Gromov hyperbolic spaces: general theory and applications to complex geometry

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Introduction

Gromov hyperbolic spaces are a generalization of negative curved spaces in the context of metric spaces, and this concept fits very well to complex geometry problems since the invariant metrics are in general not Riemannian.

Menù

1. General theory of Gromov spaces
   1. Definitions
   2. Shadowing Lemma
   3. Gromov compactification
2. Applications to complex geometry

Books:

M.R. Bridson, A. Haefliger, **Metric Spaces of Non-Positive Curvature.**

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Basic notions

Definition

- A metric space is **proper** if the closed balls are compact.
- A metric space (X, d) is a **length space** if $\forall p, q \in X$

$$d(p,q) = \inf_{\gamma:[0,1]\to X, \mathcal{C}^0, \gamma(0)=p, \gamma(1)=q} \ell(\gamma)$$

where

$$\ell(\gamma) = \sup_{0=t_0<t_1\cdots,t_{n-1}<t_n=1} \sum_{j=0}^{n-1} d(\gamma(t_j), \gamma(t_{j+1})).$$

Definition

- Let $I \subseteq \mathbb{R}$ be an interval, then $\gamma : I \longrightarrow X$ is a **geodesic** if

$$d(\gamma(s), \gamma(t)) = |t - s|, \quad \forall s, t \in I.$$

  We use the word **ray** if $I = [0, +\infty)$, and **line** if $I = \mathbb{R}$.
- (X, d) is a **geodesic space** if for any two points there is a geodesic that connects them.

Theorem (Hopf-Rinow)

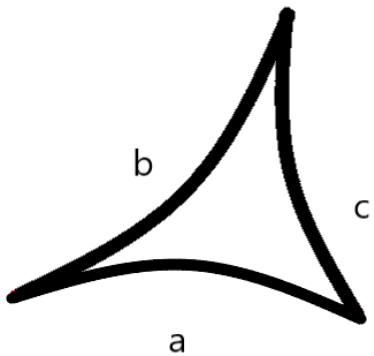
Let (X, d) be a length space. Assume (X, d) is complete and locally compact. Then

1. *(X, d) is proper.*
2. *(X, d) is geodesic.*

A **geodesic triangle** is a choice of three points in X and geodesic segments connecting these points.

Definition (Gromov hyperbolicity)

A proper geodesic metric space (X, d) is called δ-**hyperbolic** if every geodesic triangle δ-**slim**, i.e. any point on any of the sides of the triangle is within distance δ of the other two sides. Finally, a metric space is **Gromov hyperbolic** if it is δ-hyperbolic for some $\delta \geq 0$.



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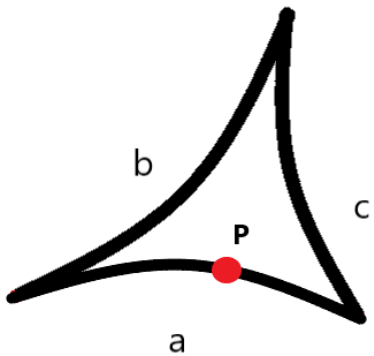

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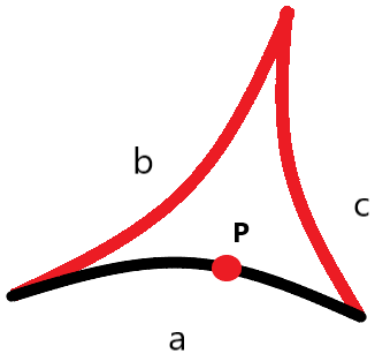

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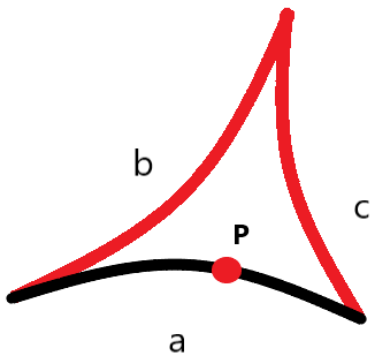
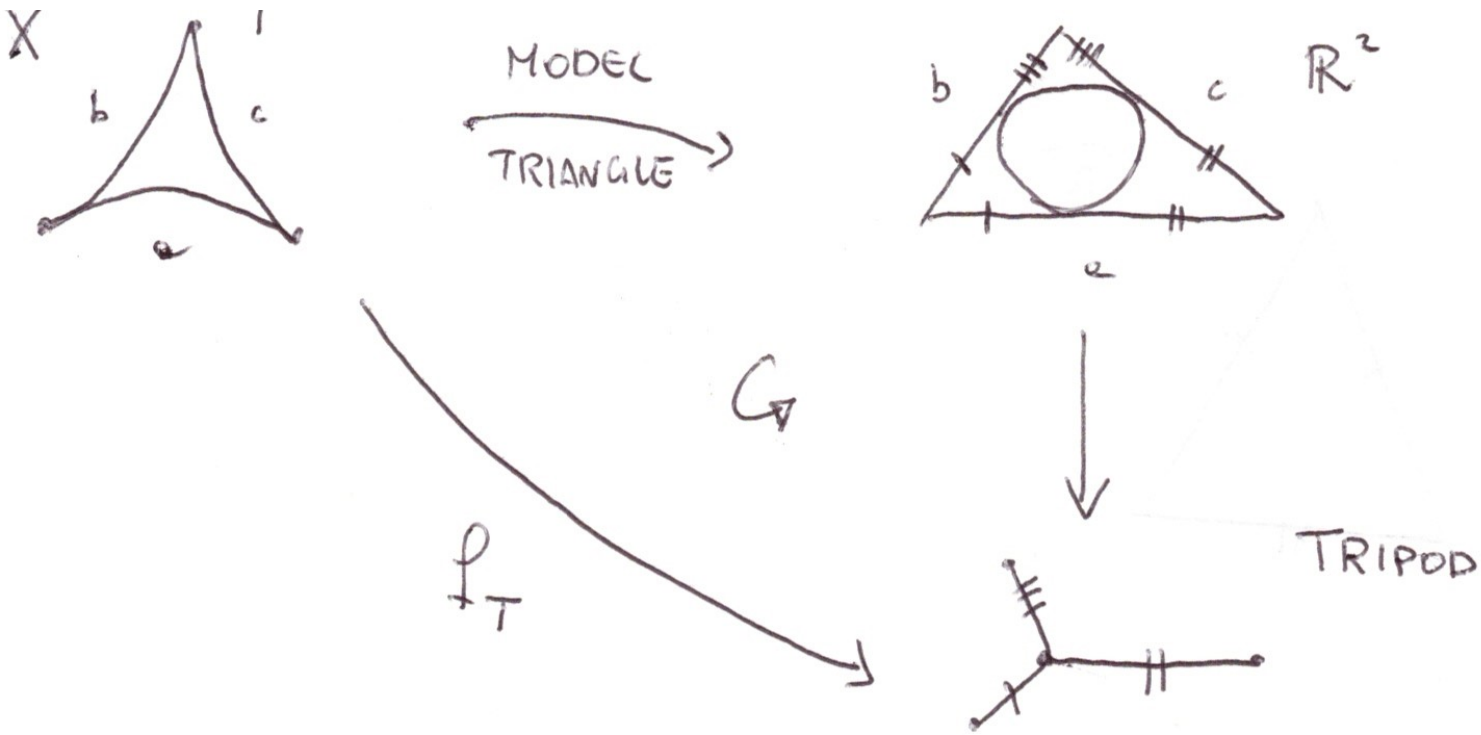


$$d(P, b \cup c) \leq \delta$$

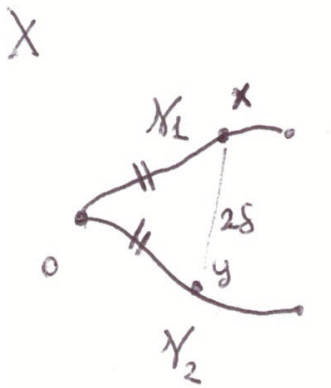


Definition

A geodesic triangle T is δ-**thin** if $\forall x, y \in T$ with $f_T(x) = f_T(y)$ then $d(x, y) \leq \delta$.

Lemma

Let (X, d) be a metric space. Let γ_1, γ_2 be two geodesics with same initial point $o \in X$ and take $x \in \gamma_1$. Denote $\delta := d(x, \gamma_2)$ and $y \in \gamma_2$ such that $d(o, x) = d(o, y)$, then $d(x, y) \leq 2\delta$.



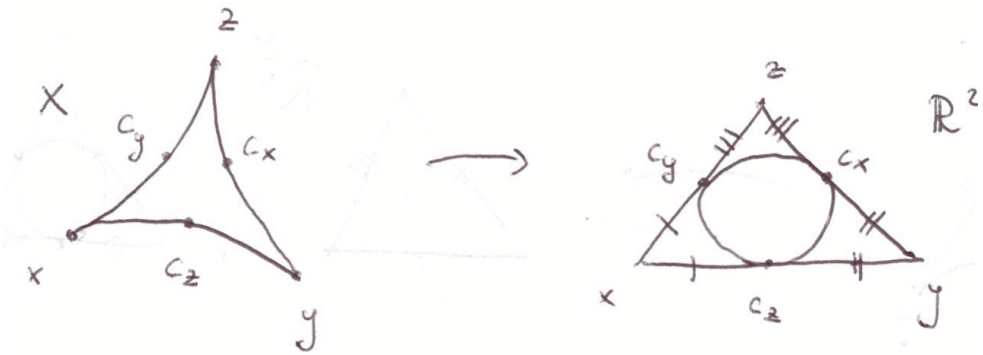
Proposition

Let T be a geodesic triangle

- *T is δ-thin $\Rightarrow$ T is δ-slim.*
- *T is δ-slim $\Rightarrow$ T is 2δ-thin.*

Let $x, y, o \in X$ we define the **Gromov product**

$$(x|y)_o := \frac{1}{2}[d(x,o) + d(y,o) - d(x,y)]$$



$$(y|z)_x := \tfrac{1}{2}[d(x,y) + d(x,z) - d(y,z)] = d(x,c_y) = d(x,c_z)$$

Definition (4-point condition)

(X,d) has the **4-point condition** (wrt $\delta \geq 0$) if for every $x, y, z, o \in X$ we have

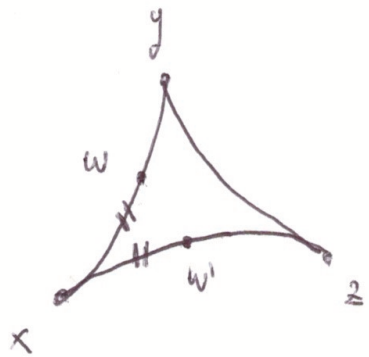
$$(x|y)_o \geq \min\{(x|z)_o, (y|z)_o\} - \delta.$$

Proposition

Let (X, d) be a proper geodesic metric space.

- *(X, d) has the 4-pts. cond. (wrt δ) $\Rightarrow \forall$ geodesic triangle is 4δ-thin.*
- *$\forall$ geodesic triangle is δ-thin $\Rightarrow (X, d)$ has the 4-pts. cond. (wrt δ).*

Proof: •



We want to prove $f_T(w) = f_T(w') \Rightarrow d(w, w') \leq 4\delta$.

$$(w|w')_x \geq \min\{(w'|y)_x, (w|y)_x\} - \delta.$$

and

$$(w'|y)_x \geq \min\{(w'|z)_x, (y|z)_x\} - \delta.$$

So

$$(w|w')_x \geq \min\{(w|y)_x, (w'|z)_x, (y|z)_x\} - 2\delta.$$

Now $(w|y)_x = (w'|z)_x = d(x,w) = d(x,w') \leq (y|z)_x$ which implies

$$(w|w')_x \geq d(x,w) - 2\delta$$

but $(w|w')_x = \frac{1}{2}[d(x,w) + d(x,w') - d(w,w')] = d(x,w) - \frac{1}{2}d(w,w')$.
Finally

$$-\frac{1}{2}d(w,w') \geq -2\delta \quad \Rightarrow \quad d(w,w') \leq 4\delta.$$

- Let $x, y, z, o \in X$, we want to prove

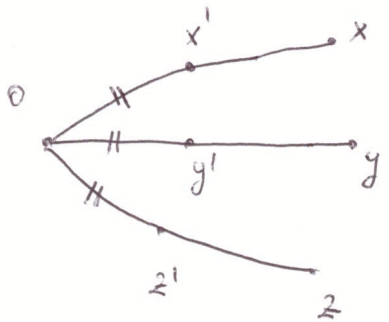
$$(x|y)_o \geq \min\{(x|z)_o, (y|z)_o\} - \delta.$$

$$(x|z)_o \leq \min\{d(o,x), d(o,z)\}, \quad (y|z)_o \leq \min\{d(o,y), d(o,z)\}$$

$$A := \min\{(x|z)_o, (y|z)_o\} \leq \min\{d(o,x), d(o,y), d(o,z)\}$$

so there exist $x' \in [o,x], y' \in [o,y], z' \in [o,z]$ such that

$$d(o,x') = d(o,y') = d(o,z') = A.$$



Consider respectively the geodesic triangles $[o, x, z]$ and $[o, y, z]$ we obtain $d(x', z') \leq \delta$ and $d(y', z') \leq \delta$ (and consequently $d(x', y') \leq 2\delta$).

$$(x|y)_o := \frac{1}{2}[d(x, o) + d(y, o) - d(x, y)] \geq$$

$$\frac{1}{2}[d(x, o)+d(y, o)-d(x, x')-d(x', y')-d(y', y)] \geq \frac{1}{2}[d(o, x')+d(o, y')]-\delta$$

$$= A - \delta = \min\{(x|z)_o, (y|z)_o\} - \delta.$$

Example

- Compact spaces are Gromov hyperbolic.
- $(\mathbb{R}, d_E)$ is 0-hyperbolic.
- Trees are 0-hyperbolic.
- $(\mathbb{R}^d, d_E)$ is not Gromov hyperbolic ($d \geq 2$).
- $(\mathbb{D}, \omega)$ is Gromov hyperbolic.
- $(\mathbb{B}^d, d^K_{\mathbb{B}^d})$ is Gromov hyperbolic.
- $(\mathbb{D}^d, d^K_{\mathbb{D}^d})$ is not Gromov hyperbolic ($d \geq 2$).

Proposition

Let (X, d_X) and (Y, d_Y) be two non-compact proper geodesic spaces. Then $(X \times Y, d_X \times d_Y)$ is not Gromov hyperbolic.

• **Sketch of the proof** $(\mathbb{D}, \omega)$: By contradiction there exist $T_n = [x_n, y_n, z_n]$ and $u_n \in [x_n, y_n]$ such that

$$\omega(u_n, [x_n, z_n] \cup [y_n, z_n]) \to \infty.$$

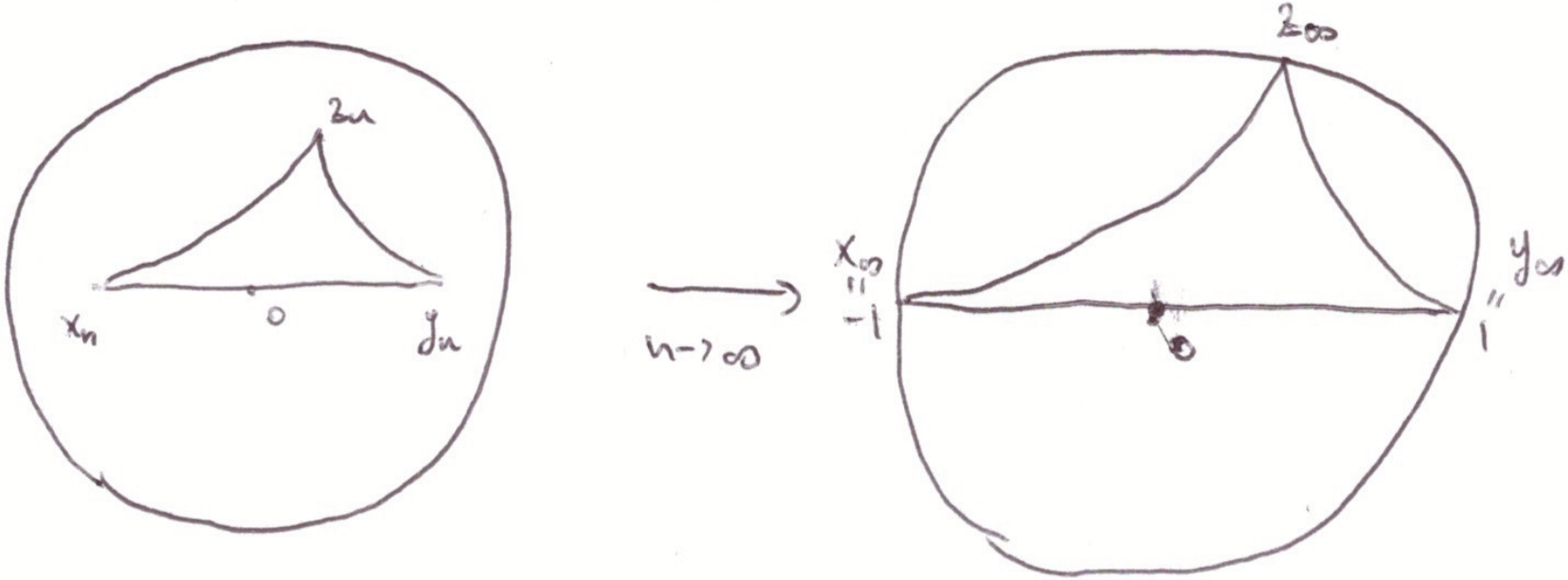
Using the automorphism group of $\mathbb{D}$ we can assume $u_n = 0 \in \mathbb{D}$ and $[x_n, y_n] \subset \mathbb{D} \cap \mathbb{R}$. Moreover

$$\omega(0, \{x_n, y_n, z_n\}) \geq \omega(0, [x_n, z_n] \cup [y_n, z_n]) \to \infty$$

so $x_n \to x_\infty \in \partial\mathbb{D}$, $y_n \to y_\infty \in \partial\mathbb{D}$, $z_n \to z_\infty \in \partial\mathbb{D}$ (in particular $x_\infty = -1, y_\infty = 1$). Now T_n converges to the "geodesic triangle" $T = [-1, 1, z_\infty]$ and

$$\infty > \omega(0, [-1, z_\infty] \cup [1, z_\infty]) = \lim_n \omega(0, [x_n, z_n] \cup [y_n, z_n]) = \infty$$

obtaining a contradiction.


z_n
x_n
0
y_n
$n \to \infty$
z_∞
x_∞
$= -1$
y_∞
$= 1$
0

Definition

- Let (X, d) be a metric space, $I \subset \mathbb{R}$ be an interval and $A \geq 1$ and $B \geq 0$. A function $\sigma : I \longrightarrow X$ is a (A, B) **quasi-geodesic** if for each $s, t \in I$

$$A^{-1}|t - s| - B \leq d(\sigma(s), \sigma(t))) \leq A|t - s| + B.$$

- Let (X, d_X) and (Y, d_Y) be two metric spaces and $A \geq 1$ and $B \geq 0$. A function $f : X \longrightarrow Y$ is a (A, B) **quasi-isometric embedding** if for each $x_1, x_2 \in X$

$$A^{-1} d_X(x_1, x_2) - B \leq d_Y(f(x_1), f(x_2))) \leq A d_X(x_1, x_2) + B.$$

If f is surjective, then f is called **quasi-isometry**.

Recall that the Hausdorff distance between two compact sets $K_1, K_2 \subseteq X$ is defined as

$$d_{\mathrm{Haus}}(K_1, K_2) := \max\left\{ \sup_{x \in K_1} d(x, K_2), \sup_{y \in K_2} d(y, K_1) \right\}.$$

Theorem (Shadowing lemma/ Geodesic stability)

Suppose that (X, d) is a δ-hyperbolic space. For any $A \geq 1$ and $B \geq 0$ there exists $R := R(\delta, A, B) > 0$ such that: if $\gamma : [a_1, b_1] \longrightarrow X$ is a geodesic and $\sigma : [a_2, b_2] \longrightarrow X$ is a (A, B)-quasigeodesic with $\gamma(a_1) = \sigma(a_2)$ and $\gamma(b_1) = \sigma(b_2)$, then

$$\max\left\{ \max_{t \in [a_1, b_1]} d(\gamma(t), \sigma([a_2, b_2]), \max_{t \in [a_2, b_2]} d(\sigma(t), \gamma([a_1, b_1])) \right\} \leq R$$

i.e.

$$d_{\mathrm{Haus}}(\gamma[a_1, b_1], \sigma[a_2, b_2]) \leq R$$

Corollary

A geodesic metric space (X, d) is Gromov hyperbolic if and only if, for every $A \geq 1$ and $B \geq 0$, there exists a constant M such that every (A, B)-quasi-geodesic triangle in X is M-slim (If (X, d) is δ-hyperbolic, then M depends only on δ, A and B).

Corollary

Let (X, d_X), (Y, d_Y) be two geodesic spaces. If there exists a quasi-isometry $f : X \longrightarrow Y$ then if (X, d_X) is Gromov hyperbolic then (Y, d_Y) is Gromov hyperbolic.

In particular, the result holds when $X = Y$, $f = \mathrm{Id}_X$ and $d_1, d_2 : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ are two distances of X such that there exists $A \geq 1, B \geq 0$

$$A^{-1} d_1(x, y) - B \leq d_2(x, y) \leq A d_1(x, y) + B, \quad \forall x, y \in X.$$

Gromov compactification

Let $\mathscr{R}(X)$ the set of geodesic rays of X, $\sigma_1, \sigma_2 \in \mathscr{R}(X)$ are *asymptotic* ($\sigma_1 \sim \sigma_2$) if

$$\sup_{t \geq 0} d(\sigma_1(t), \sigma_2(t)) < +\infty.$$

We define the **Gromov boundary** as $\partial_G X := \mathscr{R}(X)/\sim$, and the **Gromov compactification** as

$$\overline{X}^G := X \sqcup \partial_G X.$$

It is possible to define a Hausdorff topology on $\overline{X}^G$ such that the subspace topology of $X \subset \overline{X}^G$ is the same of (X, d).

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Definition (Topology of $\overline{X}^G$)

Fix a base point $p \in X$. A sequence (x_n) in $\overline{X}^G$ converges to a point $\xi \in \partial_G X$ if and only if for any subsequence of (γ_n) that converges uniformly on compact subsets, its limit belongs to the equivalence class ξ. Here γ_n is defined as follows: if $x_n \in X$, then $\gamma_n \colon [0, T_n] \to X$ is a geodesic segment connecting p to x_n, while if $x_n \in \partial_G X$, then $\gamma_n \colon \mathbb{R}_{\geq 0} \to X$ is a geodesic ray representing x_n with $\gamma_n(0) = p$.

Remark

- *Let $\gamma \colon \mathbb{R}_{\geq 0} \to X$ be a geodesic ray, then*

$$\gamma(+\infty) := \lim_{t \to +\infty} \gamma(t) = [\gamma] \in \partial_G X.$$

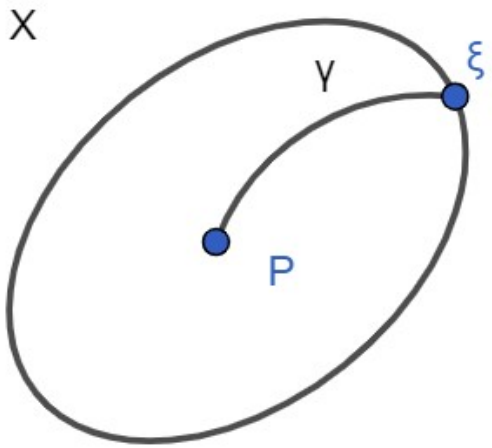
- *Let $\gamma \colon \mathbb{R} \to X$ be a geodesic line, then*

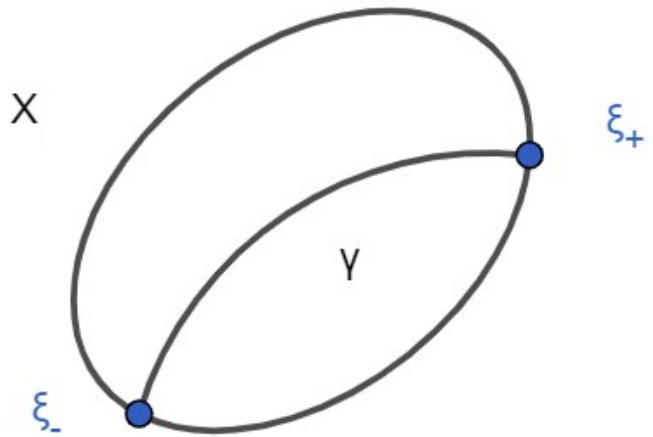
$$\lim_{t \to -\infty} \gamma(t) \neq \lim_{t \to +\infty} \gamma(t).$$

Proposition

- *For each $p \in X$ and $\xi \in \partial_G X$ there exists $\gamma\colon \mathbb{R}_{\geq 0} \to X$ a geodesic ray such that $\gamma(0) = p$ and $[\gamma] = \xi$ ($\lim_{t\to+\infty} \gamma(t) = \xi$).*
- *(Visibility) Let $\xi_+ \neq \xi_- \in \partial_G X$, then there exists a geodesic line $\gamma\colon \mathbb{R} \to X$ such that*

$$\lim_{t\to-\infty} \gamma(t) = \xi_-, \qquad \lim_{t\to+\infty} \gamma(t) = \xi_+.$$





Remark

There is an alternative description of the Gromov boundary. Fix $p \in X$. We say that a sequence (x_n) ***goes to infinity in the Gromov sense*** *if*

$$(x_n|x_m)_p \underset{m,n\to+\infty}{\longrightarrow} +\infty.$$

We denote by $\mathcal{I}^G$ the set of sequences going to infinity in the Gromov sense. Since X is Gromov hyperbolic, the relation

$$(x_n) \sim_s (y_n) \iff (x_n|y_n)_p \underset{n\to+\infty}{\longrightarrow} +\infty$$

is an equivalence relation on $\mathcal{I}^G$. Both definitions do not depend on the point p chosen ($|(x|y)_p - (x|y)_q| \leq d(p,q)$). We define

$$\partial_G^s X := \mathcal{I}^G / \sim_s.$$

The set $\partial_G^s X$ admits a canonical bijection with $\partial_G X$.

$$\partial_G X \longrightarrow \partial_G^s X$$

$$[\gamma] \longrightarrow [(\gamma(n))_{n\in\mathbb{N}}]_s.$$

Proposition

Let $(x_n), (y_n)$ be two sequences such that

$$x_n \to \xi \in \partial_G X, \quad y_n \to \eta \in \partial_G X.$$

Fix $p \in X$, then $\xi = \eta$ iff $(x_n|y_n)_p \to \infty$.

Proposition

Let $\gamma\colon \mathbb{R}_{\geq 0} \to X$ be a (A, B) quasi-geodesic ray, then

$$lim_{t\to+\infty}\gamma(t) \in \partial_G X$$

exists.

Theorem

Let (X, d_X) and (Y, d_Y) be two Gromov hyperbolic spaces, and $f : X \longrightarrow Y$ be a continuous quasi-isometry. Then f induces a homeomorphism $\partial f : \partial_G X \longrightarrow \partial_G Y$.

Sketch of the proof: If $\gamma\colon \mathbb{R}_{\geq 0} \to X$ is a geodesic ray, then $f \circ \gamma\colon \mathbb{R}_{\geq 0} \to Y$ is a quasi-geodesic ray, and so there exists $lim_{t\to+\infty}(f \circ \gamma)(t) \in \partial_G Y$.

$$\partial f(\xi) = \lim_{t\to+\infty}(f \circ \gamma)(t), \qquad \text{if } \xi = [\gamma] \in \partial_G X$$

Theorem (Denjoy-Wolff)

Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic map. Then either:

1. *f has a fixed point in $\mathbb{D}$; or*
2. *there exists a point $\xi \in \partial\mathbb{D}$ so that for any $z \in \mathbb{D}$,*

$$\lim_{n\to\infty} f^n(z) = \xi$$

Theorem (Karlsson)

Suppose (X, d) is a proper geodesic Gromov space. If $f : X \longrightarrow X$ is non-expanding (i.e. 1-Lipschitz), then either:

1. *for every $x \in X$, the orbit $\{f^n(x) : n \in \mathbb{N}\}$ is bounded in (X, d), or*
2. *there exists a point $\xi \in \partial_G X$ so that for any $x \in X$*

$$\lim_{n\to\infty} f^n(x) = \xi.$$

Invariant metrics in higher dimensions

Definition (Kobayashi distance)

For each $p \in \Omega \subseteq \mathbb{C}^d$ and $v \in \mathbb{C}^d$ we define the **Kobayashi metric**

$$K_\Omega(p, v) := \inf\{|\lambda| : f : \mathbb{D} \longrightarrow \Omega \text{ holomorphic}, \ f(0) = p, df_0(\lambda) = v\}$$

and the **Kobayashi** (pseudo)-**distance**

$$d_\Omega^K(z, w) = \inf\{ \int_0^1 K_\Omega(\gamma(t), \gamma'(t))dt : \gamma : [0, 1] \longrightarrow \Omega, \\ \mathcal{C}^1 \text{ piecewise}, \ \gamma(0) = z, \gamma(1) = w\}.$$

Ω is **Kobayashi hyperbolic** if d_Ω^K is a distance.
Ω is **Kobayashi complete** if d_Ω^K is a complete distance.

Remark

$$d_{\mathbb{D}}^K = \omega$$

Proposition

Let $f : \Omega_1 \subseteq \mathbb{C}^d \longrightarrow \Omega_2 \subseteq \mathbb{C}^q$ *be holomorphic, then*

$$K_{\Omega_2}(f(z), df_z v) \leq K_{\Omega_1}(z, v), \quad \forall z \in \Omega_1, v \in \mathbb{C}^d,$$

and

$$d^K_{\Omega_2}(f(z), f(w)) \leq d^K_{\Omega_1}(z, w), \quad \forall z, w \in \Omega_1,$$

i.e. f *is 1-Lipschitz with respect to the Kobayashi distance.*

Gromov spaces in complex geometry

It is natural to ask yourself the question:

Question

Which domains of $\mathbb{C}^d$ are Gromov hyperbolic with respect to the Kobayashi distance? Can we say something about their Gromov compactification?

We give a list of the results know so far.

Theorem (Balogh-Bonk)

Let $D \subset \mathbb{C}^d$ be a bounded strongly pseudoconvex domain with C^2 boundary, then (D, d_D^K) is Gromov hyperbolic. Moreover $\overline{D}^G$ is equivalent to $\overline{D}$.

"Equivalent" means that the identity map $\mathrm{Id} : D \longrightarrow D$ extends to an homeomorphism $\mathrm{Id}^* : \overline{D}^G \longrightarrow \overline{D}$.

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Idea of the Proof: Let (Z, d_Z) be a compact metric space and $D(Z)$ its diameter. Define $Con(Z) := Z \times (0, D(Z)]$ and

$$r((x,s),(y,t)) = 2\log\left[\frac{d_Z(x,y) + s \vee t}{\sqrt{st}}\right].$$

Now $(Con(Z), r)$ is Gromov hyperbolic and $\partial_G Con(Z)$ is canonical homeomorphic to Z [Bonk, Schramm].
If $D \subset \mathbb{C}^d$ $(d \geq 2)$ is a bounded strongly pseudoconvex domain with C^2 boundary then ∂D is a connected manifold. For a point $z \in \mathbb{C}^d$ let $\delta(z) = dist(z, \partial D)$ and the defining function ρ as

$$\rho(z) := \begin{cases} -\delta(z), & \text{if } z \in D, \\ \delta(z), & \text{if } z \notin D \end{cases}$$

$D = \{\rho < 0\}$.

Fix $p \in \partial D$, the tangent space $T_p\partial D = \{v \in \mathbb{C}^d : \mathrm{Re}\langle\bar{\partial}\rho(p), v\rangle = 0\}$, and the complex tangent space $T_p^{\mathbb{C}}\partial D = \{v \in \mathbb{C}^d : \langle\bar{\partial}\rho(p), v\rangle = 0\}$. By definition, the strongly pseudoconvexity of D means that the Levi form $\mathscr{L}_\rho$ defined by

$$\mathscr{L}_\rho(p, v) := \sum_{j,k=1}^{d} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(p) v_j \bar{v}_k$$

is positive definite on $T_p^{\mathbb{C}}\partial D$ for $p \in \partial D$. Consequently one can define a metric d_H-the **horizontal**, or **Carnot-Caratheodory metric** on ∂D - as follows. Call a piecewise C^1-smooth curve $\alpha : [0,1] \longrightarrow \partial D$ horizontal, if for $t \in [0,1]$ $\alpha'(t) \in T^{\mathbb{C}}_{\alpha(t)}\partial D$. It follows from the strongly pseudoconvexity that any two points $p, q \in \partial D$ can be joined by a horizontal curve. Define the Levi length of a curve by $\mathscr{L}_\rho - length(\alpha) := \int_0^1 \mathscr{L}_\rho(\alpha(t); \alpha'(t))^{1/2} dt$, and for $p, q \in \partial D$ let

$$d_H(p, q) = inf\{\mathscr{L}_\rho - length(\alpha) : \alpha : [0,1] \longrightarrow \partial D \text{ is a horizontal curve with } \alpha(0) = p, \alpha(1) = q\}$$

Let $\pi : D \longrightarrow \partial D$ such that $d(x, \pi(x)) = \delta(x)$, and $h(x) = \delta(x)^{1/2}$, then

$$g(x, y) := 2 \log \left[\frac{d_H(\pi(x), \pi(y)) + h(x) \vee h(y)}{\sqrt{h(x)h(y)}} \right]$$

Theorem

There exists $C \geq 0$, such that for each $x, y \in D$

$$g(x, y) - C \leq d_D^K(x, y) \leq g(x, y) + C.$$

Using the proof of $Con(Z)$ we can prove that (D, g) is Gromov hyperbolic, finally we conclude using the invariance of Gromov hyperbolicity under quasi-isometries.

We recall that a convex domain is Kobayashi hyperbolic if and only if it is $\mathbb{C}$-proper, i.e. it does not contain complex affine lines.

Theorem (Gaussier-Seshadri, Nikolov-Thomas-Trybula, Zimmer)

Suppose $D \subset \mathbb{C}^d$ is a $\mathbb{C}$-proper convex domain. If (D, d_D^K) is Gromov hyperbolic then ∂D does not contain any non-trivial holomorphic disks.

Theorem (Zimmer)

Suppose $D \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary. Then (D, d_D^K) is Gromov hyperbolic if and only if it has finite type in the sense of D'Angelo.

Theorem (Bracci, Gaussier, Zimmer)

Let $D \subset \mathbb{C}^d$ be a bounded convex domain. If (D, d_D^K) is Gromov hyperbolic, then $\overline{D}^G$ is equivalent to $\overline{D}$.

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Let $D \subset \mathbb{C}^d$ be a domain and ρ be its defining function i.e. $\rho : U \subseteq \mathbb{C}^d \longrightarrow \mathbb{R}$ is C^∞, $D := \{\rho < 0\}$ and $\nabla\rho \neq 0$ on $\{\rho = 0\}$.

Definition (D'Angelo type)

A point $p \in \partial D$ is of **D'Angelo type** m if

$$m = \sup\left\{\frac{\nu(\rho \circ f)}{\nu(f)} : f : \mathbb{D} \longrightarrow \mathbb{C}^d \text{ holomorphic, } f(0) = p\right\}$$

where $\nu(\cdot)$ indicates the order of vanishing, and ∂D is of **finite type** is every point has finite D'Angelo type.

Remark

Pseudoconvex + D'Angelo type is 2 $\Leftrightarrow$ Strongly pseudoconvex

Example

$$T := \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^{2p} < 1\}, \quad p \geq 2$$

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Theorem (F.)

Let $D \subset \mathbb{C}^2$ be a bounded smooth pseudoconvex finite type domain. Then (D, d_D^K) is Gromov hyperbolic. Moreover, $\overline{D}^G$ is equivalent to $\overline{D}$.

This theorem can be seen as a partial generalization (in $\mathbb{C}^2$) of Zimmer's analogous result for convex finite type domains.

Applications

Theorem

Let $D \subset \mathbb{C}^2$ be a bounded smooth finite type domain and $f : D \longrightarrow D$ be holomorphic, then either:

1. *for every $z \in D$, the orbit $\{f^n(z) : n \in \mathbb{N}\}$ is bounded in (D, d_D^K), or*
2. *there exists a point $\xi \in \partial D$ so that for any $z \in X$*

$$\lim_{n\to\infty} f^n(z) = \xi.$$

Theorem (F.)

Let $D_1 \subset \mathbb{C}^2$ be a bounded smooth finite type domain and $D_2 \subset \mathbb{C}^2$ be a bounded convex domain. Let $f : D_1 \longrightarrow D_2$ be a biholomorphic map, then f extends to a homeomorphism between $\overline{D}_1$ and $\overline{D}_2$.

New results involving Gromov hyperbolicity

- **Arosio, F., Gontard, Guerini:** Existence of horospheres in new classes of domains.
- **Zimmer:** Example of Kobayashi hyperbolic convex domain not biholomorphic to bounded convex domain.
- **Bracci, Contreras, Diaz-Madrigal, Gaussier, Zimmer:** Asymptotic behavior of orbits of holomorphic semigroups of the unit disc.
- **Bracci, Contreras, Diaz-Madrigal, Gaussier:** Non-tangential limits and the slope of trajectories of holomorphic semigroups of the unit disc.