Exercises 1

The boundary of the dyadic tree.

1. Let T_2 be the infinite dyadic tree and ∂T be its boundary; $\zeta \in \partial T$, $\zeta = \{\zeta_j\}_{j=0}^{\infty}$, with $\zeta_j \in E(T)$, $\zeta_0 = \omega$ being the pre-root. Show that

$$\rho_T(\zeta, \xi) = 2^{-\max\{m \geqslant 1 : \zeta_j = \xi_j \text{ for all } j \leqslant 1\}}$$

is the distance on ∂T which is associated to the weight $w(\alpha_j) = 2^{-j-1}$.

Show that $(\partial T, \rho_T)$ is a perfect, totally disconnected set.

Show that it has Hausdorff dimension 1 and that for each $\alpha \in E(T)$:

$$\mathcal{H}^1(\{\zeta : \alpha \in [o, \zeta)\}) = \mathcal{H}^1(\{S(\alpha)\}) = 2^{-d(b(\alpha), o)},$$

where $d(b(\alpha), o)$ is the level of α (prove it for $\alpha = \omega$, then use isometries).

2. To each $\zeta \in \partial T$, $\zeta = \{\zeta_j\}_{j=0}^{\infty}$, associate $\epsilon(\zeta) = \epsilon = \{\epsilon_j\}_{j=1}^{\infty}$, where $\epsilon_j = 0$ if α_j points to the left, and $\epsilon_j = 1$ if α_j points to the right, and let

$$\Lambda(\zeta) = \sum_{n=1}^{\infty} \frac{\epsilon_n}{2^n}.$$

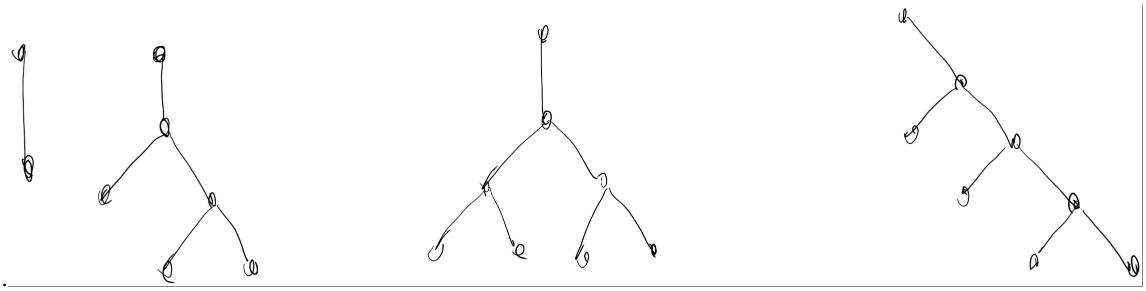
Show that $\Lambda : \partial T \to [0, 1]$ is a 1-Lipschitz map, which is onto, but not $1 - 1$ (hence, not biLipschitz). Where does it it fail to be injective?

Show that $\Lambda : (\partial T, \mathcal{H}^1) \to ([0, 1], dx)$ is measure preserving.

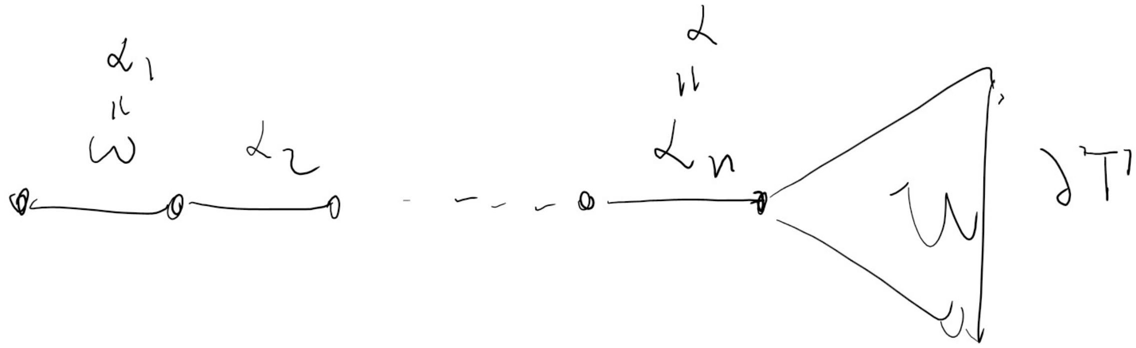
3. Suppose that $\mu \geqslant 0$ is an atomless, Borel measure on $[0, 1]$, and let $\Lambda^*(\mu)(E) = \mu(\Lambda(E))$ for any Borel subset of ∂T. Show that $\Lambda^*(\mu)$ is a well defined Borel measure on ∂T.

Capacity of ∂T for a finite tree T.

* Compute the capacity of ∂T for the following trees, and draw the corresponding square tilings.



* Compute the capacity of ∂T w.r.t. the edge ω knowing its capacity w.r.t. the edge α:



* Compute the capacity of the (infinite) "hairy stick":