GEODESICS AND OTHER GEOMETRIC OBJECTS IN THE HEISENBERG GROUP

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1. Geodesics and other geometric objects

The restriction to the the leaves $\mathbb{H} \times \{h\}$ of the (rescaled) Bergman-Kobayashi metric gave us a way to compute, for each $h \geq 0$, the length of a smooth curve γ in $\mathbb{H}$. We will forget here where these metrics came from, and we will consider the metric structures on the Heisenberg group *per se*. We will consider the case $n = 1$.

Let's summarize the state of things at the moment. Set $\mathbb{H} = \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3 \ni [z, t] \equiv [x, y, t]$, endowed with the Lie group structure induced by the product

$$[z, t] \cdot [w, s] = \left[z + w, t + s + \frac{Im(z\overline{w})}{2}\right] = \left[x + u, y + v, t + s + \frac{yv - xu}{2}\right]. \tag{1.1}$$

Its Lie algebra of left-invariant vector fields is spanned by the vectors

$$X = \partial_x + \frac{y}{2}\partial_t,\ Y = \partial_y - \frac{x}{2}\partial_t,\ T = \partial_t,\ \text{where } [X, Y] = -T \tag{1.2}$$

is the only nontrivial relation.

To enrich the structure in an important and useful way, we have a group of dilations,

$$D_\lambda[z, t] = [\lambda z, \lambda^2 t],\ \lambda > 0, \tag{1.3}$$

and the group isomorphisms

$$R_s[z, t] = [e^{is} z, t],\ s \in \mathbb{R}. \tag{1.4}$$

Finally, we have a family of metrics g_ϵ ($\epsilon \geq 0$), defined on the Lie algebra by the condition that X, Y, T are mutually orthogonal w.r.t. g_ϵ, and

$$1 = g_\epsilon(X, X) = g_\epsilon(Y, Y) = g_\epsilon(\epsilon T, \epsilon T). \tag{1.5}$$

The parameter ϵ is related to the parameter h we had on the leaves of the Siegel domain by $h = \epsilon^2$.

1.1. **Mnemonics for geodesics.** Suppose you are on a desert island with no internet connection and no geometry book, and you want to compute the geodesics in a Lie group G endowed with a metric g. Consider the (unit-speed) geodesics γ starting at the unit 1, and let $d(P) = d(P, 1)$ be the distance function associated with g. The maximum growth of d is along geodesics leaving 1, hence,

$$\nabla_g d(\gamma(\tau)) = \dot{\gamma}(\tau),\ \text{with } |\gamma(\tau)|_g = 1. \tag{1.6}$$

(This is *Gauss'Lemma*: geodesics leaving a point are orthogonal to the metric spheres centered at it). Assuming (1.6) as a starting point, the calculation of the geodesics can be carried out rather smoothly.

We will do that below in a more general framework, which does not just includes Lie groups, nor just Riemannian metrics, but more general *sub-Riemannian* ones.

Disclaimer: these are rather formal calculations, which lead to the determination of the *normal* geodesics.[1]. The nature of geodesics in sub-Riemannian structures is still not very well understood, even in very specific and natural subclasses of Lie groups.

On a manifold M having dimension $m+n$ consider linearly independent vector fields $V = X_1, \dots, X_m, Y_1, \dots, Y_n$[2] and suppose there is a positive definite two-form g which makes $X_1, \dots, X_m$ into a orthonormal system: $g(X_i, X_j) = \delta_{i,j}$. Define $H = \text{span}\{X_1, \dots, X_m\}$ to be the corresponding *horizontal space*. For $f : M \to \mathbb{R}$ define

$$\nabla f = \sum_{i=1}^{m} X_i f \cdot X_i \in H. \tag{1.7}$$

Suppose that $\gamma : I \to M$ is a *horizontal curve*,

$$\dot\gamma \in H, \tag{1.8}$$

Then, ∇f is the gradient of f in the usual sense, but for the fact that it is restricted to H. If $X = \dot\gamma \in H$, and we define ∇f by

$$X(f) = df(X) = \langle \nabla f, X \rangle_g, \tag{1.9}$$

we have

$$\nabla f = \sum_{j=1}^{m} \langle \nabla f, X_j \rangle_g X_j = \sum_{j=1}^{m} X_j(f) X_j,$$

as in (1.7).

Suppose then that $d : M \to \mathbb{R}$ is solution of the *eikonal equation*

$$|\nabla d| = 1. \tag{1.10}$$

Look now for integral curves γ of ∇d: $\gamma : I \to M$ and

$$\dot\gamma = \nabla d \in H. \tag{1.11}$$

With these in place, we compute

$$\begin{aligned}
\frac{d}{d\tau}[V d \circ \gamma](\tau) &= \langle (\nabla V d) \circ \gamma, \dot\gamma \rangle_g \\
&= \langle (\nabla V d) \circ \gamma, \nabla d \rangle_g \text{ (by (1.11))} \\
&= \sum_{j=1}^{m} X_j V d \cdot X_j d \\
&= \sum_{j=1}^{m} \left([X_j, V] d + V X_j d \right) X_j d \\
&= \sum_{j=1}^{m} [X_j, V] d \cdot X_j d + \frac{1}{2} V \left(\sum_{j=1}^{m} (X_j d)^2 \right) \\
&= \sum_{j=1}^{m} [X_j, V] d \cdot X_j d \text{ (by (1.10))}. \qquad (1.12)
\end{aligned}$$

[1] Cite MOntgomery and Monti

[2] One could relax these assumptions to cover *Gruschin-like* structures. The general requirement is that at each point of M the fields $X_1, \dots, Y_m$ span the tangent space.

Set the notation

$$v = Vd(\gamma) : \mathbb{R} \supseteq I \to \mathbb{R},\ [u,v] = [U,V]d(\gamma). \tag{1.13}$$

Then, (1.12) is a ODE system

$$\dot{v} = \sum_{j=1}^{m} [x_j, v] \cdot x_j,\ v = x_1, \ldots, x_m, y_1, \ldots, y_n, \tag{1.14}$$

whose solutions are the solutions of

$$\dot{\gamma} = \nabla d(\gamma),\ |\nabla d|_g = 1,$$

and are called the *geodesics* for g. In fact, the solutions of (1.14) allow us to compute $\dot{\gamma}$,

$$\dot{\gamma} = \nabla d = \sum_{j=1}^{m} x_j X_j, \tag{1.15}$$

hence, by integration, γ itself.

Observe that (M, H, g) is here the relevant structure, and that the choice of the vector fields $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ is irrelevant. Also, some assumptions might be relaxed to make room for Grushin metrics and other extensions.

1.2. The Heisenberg group, Riemannian and not.

We work in $\mathbb{R}^3 \ni (x, y, t)$ with the vector fields

$$X = (1, 0, y/2) = \partial_x + y/2\partial_t,\ Y = (0, 1, -x/2) = \partial_y - x/2\partial_t,\ T = (0,0,1) = \partial_t,\ [Y, X] = T, \tag{1.16}$$

and the metric g_ϵ induced by Bergman-Kobayashi one on horocycles $\mathbb{H} \times \{h\}$, $h = \epsilon^2$,

$$|X|_{g_\epsilon} = |Y|_{g_\epsilon} = |\epsilon T|_{g_\epsilon} = 1, \tag{1.17}$$

and X, Y, T are orthogonal. When $\epsilon = 0$, the metric is defined on the *horizontal space* $H = \text{span}\{X, Y\}$ only.

The nice feature of systems of this sort, coming from *Carnot type Lie groups*, is that the system of differential equations for $\tilde{x} = Xd \circ \gamma$, $\tilde{y} = Yd \circ \gamma$, $\tilde{t} = Td \circ \gamma$ do not depend on ϵ (we used here letters $\tilde{x}$, $\tilde{y}$, $\tilde{t}$ instead of x, y, t to avoid confusion).

$$\begin{cases} \dot{\tilde{x}} = [\tilde{y}, \tilde{x}]\tilde{x} = \tilde{t}\tilde{y} \\ \dot{\tilde{y}} = [\tilde{x}, \tilde{y}]\tilde{y} = -\tilde{t}\tilde{x} \\ \dot{\tilde{t}} = 0 \end{cases} \tag{1.18}$$

Integrating,

$$\begin{aligned} \tilde{t} &= \sigma \in \mathbb{R} \\ \frac{d}{d\tau}[\tilde{x}^2 + \tilde{x}^2] &= 2\tilde{t}(\tilde{y}\tilde{x} - \tilde{x}\tilde{y}) = 0, \text{ hence,} \\ \tilde{x}^2 + \tilde{y}^2 &= R^2 \text{ is a constant to be chosen later, and} \\ \tilde{x} &= R\sin(\sigma\tau + \alpha), \\ \tilde{y} &= R\cos(\sigma\tau + \alpha). \end{aligned}$$

We will set $\alpha = 0$ (it's just the effect of a rotation). Now the metric comes into the picture, since

$$\begin{aligned} \dot{\gamma} &= (\nabla d) \circ \gamma = Xd \cdot X + Yd \cdot Y + \epsilon T d \cdot \epsilon T \\ &= \tilde{x}X + \tilde{y}Y + \epsilon^2 \tilde{t} T. \end{aligned} \tag{1.19}$$

Hence,

$$\begin{aligned}\dot\gamma &= [\dot x, \dot y, \dot t] \\ &= R\sin(\sigma\tau)X + \cos(\sigma\tau)Y + \sigma\epsilon^2 T \\ &= R\sin(\sigma\tau)\partial_x + R\sin(\sigma\tau)\partial_y + \left(\frac{R\sin(\sigma\tau)y - R\cos(\sigma\tau)x}{2} + \sigma\epsilon^2\right)\partial_t,\end{aligned}$$

and we have to integrate:

$$\begin{cases}\dot x = R\sin(\sigma\tau) \\ \dot y = R\cos(\sigma\tau) \\ \dot t = \frac{R\sin(\sigma\tau)y - R\cos(\sigma\tau)x}{2} + \sigma\epsilon^2.\end{cases} \tag{1.20}$$

This is easily done. We normalize solutions to have $\gamma(0) = [0,0,0]$ (the other geodesics can be obtained by left-translation), so

$$\begin{cases}x(\tau) = \frac{R}{\sigma}(1-\cos(\sigma\tau)), \\ y(\tau) = \frac{R}{\sigma}\sin(\sigma\tau).\end{cases} \tag{1.21}$$

This gives:

$$\begin{aligned}\dot t &= \frac{R\sin(\sigma\tau)y - R\cos(\sigma\tau)x}{2} + \sigma\epsilon^2 \\ &= \frac{R^2\sin^2(\sigma\tau) - R^2(1-\cos(\sigma\tau))\cos(\sigma\tau)}{2\sigma} + \sigma\epsilon^2 \\ &= \frac{R^2 - R^2\cos(\sigma\tau)}{2\sigma} + \sigma\epsilon^2,\end{aligned}$$

hence,

$$t(\tau) = \frac{R^2}{2\sigma^2}\left(\sigma\tau - \sin(\sigma\tau)\right) + \sigma\epsilon^2\tau. \tag{1.22}$$

Finally, we have to normalize in such a way that γ has unit speed, and this can be done at time $\tau = 0$:

$$1 = |\nabla d|^2 = \tilde x(0)^2 + \tilde y(0)^2 + \epsilon^2\tilde t(0)^2 = R^2 + \epsilon^2\sigma^2,$$

i.e. $R = \sqrt{1-\epsilon^2\sigma^2}$. We finally have the equations for the geodesics starting at $[0,0,0]$,

$$[x,y,t] = \begin{cases}\frac{\sqrt{1-\epsilon^2\sigma^2}}{\sigma}(1-\cos(\sigma\tau)) \\ \frac{\sqrt{1-\epsilon^2\sigma^2}}{\sigma}\sin(\sigma\tau) \\ \frac{1-\epsilon^2\sigma^2}{2\sigma^2}\left(\sigma\tau - \sin(\sigma\tau)\right) + \sigma\epsilon^2\tau.\end{cases} \tag{1.23}$$

Here we have one degree of freedom, which is given by σ. Recall that we had another one, α, that we fixed to 0. Two degrees of freedom for the geodesics starting at one point is what we expect in a three dimensional manifold.

The initial speed is

$$\dot\gamma(0) = [\dot x(0), \dot y(0), \dot t(0)] = [0, \sqrt{1-\epsilon^2\sigma^2}, \sigma\epsilon^2]. \tag{1.24}$$

In the Riemannian case, when $1 = \epsilon^2\sigma^2$, we see that the t axis is a geodesic.

There is an important difference between the Riemannian ($\epsilon > 0$) and the sub-Riemannian one ($\epsilon = 0$). In the former the tangent vectors range over all vectors, while in the latter they are all in the plane $t = 0$. This implies (and it can made precise by direct calculation), that for each vector $[a,b,0]$ there are infinitely many geodesics having it as initial speed.

1.2.1. *The cut-locus.* Consider the Riemannian case. If in (1.23) you fix $\sigma > 0$ and pick τ such that $\sigma\tau = 2\pi$, you find

$$\gamma(2\pi/\sigma) = \left[0, 0, 2\pi\left(\frac{1-\epsilon^2\sigma^2}{2\sigma^2} + \epsilon^2\right)\right].$$

But there is another geodesic joining $[0, 0, 0]$ and the same point, namely $\gamma_0(\tau) = [0, 0, \epsilon\tau]$, which reaches it at time

$$\tau = \frac{1}{\epsilon} 2\pi \left(\frac{1-\epsilon^2\sigma^2}{2\sigma^2} + \epsilon^2\right).$$

This shows that in $\mathbb{H}$ most geodesics are not length-minimizing for an infinite time.