{T} \setminus V(T)$. Then $(\partial T, \rho)$ is a complete, totally disconnected, compact set and $\rho(\xi, \zeta) \leq \max\{\rho(\xi, \eta), \rho(\eta, \zeta)\}$ (<u>ultrametric</u> property).
- Two pictures to be explained, providing two models to keep in mind.