THE HEISENBERG GROUP: A TOOLBOX

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Contents

1. The hyperbolic disc and half-plane

Before moving to higher dimension, it is useful to have a picture of some basic objects and phenomena in the complex one-dimensional case for comparison, inspiration, and intuition. Another, deeper reason is that the complex hyperbolic metric in higher dimensions (the Bergman-Kobayashi metric) bears a strong resemblance with one model of non-Euclidean geometry (the Beltrami-Klein model), and contains many copies of a different on (the Riemann-Beltrami-Poincaré one). Knowing these models in some details allows us to skip those proofs and calculations which are just like in the planar case, and concentrate on some few, important features, which really are different.

1.1. **The unit disc, its automorphisms, and the hyperbolic distance.**

1.1.1. *The automorphism group of the complex unit disc.* Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc and $\mathbb{T} = \partial\mathbb{D}$ be the unit circle. The biholomorphic maps from $\mathbb{D}$ to itself (the *automorphisms* of $\mathbb{D}$) have the form:

$$\varphi(z) = e^{is}\frac{a - z}{1 - \overline{a}z}, \tag{1.1}$$

where $a \in \mathbb{D}$ and $s \in \mathbb{R}$. We denote by $\mathcal{M}$ (the *Moebius group*) their family. We denote $\varphi_a = \varphi$ the map in (1.1) when $s = 0$: $\varphi_a \circ \varphi_a = I$ is the identity and $\varphi_a(a) = 0$, $\varphi_a(0) = a$.

A proof that such maps are automorphisms follows from (M1), the first of the two *magic relations* below. The fact that all automorphisms have the form (1.1) will be proved later on.

(M1) We have:
$$1-|\varphi(z)|^2=\frac{|1-\overline{a}z|^2-|z-a|^2}{|1-\overline{a}z|^2}=\frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}.$$
Observe the unexpected symmetry with respect to a,z, that will soon find e geometric explanation.

(M2) About derivatives,
$$\frac{|\varphi'(z)|^2}{1-|\varphi(z)|^2}=\frac{1-|a|^2}{|1-\overline{a}z|^2}\cdot\frac{|1-\overline{a}z|^2}{(1-|a|^2)(1-|z|^2)}=\frac{1}{1-|z|^2}.$$

By (M1), $|\varphi(z)|<1$ for $|z|<1$ and $|\varphi(z)|=1$ for $|z|=1$, showing that φ (injectively) maps $\mathbb{D}$ onto itself.

1.1.2. *The Poincaré metric.* Relation (M2) says that the Riemannian metric
$$ds^2=\frac{4|dz|^2}{(1-|z|^2)^2} \tag{1.2}$$
is invariant under such automorphisms,
$$\frac{|d\varphi(z)|^2}{(1-|\varphi(z)|^2)^2}=\frac{|dz|^2}{(1-|z|^2)^2}. \tag{1.3}$$
It is called the *hyperbolic metric* in $\mathbb{D}$. The corresponding Riemannian distance, the *hyperbolic distance*, is denoted by d,
$$\begin{aligned} L(\gamma) &= \int_p^q\frac{2|dz|}{1-|z|^2},\ \text{where } \gamma:[p,q]\to\mathbb{D},\ \gamma:[p,q]\to\mathbb{D}); \\ d(a,b) &= \inf\{L(\gamma):\ \gamma:[p,q]\to\mathbb{D}\text{ such that }\gamma(p)=a,\ \gamma(q)=b\}. \end{aligned}\tag{1.4}$$
The quantity $L(\gamma)$ is the *hyperbolic length* of γ.

We mention here that the area form associated with the hyperbolic metric, the one which is invariant under automorphisms, is
$$dA(z)=\frac{1}{\pi}\frac{dxdy}{(1-|z|^2)^2}. \tag{1.5}$$
The constant is normalized the way it is usually done in holomorphic function theory. Formula (1.5) follows from a general formula in Riemannian geometry, but it can also be found by invariance alone.

There various geometric ways to parametrize automorphisms.

(i) Given points a,a' in $\mathbb{D}$ and ζ,ζ' in $\mathbb{T}$, there exists a unique $\varphi\in\mathcal{M}$ such that $\varphi(a)=a'$ and $\varphi(\zeta)=\zeta'$.

(ii) Given two triples α,β,γ and α',β',γ' of distinct points on $\mathbb{T}$, following each other in counterclockwise order, there is a unique $\varphi\in\mathcal{M}$ such that $\varphi(\alpha)=\alpha'$, $\varphi(\beta)=\beta'$, $\varphi(\gamma)=\gamma'$.

(iii) Given points a,a' in $\mathbb{D}$ and nonzero vectors v,v' based at a,a', respectively, there a unique $\varphi\in\mathcal{M}$ such that $\varphi(a)=a'$ and $\varphi_*(z)v=\lambda v'$ for a positive constant λ. Here $\varphi_*(z)=J\varphi(z)$ is the Jacobian of φ when considered as a map on $\mathbb{R}^2$. All this can be rephrased in the language of manifolds, that we will keep in the background whenever it is possible.

Property (iii) says that $\mathbb{D}$ is, from the viewpoint of the action of $\mathcal{M}$ on it, *homogeneous and isotropic* (all points and all directions are indistinguishable). This and the fact that $\mathcal{M}$ is a group of isometries for the metric ds^2, implies that $(\mathbb{D},ds^2)$ has constant curvature, a fact that we will verify directly later in this section.

1.2. **Schwarz Lemma.** We state without proof the well known

Lemma 1. *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, $f(0) = 0$. Then,*

(i) $|f(z)| \leq |z|$ *and equality holds for some $z \neq 0$ only if $f(z) = \alpha z$ for some α in $\mathbb{T}$.*
(ii) $|f'(0)| \leq 1$, *with equality if and only if $f(z) = \alpha z$ for some α in $\mathbb{T}$.*

By moving around points by means of automorphisms, we can remove the special role of the origin. For holomorphic $f : \mathbb{D} \to \mathbb{D}$.

(i) For $z, w \in \mathbb{D}$,

$$\left| \frac{f(z) - f(w)}{1 - \overline{f(w)} f(z)} \right| \leq \left| \frac{z - w}{1 - \overline{w} z} \right|, \tag{1.6}$$

and equality holds for a couple of distinct z, w if and only if $f \in \mathcal{M}$ is an automorphism.

(ii) For $z \in \mathbb{D}$ we have

$$\frac{|f'(z)|^2}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}, \tag{1.7}$$

with equality for some z if and only if $f \in \mathcal{M}$ is an automorphism.

To see why (1.6) holds, consider automorphisms $\varphi = \varphi^{-1}$ switching w and 0, and $\psi = \psi^{-1}$ switching $f(w)$ and 0, so that $|\psi(f(\varphi(\xi)))| \leq |\xi|$ by Schwarz Lemma. Replace $\xi = \varphi^{-1}(z) = \frac{z - w}{1 - \overline{w} z}$, so that $f(\varphi(\xi)) = f(z)$, and

$$\begin{aligned} \left| \frac{f(w) - f(z)}{1 - \overline{f(w)} f(z)} \right| &= |(\psi \circ f \circ \varphi)(\xi)| \\ &\leq |\xi| = \left| \frac{z - w}{1 - \overline{w} z} \right|. \end{aligned}$$

In case of equality, $(\psi \circ f \circ \varphi)(\xi) = \alpha \xi =: h(\xi)$, for some $\alpha \in \mathbb{T}$, then $f = \psi \circ h \circ \varphi$ belongs to $\mathcal{M}$.

Inequality (1.7) follows by taking the limit $w \to z$. It can be rephrased in geometric terms.

Theorem 1. *Holomorphic functions $f : \mathbb{D} \to \mathbb{D}$ contract the hyperbolic metric, $d(f(z), f(w)) \leq d(z, w)$.*

Moreover, if equality holds for a couple of distinct points, then f is an automorphism (hence, an isometry).

Using the expression for the hyperbolic metric, in fact, we see that

$$f_*(ds^2) := \frac{4|df(z)|^2}{(1 - |f(z)|^2)^2} = \frac{4|f'(z)|^2 |dz|^2}{(1 - |f(z)|^2)^2} \leq \frac{4|dz|^2}{(1 - |z|^2)^2}.$$

If γ is the geodesic joining z and w,

$$d(f(z), f(w)) \leq L(f \circ \gamma) \leq L(\gamma) = d(z, w).$$

The equality case will be best discussed after we have a geometric interpretation for (1.6).

An application of Schwarz Lemma is the characterization of the automorphisms, which must all have the form (1.1). Let in fact f be a biholomorphism of $\mathbb{D}$. We can assume that $f(0)=0$. Since f^{-1} is similarly a biholomorphisms,

$$1=(f^{-1}\circ f)'(0)=(f^{-1})'(0)f'(0),$$

so that, by Schwarz Lemma, $|f'(0)|=|f^{-1}(0)|=1$, and by the equality case, $f(z)=\alpha z$.

1.3. **Geodesics and hyperbolic distance.**

1.3.1. *Geodesics and distance.* We compute first the distance between 0 and $r\in(0,1)$. Let γ be a curve joining them,

$$\begin{aligned} L(\gamma) &= \int_\gamma \frac{2|dz|}{1-|z|^2} \le \int_\gamma \frac{d|dx|}{1-x^2} \le \int_0^r \frac{2dx}{1-x^2} = \int_0^r \left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx \\ &= \log\frac{1+r}{1-r} \\ &= L([0,r]), \end{aligned}$$

where $[0,r]$ is the segment joining 0 and r. Moving things around by automorphisms, and recalling that (i) fractional transformations map straight lines and circles to straight lines and circles; (ii) they are obviously conformal, we immediately find all geodesics and an explicit expression for the distance function.

Theorem 2. (i) *The geodesics for the hyperbolic geometry in the disc,* $(\mathbb{D},ds^2)$, *are segment of straight lines and arcs of circles whhich are perpendicular to* $\mathbb{T}$.

(ii) *The distance between* $z,w\in\mathbb{D}$ *is*

$$d(z,w)=\log\frac{1+\left|\frac{z-w}{1-\overline{w}z}\right|}{1-\left|\frac{z-w}{1-\overline{w}z}\right|}. \tag{1.8}$$

1.3.2. *Some history.* The hyperbolic disc and its higher dimensional analogs are a central objects in the history of non-Euclidean geometries. [1] The hyperbolic metric first appears in Riemann's 1854 Habilitationsschrift (in fact, it is the only formula which appears in it!), and it was carefully shown by Beltrami in 1868 that it provided a model for Lobachevskyan non-Euclidean geometry:

- the hyperbolic plane is homogeneous and isotropic;
- it is orientable;
- given a geodesic γ and a point P outside it, there are *infinitely many* geodesics passing through P and not intersecting γ.

In 1882 Poincaré used the disc model to study Fuchsian groups, and popularized it in his influential La Science et l'Hypothése.

[1] Riemann, https://www.maths.tcd.ie/pub/HistMath/People/Riemann/Geom/; Beltrami, Teoria fondamentale degli spazii di curvatura costante; Poincaré, Théorie des groupes fuchsiens

1.3.3. *The pseudo hyperbolic metric.* Let us go back to the distance. The quantity

$$\delta(z, w) := |\varphi_w(z)| = \left|\frac{z-w}{1-\overline{w}z}\right| \tag{1.9}$$

appearing in the invariant Schwarz Lemma, is related to the hyperbolic distance via the relations

$$d = \log\frac{1+\delta}{1-\delta};\ \delta = \tanh(\delta/2). \tag{1.10}$$

The function tanh is increasing, concave, $\tanh(0) = 0$, hence, by a basic general result, δ defines a distance because d does.

Lemma 2. *The quantity $\delta(z, w)$ defines a distance on $\overline{\mathbb{D}}$.*

The distance δ is named *pseudo-hyperbolic distance*: it is invariant under the action of $\mathcal{M}$, two points in $\mathbb{D}$ have distance which is bounded by one, and it is clearly not a Riemannian distance. As a matter of fact, d is (one half of) the *length distance* associated to δ. If $\gamma : [p, q] \to \mathbb{D}$ is a curve, in fact,

$$L(\gamma) = 2\sup\left\{\sum_{j=1}^{n}\delta(\gamma(t_j), \gamma(t_{j-1})) :\ p = t_0 < t_1 < \cdots < t_{n-1} < t_n = q\right\}.$$

We will see analogous phenomena in higher dimensions.

1.3.4. *Hyperbolic discs.* Let us close with some remarks on metric circles ("intrinsic circles"). The ones having center in the origin are

$$\partial B(0, R) = \left\{z :\ R = d(0, z) = \frac{1+|z|}{1-|z|}\right\}, \tag{1.11}$$

which is in fact a circle in the disc (an "extrinsic circle"), having center at 0. Since automorphisms map Euclidean circles to Euclidean circles, we have all metric circles are Euclidean circles, although their metric center if not the Euclidean center. Euclidean and metric centers are however not too far, as the following calculation shows. Fix $R > 0$ (the hyperbolic radius) and find $r \in (0, 1)$ such that $R = \frac{1+r}{1-r}$. Consider then $\varphi(z) = \frac{z+r}{1-rz}$, $\varphi^{-1}(z) = \frac{z-r}{1-rz}$, mapping 0 to r and $-r$, respectively. Fix now $c \in (0, 1)$. The (extrinsic) diameter of $B(c, R)$ on the real line is $[\varphi^{-1}(c), \varphi(c)] = \left[\frac{c-r}{1-rc}, \frac{c+r}{1+rc}\right]$, having Euclidean center

$$x(c) = \frac{1}{2}\left(\frac{c+r}{1+rc} + \frac{c-r}{1-rc}\right) = \frac{1-r^2}{1-r^2c^2}c \sim c \text{ as } c \to 0, 1,$$

which is not too far from the metric center. About the Euclidian radius, it is

$$\rho(c) = \frac{1}{2}\left(\frac{c+r}{1+rc} - \frac{c-r}{1-rc}\right) = \frac{r}{1-r^2c^2}(1-c^2) \sim \frac{2r}{1-r^2}(1-c) \text{ as } c \to 0,$$

and the Euclidean distance from the metric disc and $\mathbb{T}$ is

$$1 - \frac{c+r}{1+rc} = \frac{1-r}{1+rc}(1-c) \sim \frac{1-r}{1+r}(1-c) \text{ as } c \to 0.$$

1.3.5. *Summary.* At the end of this subsection, we have-learned a few things.

- There is a metric which is invariant under automorphisms, and we can explicitly compute the distance between two points by (1.8).
- The set of geodesics is the set of the Euclidean straight lines and circles which are perpendicular to $\mathbb{T}$.
- The set of the metric circles coincide with the Euclidean circles contained in $\mathbb{D}$.
- Metric discs having fixed radius $R = \log\frac{1+r}{1-r}$ and center c are approximatively Euclidean discs having the same center c and Euclidean radius and distance to $\mathbb{T}$ comparable to $1-c$.

We will later see that the other Euclidean straight lines and circles have an intrinsic geometric interpretation as well.

1.4. **The pseudo-hyperbolic distance and the Hardy space.** It is noteworthy that the pseudo-hyperbolic metric arises in connection with Hardy space theory. The *Hardy space* $H^2 = H^2(\mathbb{D})$ is populated by complex power series

$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$$

such that

$$\|f\|_{H^2}^2 := \sum_{n=0}^{\infty} |\widehat{f}(n)|^2 < \infty. \tag{1.12}$$

If $f \in H^2$, the series converges in $\mathbb{D}$ and by L^2 Fourier theory

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \frac{1}{2\pi}\int_{-\pi}^{+\pi} |f(e^{it})|^2 dt = \lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{+\pi} |f(e^{it})|^2 dt.$$

The inner product in H^2 is defined by

$$\langle f, g\rangle_{H^2} = \sum_{n=0}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)} = \lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(e^{it})\overline{g(e^{it})} dt. \tag{1.13}$$

The Hardy space is a *reproducing kernel Hilbert space* (RKHS): for each z in $\mathbb{D}$ there is k_z in $H = H^2$, the *kernel function* at z, such that

$$f(z) = \langle f, k_z\rangle_H. \tag{1.14}$$

The *reproducing kernel* $k = k(z,w)$ is defined as

$$k(z,w) = k_w(z) = \langle k_w, k_z\rangle_H. \tag{1.15}$$

Many Hilbert function spaces H have reproducing kernel. We deduce the explicit expression of k_z for H^2 from a simple calculation,

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \widehat{f}(n) z^n = \sum_{n=0}^{\infty} \widehat{f}(n)\overline{\overline{z}^n} \\ &= \langle f, k_z\rangle_{H^2}, \end{aligned}$$

with

$$k_z(w) = \sum_{n=0}^{\infty} \widehat{k_z}(n) w^n$$

$$= \sum_{n=0}^{\infty} \overline{z}^n w^n$$
$$= \frac{1}{1-\overline{z}w}.$$

To any RKHS H of functions defined on some space X one can associate the *Gleason distance* D_H on X,

$$D_H(z,w) := \sqrt{1 - \frac{|k(z,w)|^2}{k(z,z)k(w,w)}}. \tag{1.16}$$

For some properties and applications of the Gleason distance (including a proof of the triangle property) see [ARSW] and the references therein.

In the case of the Hardy space, using the relation (M1),

$$D_{H^2}(z,w)^2 = 1 - \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} = |\varphi_z(w)|^2 = \delta(z,w)^2, \tag{1.17}$$

the Gleason distance the pseudo-hyperbolic distance!

1.5. The upper half-plane, cones and horocycles.

1.5.1. *Automorphisms and the hyperbolic metric in $\mathbb{C}_+$ coordinates.* Some geometric features of hyperbolic geometry are best seen after changing coordinates in $\mathbb{C}_+ = \{w = u + iv : \ v > 0\}$, having as boundary $\mathbb{R} \cup \{\infty\}$. The change of coordinates is by means of the *Caley map* ψ,

$$\psi(z) = i\frac{1-z}{1+z}, \ z = \pi^{-1}(w) = \frac{i-w}{i+w}, \tag{1.18}$$

and $\psi : 0, 1, -1 \mapsto i, 0, \infty$, respectively, $\psi(e^{it}) = \tan(t/2)$.

We list the new form of the hyperbolic objects we have seen so far after the change of coordinates, .

- The automorphism group of $\mathbb{C}_+$ is $SL(2,\mathbb{R})$, the group of the Moebius maps $\xi = \frac{aw+bw}{cw+dw}$ with real entries such that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$. The group is generated by
  - (Tr) the group of translations $w \mapsto w + b$, which is $(\mathbb{R}, +)$;
  - (Dil) the group of dilations $w \mapsto \lambda w$, which is multiplicative $((0,+\infty), \cdot)$;
  - (In) the group $\{I, j\}$, where $j(w) = -1/w$ is the inversion with respect to the origin, which is isomorphic to $\mathbb{Z}_2$.
- Since $\frac{dz}{dw} = \frac{-2i}{(i+w)^2}$, the hyperbolic metric becomes, with $w = u + iv$,

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2} = \frac{|dw|^2}{v^2}. \tag{1.19}$$

  Observe that the hyperbolic metric restricted to the *horocycles* is a rescaled version of the Euclidean metric, $\frac{du}{v}$.
- Hyperbolic geodesics are semicircles orthogonal to $\mathbb{R}$, or half lines $\tau \mapsto a + i\tau$, $\tau > 0$.
- The pseudo-hyperbolic distance between $z, w \in \mathbb{C}_+$ has an especially geometric expression,

$$\delta(z,w) = \left|\frac{z-w}{z-\overline{w}}\right|, \tag{1.20}$$

so that

$$d(z,w)=\log\frac{1+\left|\frac{z-w}{z-\overline{w}}\right|}{1-\left|\frac{z-w}{z-\overline{w}}\right|}.$$

- The invariant form of Schwarz Lemma for holomorphic maps $f:\mathbb{C}_+\to\mathbb{C}_+$, is

$$\frac{|f'(w)|}{Im(f'(w))}\le\frac{1}{Im(w)},\ \left|\frac{f(z)-f(w)}{f(z)-\overline{f(w)}}\right|\le\left|\frac{z-w}{z-\overline{w}}\right|. \tag{1.21}$$

- The invariant area form is $\frac{1}{\pi}\frac{dudv}{v^2}$.

All these properties easily follow from the corresponding ones in the disc model, just by changing coordinates $w=\psi(z)$.

1.6. **The boundary of the upper-half plane, and of the disc, from the intrinsic viewpoint.** The boundary $\mathbb{T}$ of the unit disc $\mathbb{D}$ is well visible to the Euclidean observer, but it is at infinite distance from the hyperbolic one. Nonetheless, the circle is within reach of the pseudo-hyperbolic observer, which is a hyperbolic observer with high acceleration, in a way. In fact, $\overline{\mathbb{D}}$ is clearly the completion of $\mathbb{D}$ with respect to the pseudo-hyperbolic metric. After changing coordinates, we might replace $\mathbb{D}$ by $\mathbb{C}_+$, and $\mathbb{T}$ by $\mathbb{R}\cup\{\infty\}$.

A more metric viewpoint is the following. Consider the set Γ of the oriented geodesic in $(\mathbb{C}_+,ds^2)$, parametrized by arclength, and say that $\gamma\sim\eta$ if

$$\limsup_{t\to+\infty}d(\gamma(t),\eta(t))<\infty.$$

Observe that this relation does not depend on the position of η,γ at $t=0$, since

$$|d(\gamma(t+t_0),\eta(t))-d(\gamma(t),\eta(t))|\le d(\gamma(t+t_0),\gamma(t))=t_0.$$

The relation $\sim$ is clearly an equivalence one.

Theorem 3. *There is a bijection between the equivalence classes in $\Gamma/\sim$ and the points of $\mathbb{T}$:*

$$\gamma\sim\eta \text{ if and only if } \lim_{t\to+\infty}\gamma(t)=\lim_{t\to+\infty}\gamma(t)\in\mathbb{T}.$$

Moreover, there is a dychotomy:

$$\begin{aligned}\gamma\sim\eta\quad&\iff\quad\lim_{t\infty}d(\gamma(t),\eta(t))=0;\\ \gamma\nsim\eta\quad&\iff\quad\lim_{t\infty}d(\gamma(t),\eta(t))=+\infty.\end{aligned} \tag{1.22}$$

In fact, $d(\gamma(t),\eta(t))$ decreases or increases exponentially.

For the proof, after moving things by automorphisms, we might suppose the $\gamma(\tau)=\tau$, $\gamma:(0,\infty)\to\mathbb{C}_+$. Suppose first that $\eta(\tau)=a+i\tau$, with $a\in\mathbb{R}$, so that η and γ have the same (extrinsic) endpoint $\gamma(+\infty)=\eta(+\infty)=\infty\in\partial\mathbb{C}_+$. We do not have parametrized γ and η by hyperbolic arclength, but this can be done by replacing $\tau=\log(t)$, since $d(\gamma(t),\gamma(1))=\int_1^t\frac{dv}{v}=\log(t)=\tau$.

Then,

$$\begin{aligned}\delta(\eta(t),\gamma(t))&=\left|\frac{(a+it)-(it)}{(a+it)-(-it)}\right|\\&=\left|\frac{a}{\sqrt{a^2+4t^2}}\right|\end{aligned}$$

$$\sim \quad \frac{a}{2t} \to 0 \textbf{ as } t \to \infty,$$

hence, $d(\eta(t), \gamma(t)) = 2\mathrm{ArcTanh}(\delta(\eta(t), \gamma(t))) \to 0$. We also obtain the exponential decrease,

$$d(\eta(t), \gamma(t)) \sim \frac{a}{t} = ae^{-\tau}.$$

If γ is the same and η has an extrinsic endpoint in $\mathbb{R}$, then

$$\delta(\gamma(t), \eta(t)) = \left| \frac{\eta(t) - (it)}{\eta(t) - (-it)} \right| \to 1 \text{ as } t \to \infty,$$

hence, $\lim_{t\to\infty} d(\gamma(t), \eta(t)) = \infty$. It is easy to see that the distance between the geodesics increase exponentially.

A weak point of the considerations above is that we do not provide a good topology on $\mathbb{T}$. In fact, the restriction of the pseudohyperbolic metric to $\mathbb{T}$ is trivial. For $z, w \in \mathbb{T}$,

$$\delta(z, w) = 0 \textbf{ if } z = w, \ \delta(z, w) = 1 \textbf{ if } z \neq w.$$

This is to be expected, since δ is invariant under an unreasonably large family of transformations ($\mathbb{T}$ has dimension one, while the automorphism group of $\mathbb{D}$ is three-dimensional). There is another way to define a *boundary* for the hyperbolic space $(\mathbb{D}, ds^2)$, or for its "flat model" $(\mathbb{C}_+, ds^2)$, which is based on the introduction of Gromov's *visual metric*, which in our case coincides with the Euclidean one. It requires fixing a base point, or a base (oriented) geodesic, hence it is not biholomorphically invariant as a metric (it is in topological terms).

1.6.1. *Metric cylinders are extrinsic cones.* Consider now $c > 0$ fixed, and the half lines $v = \pm cu$ in $\mathbb{C}_+$, and their bisectrix $u = 0$. The dilation group leaves all of them invariant, and maps circles centered at $u = 0$ and tangent to $v = \pm cu$ to circles of the same kind; in particular, all such circles are isometric to each other. As a consequence, there is $d(c) > 0$ (which can be explicitly calculated) so that the points on $v = \pm cu$ are those having hyperbolic distance $d(c)$ from the half-line $u = 0$,

$$\{w = u + ic|u| : \ u \in \mathbb{R}\} = \{w \in \mathbb{C}_+ : \ d(w, \{iv : \ v > 0\}) = d(c)\}.$$

Moving the statement around by automorphisms, and back to the unit circle through ψ^{-1}, we have an interpretation for arcs of circles meeting $\mathbb{T}$ at a fixed angle α.

Proposition 1. *Let γ be a geodesic in $\mathbb{D}$ or in $\mathbb{C}_+$, fix $0 < \alpha < \pi/2$. Then, there is a constant $d(\alpha) > 0$, independent of γ with the following property.*

Consider two arcs $\gamma_\pm(\alpha)$ of circles/straight lines meeting the boundary at the same points where it meets γ, and making an angle α with γ.

Then, $\gamma_-(\alpha) \cup \gamma_+(\alpha)$ is the set of the points having distance $d(\alpha)$ from γ,

$$z \in \gamma_-(\alpha) \cup \gamma_+(\alpha) \textbf{\textit{ if and only if }} \inf\{d(z, w) : \ w \in \gamma\} = d(\alpha).$$

The function $d(\alpha)$ is continuous and strictly increasing, $d(\alpha) \to 0$ as $\alpha \to 0$, and $d(\alpha) \to \infty$ as $\alpha \to \pi/2$.

In other terms, extrinsic "cones" in $\mathbb{C}_+$ are intrinsic "cylinders". This fact, in a more or less direct way, underlies a number of concepts, objects, and results in analysis. Nontangential convergence, for instance, means convergence within a hyperbolic cylinder.

1.6.2. *The Busemann function, and horocycles.* We noted that the hyperbolic metric on $\mathbb{C}_+$, when restricted to the line $\{w = u + ih : u \in \mathbb{R}\}$, is a rescaling of the Euclidean metric. This might be mere coincidence: any (rectifiable) curve starting and ending at the boundary has infinite length, hence it can be parametrized unit speed on the whole real line. The same phenomenon occurs in n real dimensions, where we deal with an $(n-1)$-dimensional surface, which is generically non-Euclidean, so we start suspecting that there must be some intrinsic geometric explanation.

Fix a an oriented, unit speed geodesic $\gamma : \mathbb{R}$, and the point $\gamma(0)$ on it, and for $z \in \mathbb{C}_+$ define the *Busemann function* B_γ as

$$B_\gamma(z) = \lim_{t\to\infty}[d(\gamma(t), x) - d(\gamma(t), \gamma(0))] = \lim_{t\to\infty}[d(\gamma(t), x) - t)]. \tag{1.23}$$

The rough idea is that we have a degenerate metric circle having center $\gamma(+\infty)$ and containing $\gamma(0)$, then we ask what if z lies inside or outside the circle, and at which distance from it.

Theorem 4. *In the disc model, the Busemann function is constant on circles which are tangent to* $\mathbb{T}$*.*

We call such circles *horocycles* for the hyperbolice metric. Actually, the proof will say more. We move to half-plane coordinates, and we can assume that the geodesic is $\gamma(t) = it$ and $\gamma(1) = i$. It is not unit speed, but we can recover the length parameter as $\tau = \log(t)$.

$$\begin{aligned} \left|\frac{a+ib-it}{a+ib+it}\right| &= \sqrt{\frac{a^2+(b-t)^2}{a^2+(b+t)^2}} \\ &= \sqrt{\frac{1-\frac{2b}{t}+O(\frac{1}{t^2})}{1+\frac{2b}{t}+O(\frac{1}{t^2})}} \\ &= \frac{1-\frac{b}{t}+O(\frac{1}{t^2})}{1+\frac{b}{t}+O(\frac{1}{t^2})} = \\ &= 1-\frac{2b}{t}+O(\frac{1}{t^2}), \end{aligned}$$

Using the formula for the hyperbolic distance in $\mathbb{C}_+$,

$$\begin{aligned} d(a+ib,t) &= \log\frac{1+1-\frac{2b}{t}+O(\frac{1}{t^2})}{1-\left(1-\frac{2b}{t}+O(\frac{1}{t^2})\right)} \\ &= \log 2 - \log\left(\frac{2b}{t}\right) + o(1) \\ &= \log(t) - \log(b) + o(1), \end{aligned}$$

i.e.

$$B_{\gamma(t)=it}(a+ib) = \log(1/b). \tag{1.24}$$

We list here a few consequences of the above calculations together with invariance of the metric.

- In $\mathbb{D}$ the horocycles are the circles tangent to $\mathbb{T}$; in $\mathbb{C}_+$, we add the straight lines which are parallel to $\mathbb{R}$. We say the point of tangence is the *center* of the horocycle.

- The distance between two horocycles having the same center is constant. Moreover, all geodesics having $\zeta \in \mathbb{T}$ as endpoint at infinity are orthogonal to all horocycles having center at ζ.

1.6.3. *The Hardy space $H^2(\mathbb{C}_+)$.* The *Hardy space* in the upper-half plane contains those holomorphic extensions $f : \mathbb{C}_+ \to \mathbb{C}$ of functions $f : \mathbb{R} \to \mathbb{C}$ in $L^2(\mathbb{R})$ such that $\widehat{f}(\omega) = 0$ for $\omega < \infty$. Here we use the normalization

$$\widehat{f}(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx. \tag{1.25}$$

The values of $f \in H^2$ can be reconstructed by:

$$f(z) = \frac{1}{2\pi} \int_0^{+\infty} \widehat{f}(\omega) e^{i\omega z} d\omega, \text{ with } \frac{1}{2\pi} \int_0^{+\infty} |\widehat{f}(\omega)|^2 d\omega. \tag{1.26}$$

The reason why the requirement that $\widehat{f}$ is supported in $[0.+\infty)$ is that $\left|e^{i\omega z}\right| = e^{-\omega y}$ decays exponentially in $y > 0$ for $\omega > 0$, while it increases exponentially for $\omega < 0$. We set

$$\begin{aligned} \|f\|^2_{H^2(\mathbb{C}_+)} &= \frac{1}{2\pi} \int_0^{+\infty} |\widehat{f}(\omega)|^2 d\omega \\ &= \sup_{y>0} \frac{1}{2\pi} \int_0^{+\infty} |\widehat{f}(\omega)|^2 e^{-2\omega y} d\omega = \sup_{y>0} \int_{-\infty}^{+\infty} |f(x+iy)|^2 dx. \end{aligned} \tag{1.27}$$

It is also true that if the last term in the chain of equalities (1.27) is finite, then $f \in H^2$ and its norm coincides with the second term in the chain.

The space $H^2(\mathbb{C}_+)$ is a reproducing kernel space with kernel

$$k_z(w) = \frac{1}{2\pi i} \frac{1}{w - \overline{z}}. \tag{1.28}$$

In fact, in analogy with what we did for the Hardy space on the disc,

$$f(z) = \int_0^{+\infty} \widehat{f}(\omega) \overline{e^{-i\omega \overline{z}}} d\omega = \langle f, k_z \rangle_{H^2}$$

where

$$\begin{aligned} k_z(w) &= \frac{1}{2\pi} \int_0^{+\infty} e^{-i\omega \overline{z}} e^{i\omega w} d\omega \\ &= \frac{i}{2\pi} \frac{1}{w - \overline{z}}. \end{aligned}$$

Observe that $\|k_z\|^2_{H^2} = k_z(z) = \frac{1}{4\pi Im(z)}$, so that the Gleason distance between $z = x + iy$ and $w = u + iv$ is:

$$D(z,w)^2 = 1 - \frac{4vy}{|w - \overline{z}|^2} = \frac{|w - z|^2}{|w - \overline{z}|^2} = \delta^2(z,w),$$

as in the disc case.

1.7. **Curvature.** For a *conformal metric* $ds^2 = e^{\sigma(z)}|dz|^2$ the Gaussian curvature is computed

$$\kappa = -\frac{\Delta\sigma}{2e^{2\sigma}}. \tag{1.29}$$

When we plug is the expression for the hyperbolic metric, we obtain $\kappa = -1$.

1.8. **Dyadic tilings.** The hyperbolic half-plane can be tiled by isometric (extrinsic) squares, which are also approximate metric discs. This picture is most beloved by harmonic analysts and other species of mathematicians.

Consider the mesh
$$\frac{m+i}{2^n},\ m,n\in\mathbb{Z}.$$
Observe that we can move from point to point of the mesh by compositions of dilations $z\mapsto z/2^n$ and $z\mapsto z+m$. In particular, the squares Q having South-West corner $\frac{m+i}{2^n}$ and side $\frac{1}{2^n}$ are isometric, they have disjoint interiors, and they tile the whole half-plane. [Add a picture here].

It is an instructive exercise proving the following facts.

- There are constants $c<C$ so that for each square Q in the tiling there are points $z(c)$ and $z(C)$ such that
$$B(z(c),c)\subset Q\subset B(z(C),C).$$
Actually, we might even choose c,C so that $z(c)=z(C)$.
- Let z,w be points of the half-plane. Then,

$$d(z,w)+1\approx(\text{ the minimun number of squares one has to cross going from } z \text{ to } w\ )+1.$$

From the tiling we have an intuition, and sometimes we can prove "by pictures", several properties of hyperbolic geometry. For instance, one can easily see that the hyperbolic area of hyperbolic disc grows exponentially with the raius.

1.9. **The projective model and its geodesics.** Earlier in 1868, Beltrami had come with a different model of nonEuclidean geometry, which was incidentally the first to be produced, and that it is known as the Kelin model. We discuss it here because we will find an analog of it, and many copies of it, when we consider the complex ball. On the unit disc $\mathbb{D}\ni z=(x,y)$, considered as a subset of $\mathbb{R}^2$, consider the Riemannian metric

$$ds^2=\frac{(z\cdot dz)^2}{(1-|z|^2)^2}+\frac{|dz|^2}{1-|z|^2}. \tag{1.30}$$

Beltrami had probably come up with this expression while classifying all two-dimensional Riemannian metrics in which all geodesics where, in some coordinate system, straight lines. He had found, back in 1865, that this property is peculiar of surfaces with constant Gaussian curvature, and although back then he had not written (1.30) in the article, we can guess that he had it in his notebooks.

The metric coincides with the Euclidean metric $|dz|^2$ at the origin, it is invariant under rotation around the origin, and changing x to y and viceversa, and also under the projective transformations:

$$\varphi(x,y)=\left(\frac{x-r}{1-rx},\frac{\sqrt{1-r^2}y}{1-rx}\right). \tag{1.31}$$

In fact, we could define the metric at z by pushing forward the Euclidean metric $|dw|^2$ from $w=0$ by a projective map like (1.31) followed by a rotation:

$$\begin{aligned} d((r,0)+dz,(r,0))^2 &= d(\varphi((r,0)+dz),\varphi((r,0)))^2= \\ &= |J\varphi((r,0))dz|^2 \\ &= \frac{dx^2}{(1-r^2)^2}+\frac{dy^2}{1-r^2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(rdx)^2}{(1-r^2)^2} + \frac{dx^2+dy^2}{1-r^2} \\
&= ds^2,
\end{aligned}
$$

with $z=(r,0)$. By rotational invariance, the same calculation holds for all z. Once we have the explicit expression, it is just a matter of calculations verifying that it is invariant under compositions of rotations, permutations of coordinates, and projective transformations of $\mathbb{D}$. Moreover, it is easy to see that the same calculations, hence, the same conclusions, hold in the unit ball of $\mathbb{R}^n$.

When we refer to (1.31) as a projective transformation of the unit disc (or ball, if we pick $y\in\mathbb{R}^{n-1}$), we mean that is maps $\mathbb{D}$ onto itself bijectively, and that it maps straight lines to straight lines. This easily verified, since if

$$
u=\frac{x-r}{1-rx},\ v=\frac{\sqrt{1-r^2}y}{1-rx},\quad \text{and } \alpha u+\beta v=\gamma, \tag{1.32}
$$

then x and y satisfy a linear equation as well,

$$
\alpha(x-r)+\beta(\sqrt{1-r^2}y)=\gamma(1-rx).
$$

Theorem 5. *The geodesics for Beltrami's projective metric are straight lines.*

We might prove this by changing coordinates to reduce to the Beltrami-Poincaré case. Beltrami showed it by writing down the geodesic equation. We show instead by a direct calculation that the diameter $y=0$ is a geodesic[2], so are all other straight chordes of the disc, by invariance under rotations and projective transformations.

Write $z=(x,y)$ in polar coordinates, $x=\rho\cos(t),\ y=\rho\sin(t)$, so that $dx=d\rho\cos(t)-\rho\sin(t)dt$ and $dy=d\rho\sin(t)+\rho\cos(t)$. Substituting in the expression for ds^2 we obtain

$$
\begin{aligned}
ds^2 &= \frac{(xdx+ydy)^2}{(1-(x^2+y^2))^2} + \frac{dx^2+dy^2}{1-(x^2+y^2)} \\
&= \frac{\rho^2 d\rho^2}{(1-\rho^2)^2} + \frac{d\rho^2+\rho^2dt^2}{1-\rho^2} \\
&\geq \frac{\rho^2 d\rho^2}{(1-\rho^2)^2} + \frac{d\rho^2}{1-\rho^2} \\
&= \frac{d\rho^2}{(1-\rho^2)^2}.
\end{aligned}
$$

Let γ be a curve joining $(0,0)$ and $(r,0)$. By the only inequality in the chain above we see that its length is reduced under the circular projection $z\mapsto|z|$, hence the geodesic must lie on the diameter $y=0$. Moreover, by the same calculation we did with the Beltrami-Poincaré metric, we find for the distance d_1 associated to ds^2 the expression

$$
d_1((0,0),(0,r))=\frac{1}{2}\log\frac{1+r}{1-r}. \tag{1.33}
$$

Using the group of isometries for this ds^2, one can easily find the expression for $d_1(z,w)$.

We end with a remark on metric discs, which this time are not extrinsic discs: flattening Poincaré geodesics, which are arcs of the unit circle, onto chords, has the effect of squeezing the metric discs in the direction which is normal to the boundary.

[2]Thanks to Michelangelo Cavina for the idea.

We can compute this precisely, without loss of generality at the point $(r,0)$, where the metric is

$$ds^2 = \frac{r^2dx^2}{(1-r^2)^2} + \frac{dx^2+dy^2}{1-r^2} = \frac{dx^2}{(1-r^2)^2} + \frac{dy^2}{1-r^2}.$$

The infinitesimal metric disc $\{(r,0)+dz \in \mathbb{D} : \ d_1((r,0)+dz,(r,0))\} = d\rho$ has equation

$$\frac{dx^2}{(1-r^2)^2} + \frac{dy^2}{1-r^2} = d\rho^2, \tag{1.34}$$

which is an ellipse having a very small semiaxis $(1-r^2)d\rho$ is the normal x-direction and a much larger semiaxis $\sqrt{1-r^2}d\rho$ in the tangential y-direction.

We will see the same phenomenon when dealing with the unit ball in $\mathbb{C}^{n+1}$, but with a new twist: $i\times($ normal direction) will be small, because multiplication times i does not affect the metric, but tangential, because i is a rotation by a right angle.