Differential Inequalities vs. Integral Inequalities

N. Arcozzi

Università di Bologna

12 December 2019
Seminario Pini

Plan

- convexity;
- sub-harmonicity and the conjugate function inequality;
- Burkholder and differential subordination of martingales;
- Bellman functions and the characterization of Carleson measures.

Jensen

CONVEX FUNCTIONS AND JENSEN'S INEQUALITY

$$\mathbb{R} \xrightarrow{\varphi} \mathbb{R}$$

DEF $\boxed{\varphi((1-t)x+ty) \le (1-t)\varphi(x) + t\varphi(y)}$ $0 \le t \le 1$

SUB-MEAN VALUE PROPERTY $\boxed{\varphi\left(\frac{x+y}{2}\right) \le \frac{\varphi(x)+\varphi(y)}{2}}$

DIFFERENTIAL INEQUALITY $\boxed{\varphi'' \ge 0}$

INTEGRAL INEQUALITY (JENSEN)

$$\begin{array}{c} \mathbb{R} \\ \uparrow \varphi \\ \mathbb{R} \\ \uparrow f \\ (X,\mu) : \mu(X) = 1 \end{array}$$

$$\boxed{\varphi\left(\int_X f \, d\mu\right) \le \int_X \varphi(f) \, d\mu}$$

CONVEX FUNCTIONS AND JENSEN'S INEQUALITY

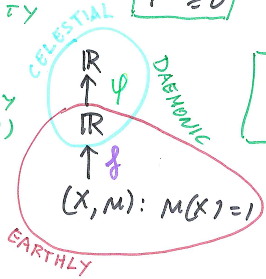
$$\mathbb{R} \xrightarrow{\varphi} \mathbb{R}$$

DEF $\boxed{\varphi((1-t)x+ty) \le (1-t)\varphi(x) + t\varphi(y)}$ $0 \le t \le 1$

SUB-MEAN VALUE PROPERTY $\boxed{\varphi\left(\frac{x+y}{2}\right) \le \frac{\varphi(x)+\varphi(y)}{2}}$

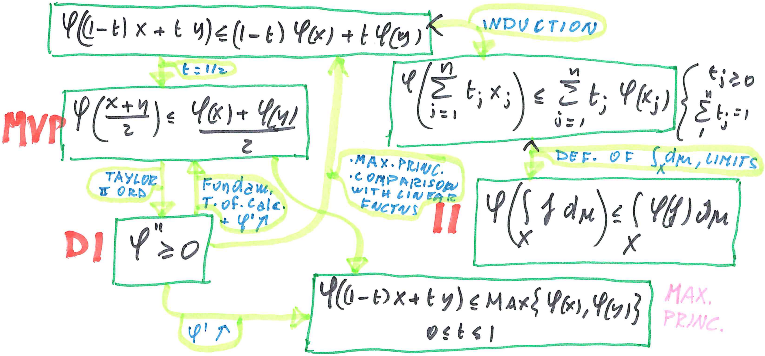
DIFFERENTIAL INEQUALITY $\boxed{\varphi'' \ge 0}$

INTEGRAL INEQUALITY (JENSEN)

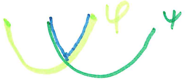


$$\boxed{\varphi\left(\int_X f\,d\mu\right) \le \int_X \varphi(f)\,d\mu}$$

Jensen



WHICH IS THE "RIGHT" REGULARITY?



$\mathrm{MAX}(\varphi,\psi) \notin C^1$ IS CONVEX

$\varphi(t) = t^2$: CAUCHY-SCHWARZ

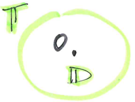
$$\left[\int f \cdot g \, dx\right]^2 = \Bigg[\int \underbrace{g^{-1} f}_{F} \cdot \underbrace{\frac{g^2}{\|g\|_2^2} dx}_{d\mu}\Bigg]^2 \cdot \|g\|_2^4$$

JENSEN

$$\leq \int \underbrace{(g^{-1} f)^2}_{F^2} \cdot \underbrace{\frac{g^2 \, dx}{\|g\|_2^2}}_{d\mu} \cdot \|g\|_2^4$$

$$= \|f\|_2^2 \cdot \|g\|_2^2$$

Conjugate L^p Function Inequality $\quad 1<p<\infty$



$$\xrightarrow{f = V + iv} \mathbb{C}$$

$$f(0)=0$$

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi}|v(e^{it})|^p dt \le C_p^p \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi}|V(e^{it})|^p dt}$$

M. RIESZ 1927
P. STEIN 1933
A. CALDERÓN 1948 ?
CALDERÓN-ZYGMUND 1952
S. PICHORIDES 1972 BEST
M. ESSEN 1984
D. BURKHOLDER
R. BAÑUELOS-G. WANG 1995
...

VIEWPOINTS ON

$v = \tilde{V} = HV \qquad z = re^{it}$

FOURIER SERIES

$$\frac{z^n + \bar{z}^n}{2} = V \longmapsto \frac{z^n - \bar{z}^n}{2i} = v$$

$$\sum_{n\neq 0} \hat{V}(n) r^{|n|} e^{int} \longmapsto \sum_{n\neq 0} \hat{V}(n)[-i\,\mathrm{sign}(n)]\cdot r^{|n|}\cdot e^{int}$$

CONVOLUTION OPERATOR (SINGULAR INTEGRAL).

$$v(e^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(e^{i(t-s)})\cdot \cot(s/2)\, ds$$

P. STEIN'S PROOF (1935) $1<p\le 2$

- $\underbrace{\mathbb{D} \xrightarrow[\text{Hol.}]{f} \mathbb{C} \xrightarrow[\text{Super Harm.}]{G} \mathbb{R}}_{\text{Super-Harm}}$

$$\boxed{\Delta(G\circ f) = \Delta G\circ f\cdot|f'|^2}$$

- G superharmonic

$\Updownarrow$

- $\Delta G \le 0$ **DI**

$\Updownarrow$

- $\fint_{\partial D(w,r)} G(w+re^{it})\cdot r\,dt \le G(w)$ **Super MVP**

$$\begin{array}{c} \mathbb{R} \\ \uparrow G \\ \mathbb{C} \\ \uparrow f \\ \mathbb{D} \end{array}$$

CELESTIAL

EARTHLY

DAEMONIC

WHAT IS THE (AN) ASSOCIATED INTEGRAL INEQUALITY?

II $\frac{1}{2\pi}\int_{-\pi}^{\pi} G(f(e^{it}))\,dt \le G(f(0))$

sub-Harmonicity

$$\boxed{\begin{array}{l} G(w) = G(u+iv) = |w|^p - c_p^p |u|^p \\ \text{is sup-Harm if } c_p^p = \frac{p}{p-1} : \end{array}}$$

$$\Delta G(w) = p \cdot \left[p |w|^{p-2} - c_p^p \cdot (p-1) |u|^{p-2} \right]$$

$$= p^2 \left[|w|^{p-2} - |u|^{p-2} \right] \le 0$$

$$\boxed{G(0) = 0}$$

More: $G(u + i \cdot 0) = |u|^p (1 - c_p^p) \le 0$

Hence: $\frac{1}{2\pi} \int_{-\pi}^{\pi} \{ |v(e^{it})|^p - c_p^p |u(e^{it})|^p \} dt$

$$\le \frac{1}{2\pi} \int_{-\pi}^{\pi} \{ |f(e^{it})|^p - c_p^p \cdot |u(e^{it})|^p \} dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} G(f(e^{it}))\, dt \le G(f(0)) = G(0) \le 0 :$$

$$\boxed{\|v\|_{L^p(\mathbb{T})} \le \|f\|_{L^p(\mathbb{T})} \le \left(\frac{p}{p-1}\right)^{1/p} \cdot \|u\|_{L^p(\mathbb{T})}}$$

sub-Harmonicity

ESSÉN'S REMARK AND NEW PROOF (1984)

P. STEIN: $\frac{1}{2\pi}\int_{-\pi}^{\pi} G(f(e^{it}))dt \le G(f(0)) = G(0) \le 0$

SUPER H. SPECIAL HP.

SUPPOSE WE HAVE: $\mathbb{C} \xrightarrow{H} \mathbb{R}$ S.T.

(A) $G \le H$

(B) H IS SUPER H.

(C) $H(0) \le 0$

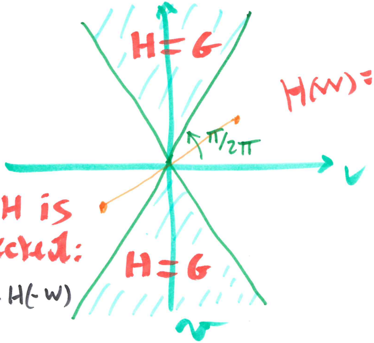
THEN: $\frac{1}{2\pi}\int_{-\pi}^{\pi}\{|f(e^{it})|^p - C_p^p |v(e^{it})|^p\} dt =$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} G(f(e^{it}))dt \overset{(A)}{\le} \frac{1}{2\pi}\int_{-\pi}^{\pi} H(f(e^{it}))dt \overset{(B)}{\le} H(f(0)) = H(0) \overset{(C)}{\le} 0$

∷ THERE IS H WITH (A-C) AND

$C_p = \frac{1}{\cos(\pi/2p)}$ BEST

THE FUNCTION H



$H(w) = -\operatorname{tg}\left(\frac{\pi}{2p}\right)\cdot|w|^p\cdot\cos(p\arg(w))$

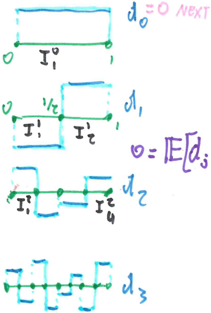
WHICH IS HARMONIC!

H is reflected:

$H(w) = H(-w)$

BURKHOLDER: L^p INEQUALITIES FOR DIFFERENTIALLY SUBORDINATE MARTINGALES

MARTINGALES (DYADIC) W/O FILTRATIONS



$f_n = \sum_{j=1}^{n} d_j$: $\{f_n\}_0^\infty$ IS A MARTINGALE

$g_n = \sum_{j=1}^{n} e_j$: ANOTHER MARTINGALE

$0 = \mathbb{E}[d_j \mid \mathcal{F}_{j-1}]$

$\{g_n\} < \{f_n\}$ ARE DIFFERENTIALLY SUBORDINATE T.E.O. IF

$|e_n| \le |d_n| \quad \forall n$

i.e. $e_n = \alpha_n \cdot d_n$ WITH $|\alpha_n| \le 1$

AND α_n IS CONSTANT ON EACH I_j^{n-1} (PREDICTABILITY: $\alpha_n \in \mathcal{F}_{n-1}$).

BUZZWORD: HAAR WAVELET

BURKHOLDER: 1984 $\quad 1<p<\infty \quad$ PROOF: 1988

$$\boxed{\begin{array}{c}\{g_n\} < \{f_n\} \Rightarrow \mathbb{E}|g_n|^p \leq (p^*-1)^p \cdot \mathbb{E}|f_n|^p \\ p^* = \mathrm{MAX}\{p, q : \frac{1}{p} + \frac{1}{q} = 1\}\end{array}}$$

BEST

$$G_1(x,y) = |y|^p - (p^*-1)^p \cdot |x|^p$$

$$\begin{tikzcd} \mathbb{R} \\ \mathbb{R}^2 \arrow[u, "{G, H_1}"'] \\ {[0,1]=\Omega} \arrow[u, "{(f_n, g_n)}"'] \end{tikzcd}$$

suppose $\exists\, \mathbb{R}^2 \xrightarrow{H_1} \mathbb{R}$:

(A) $G_1(x,y) \leq H_1(x,y)$

D1 (B) $|k| \leq |h| \Rightarrow t \mapsto H_1(x+th, y+tk)$ IS CONCAVE

(C) $H_1(0,0) \leq 0$

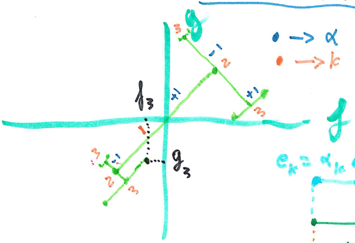
martingales

ZIG-ZAG MARTINGALES

$\alpha_n \in \{\pm 1\}$; $\{f_n\}$: MARTINGALE; $f_n = \sum_1^n d_k$

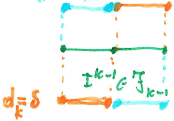
$g_n = \sum_1^n \alpha_k d_k$; $\boxed{(f_n, g_n): \Omega = [0,1] \longrightarrow \mathbb{R}^2}$

DEFINES A ZIG-ZAG MARTINGALE



S.T.P. FOR Z-Z MART.

$e_k = \alpha_k d_k = \pm\delta = \varepsilon$: $|\varepsilon| = |\delta|$



Obs. $\mathbb{E}[H_1(f_k, g_k) | \mathcal{F}_{k-1}] = \mathbb{E}[H_1(f_{k-1} + d_k, g_{k-1} + e_k) | \mathcal{F}_{k-1}] =$

II $= \frac{1}{2}\{H_1(f_{k-1} + \delta, g_{k-1} + \varepsilon) + H_1(f_{k-1} - \delta, g_{k-1} - \varepsilon)\} \overset{(B)}{\leq} H_1(f_{k-1}, g_{k-1})$

END OF THE PROOF

$$\mathbb{E}\left(|g_n|^p-(p^*-1)^p|f_n|^p\right)=\mathbb{E}\,G_1(f_n,g_n)\overset{(A)}{\le}\mathbb{E}\,H_1(f_n,g_n)$$

$$=\mathbb{E}\left\{\mathbb{E}\left[H_1(f_n,g_n)|\mathcal{F}_{n-1}\right]\right\}\overset{(B)}{\le}\mathbb{E}\,H_1(f_{n-1},g_{n-1})$$

$$\overset{(B)}{\le}\dots\overset{(B)}{\le}\mathbb{E}\,H_1(f_0,g_0)=\mathbb{E}\,H_1(0,0)\overset{(C)}{\le}0.$$

WHERE IS THE DIFFICULTY?

FINDING H_1:

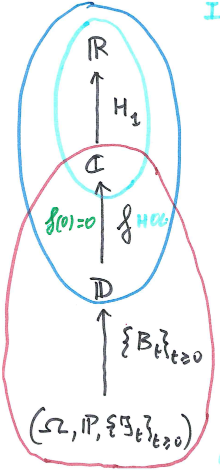
$$H_1(x,y)=p\left(1-\frac{1}{p^*}\right)^{p-1}\cdot\left(|y|-(p^*-1)|x|\right)\cdot\left(|x|+|y|\right)^{p-1}$$

I $(d,e)\,\mathrm{Hess}\,H_1(x,y)\begin{pmatrix}d\\e\end{pmatrix}\le0$ if $|d|\ge|e|$

II $\mathbb{E}\,H_1(f_n,g_n)\le H_1(0,0)$ if $\{g_n\}\prec\{f_n\}$

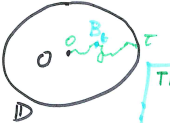
CONJUGATE FUNCTION AND DIFFERENTIAL SUBORDINATION

BURKHOLDER 1987: $\Delta H_1 \le 0$ DI



ITÔ FORMULA $\mathbb{D} \xrightarrow{\varphi} \mathbb{R}$

$$\varphi(B_t) = \varphi(B_0) + \int_0^t \nabla\varphi(B_s)\cdot dB_s + \frac{1}{2}\int_0^t \Delta\varphi(B_s)\,ds$$



C.R. $\Rightarrow$ $\Delta(H_1\circ f)\le 0$

$$\boxed{\text{THM. } \|v\|_{L^p(\mathbb{T})} \le (p^*-1)\cdot\|u\|_{L^p(\mathbb{T})}}$$

PF. $\frac{1}{2\pi}\int_{-\pi}^{\pi} |v(e^{it})|^p - (p^*-1)^p|u(e^{it})|^p\,dt$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} G_1(f(e^{it}))\,dt = \mathbb{E}^0(G_1\circ f)(B_\tau)$$

$$\le \mathbb{E}^0(H_1\circ f)(B_\tau)$$

$$= \mathbb{E}^0 H_1(f(0)) + \overbrace{\mathbb{E}^0\int_0^\tau \nabla(H_1\circ\varphi)(B_s)\cdot dB_s}^{=0}$$

$$\underbrace{+\frac{1}{2}\mathbb{E}^0\int_0^\tau \Delta(H_1\circ\varphi)(B_s)\,ds}_{0\ge} \le H_1(0) = 0 \quad \blacksquare$$

PICHORIDES 1972 $\|v\|_{L^p(\mathbb{T})} \le \cot\left(\frac{\pi}{2p^*}\right)\cdot\|u\|_{L^p(\mathbb{T})}$ (A)

ESSÉN, VERBITSKY 1984 1984 $\|f\|_{L^p(\mathbb{T})} \le \operatorname{cosec}\left(\frac{\pi}{2p^*}\right)\cdot\|u\|_{L^p(\mathbb{T})}$ (B)

ESSÉN : (B) $\Rightarrow$ (A)

BAÑUELOS-WANG: 1995 PROOFS OF (A, B) BY ITÔ + NEW "BELLMAN FUNCTIONS"

GUNDY-VAROPOULOS: 1979 PROBABILISTIC INTERPRETATION OF RIESZ TRANSFORMS

CARBONARO-DRAGIČEVIĆ: + MANIFOLDS, RICCI CURVATURE

STEFANIE PETERMICHL, KOMLA DOMELEVO, N.A. : + JUMP PROCESSES

...

CARLESON MEASURES FOR THE HARDY SPACE

$$\begin{cases} \Delta v = 0 \\ v|_{\mathbb{T}} \in L^2(\mathbb{T}) \end{cases}$$

$\mu \geq 0$ ON $\mathbb{D}$

THM. 1962 (A)$\Longleftrightarrow$(B)

(A) $\int_{\mathbb{D}} v^2 \, d\mu \leq C \cdot \|v\|^2_{L^2(\mathbb{T})}$

(B) $\mu(S(I)) \leq C' \cdot |I|$

DYADIC (EQUIVALENT) VERSION: DYADIC MARTINGALES

$\mathbb{D}$	$\longleftrightarrow$	$\mathcal{D} = \{I \subseteq [0,1]:$ DYADIC$\}$
$\mathbb{T}$	$\longleftrightarrow$	$[0,1] = I_0$
μ	$\longleftrightarrow$	$\mu \geq 0$ ON $\mathcal{D}$
I	$\longleftrightarrow$	$I \in \mathcal{D}$
$S(I)$	$\longleftrightarrow$	$\{J \in \mathcal{D} : J \subseteq I\}$
$v\|_{\mathbb{T}}$	$\longleftrightarrow$	$\varphi \in L^2[0,1]$
v	$\longleftrightarrow$	$I \mapsto \langle \varphi \rangle_I := \frac{1}{\|I\|} \int_I \varphi \, dx$

$\varphi_n = \sum_{|I| = 2^{-n}} \langle \varphi \rangle_I \chi_I$:

$\{\varphi_n\}$ IS A MARTINGALE

THM. (A) $\Longleftrightarrow$ (B)

(A) $\sum_I \mu_I \langle \varphi \rangle_I^2 \leq C \cdot \langle \varphi^2 \rangle_{I_0}$

(B) $\sum_{J \subseteq I} \mu_J \leq C' \cdot |I|$

BELLMAN FUNCTION PROOF (NAZAROV, TREIL 1995)

$\boxed{\text{THM. (B) } \sum_{J\subseteq I} \mu_J \le |I| \ \forall I \implies \text{(A) } \sum_{I\subseteq I_0} \mu_I \langle\varphi\rangle_I^2 \le C\cdot\langle\varphi^2\rangle_I}$

METHOD: (i) LIST RELEVANT QUANTITIES WITH SCALINGS
(ii) LIST THEIR RELATIONS, USING HYPOTHESIS
(iii) WRITE L.H.S. (A) AS A FUNCTION $\mathfrak{B}$ AND FIND ITS "DI"
(iv) WRITE (A) AS AN INEQUALITY FOR $\mathfrak{B}$
(v) FIND A FUNCTION AS IN (i-iv) (EXPLICIT)
(vi) "TELESCOPE" TO PROVE (A) $\Rightarrow$ (B)

"WILL'S OPTIMISM" A.G.

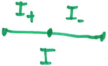
(i) $M = M_I = \frac{1}{|I|}\sum_{J\subseteq I}\mu_J$ (a)

$F = F_I = \langle\varphi^2\rangle_I = \frac{1}{|I|}\int_I \varphi^2$ (b)

$f = f_I = \langle\varphi\rangle_I = \frac{1}{|I|}\int_I \varphi$ (c)

(ii) $0 \le M \le 1$ (B)

$f^2 \le F$ (C-S)



$f_I = \frac{1}{2}(f_{I_+} + f_{I_-})$

$F_I = \frac{1}{2}(F_{I_+} + F_{I_-})$

MARTINGALES

$M_I = \frac{1}{2}(M_{I_+} + M_{I_-}) + \Delta M$ SUPER-MARTINGALE

$\Delta M = \frac{\mu_I}{|I|} \ge 0$

(iii) $\mathcal{B}(F, f, M) = \frac{1}{|I|} \sup \left\{ \sum_{J \subseteq I} M_J \langle \varphi \rangle_J^2 : (a, b, c) \right\}$

$$\frac{1}{|I|} \sum_{J \subseteq I} M_J \langle \varphi \rangle_J^2 = \frac{1}{2}\left(\frac{1}{|I_+|} \sum_{J \subseteq I_+} M_J \langle \varphi \rangle_J^2 + \frac{1}{|I_-|} \sum_{J \subseteq I_-} M_J \langle \varphi \rangle_J^2 \right) + \frac{M_I}{|I|} \cdot \langle \varphi \rangle_I^2$$

DI $\boxed{\mathcal{B}(F, f, M) \geq \frac{1}{2}\left(\mathcal{B}(F_+, f_+, M_+) + \mathcal{B}(F_-, f_-, M_-) \right) + \Delta M \cdot f^2}$

$$\mathcal{B} : \{ (F, f, M) : 0 \leq M \leq 1 ;\ f^2 \leq F \} \longrightarrow \mathbb{R}_+$$

(iv) $\mathcal{B}(F, f, M) \leq C \cdot F$

SUMMARY: WE LOOK FOR $\{ 0 \leq M \leq 1,\ f^2 \leq F \} \xrightarrow{\mathcal{B}} \mathbb{R}_+$ DOMAIN

SUCH THAT $\mathcal{B}(F, f, M) \geq \frac{1}{2}\left(\mathcal{B}(F_+, f_+, M_+) + \mathcal{B}(F_-, f_-, M_-) \right) + \Delta M \cdot f^2$ DI

IF $F = \frac{F_+ + F_-}{2}$; $f = \frac{f_+ + f_-}{2}$; $M = \frac{M_+ + M_-}{2} + \Delta M$

AND SO THAT: $\mathcal{B}(F, f, M) \leq C \cdot F$ RANGE

martingales

RUNNING $\mathcal{B}$ BACKWARD.

(vi) SUPPOSE WE HAVE $\mathcal{B}$ AS IN THE SUMMARY.

WITH $f_I = \langle \varphi \rangle_I$, $F_I = \langle \varphi^2 \rangle_I$, $\Delta M_I = \frac{M_I}{|I|}$:

DI

$$|I| \cdot \mathcal{B}(F_I, f_I, M_I) - |I_+| \mathcal{B}(F_{I_+}, f_{I_+}, M_{I_+}) - |I_-| \mathcal{B}(F_{I_-}, f_{I_-}, M_{I_-}) \geq |I| \cdot \Delta M \cdot f_I^2 = M_I \cdot \langle \varphi \rangle_I^2$$

$$\Downarrow$$

$$|I_0| \mathcal{B}(F_{I_0}, f_{I_0}, M_{I_0}) - \sum_{\substack{I \subseteq I_0 \\ |I| = 2^{-n-1}}} |I| \mathcal{B}(F_I, f_I, M_I) \geq \underbrace{\sum_{\substack{J \subseteq I_0 \\ 2^{-n} \leq |J| \leq 1}} M_J \langle \varphi \rangle_J^2}$$

$$\wedge\!\!\!\backslash$$

$$|I_0| \mathcal{B}(F_{I_0}, f_{I_0}, M_{I_0})$$

$$\wedge\!\!\!\backslash \quad \text{RANGE}$$

$$C_1 \cdot F_0 = C_1 \langle \varphi^2 \rangle_{I_0} = \underbrace{C_1 \cdot \int_{I_0} \varphi^2}$$

AND $(B) \Rightarrow (A)$ IS PROVED

THE EXPLICIT FUNCTION

(V) $\mathcal{B}(F, f, M) = 4\left(F - \frac{f^2}{1+M}\right)$ WORKS

$\mathcal{B} \leq 4 \cdot F$

(I) $0 \overset{DI}{\geq} \mathcal{B}(F+\Delta F, f+\Delta f, M+\Delta M) + \mathcal{B}(F-\Delta F, f-\Delta f, M-\Delta M)$
$- 2\mathcal{B}(F, f, M) \Rightarrow \boxed{\mathrm{Hess}\,\mathcal{B}(F, f, M) \leq 0}$ (I')

(II) $\mathcal{B}(F, f, M) - \mathcal{B}(F, f, M-\Delta M) \overset{DI}{\geq} \Delta M \cdot f^2 \Rightarrow \boxed{\frac{\partial \mathcal{B}}{\partial M} \geq f^2}$ (II')

- (I'), (II') $\Rightarrow$ DI
- $\mathcal{B}$ satisfies them (exercise)

IS THERE A <u>METHOD</u> TO FIND SUCH $\mathcal{B}$'s ?

VASYUNIN, VOLBERG: MONGE-AMPÈRE HELPS

Some readings

- 47 articles in MR with *Bellman function*in the title since 1997 (non exhaustive list).
- 92 preprints in Arxiv with *Bellman function*in the title (only artially overlapping with the above).

- Zygmund, A. Trigonometric series. Reprint of the 1979 edition. Cambridge, 1988.
- Essén, M. A superharmonic proof of the M. Riesz conjugate function theorem. Ark. Mat. 22 (1984).
- Burkholder, D. L. Boundary value problems and sharp inequalities for martingale transforms. Ann. Probab. 12 (1984).
- Bañuelos, Rodrigo, The foundational inequalities of D. L. Burkholder. Ill. J. Math. 54 (2010).
- Bañuelos, Rodrigo; Wang, Gang Orthogonal martingales under differential subordination... Ill. J. Math. 40 (1996).
- Nazarov, F. L.; Treĭl, S. R.(1-MIS) The hunt for a Bellman function... St. Petersburg Math. J. 8 (1997).
- Vasyunin, Vasily; Volberg, Alexander Monge-Ampère equation and Bellman optimization... AMS Transl. Ser. 2, 226 (2009).
- Carbonaro, A.; Dragičević, O., Bellman function and linear dimension-free estimates in a theorem of Bakry. JFA 265 (2013).
- Domelevo, K.; Petermichl, S., Differential subordination under change of law. Ann. Probab. 47 (2019).
- Gundy, R.; Varopoulos, N. Les transformations de Riesz et les intégrales stochastiques. CRAS Paris A-B 289 (1979).
- Burkholder, Donald L. Sharp inequalities for martingales and stochastic integrals. Astérisque No. 157-158 (1988).
- Nazarov, F.; Treil, S.; Volberg, A. Bellman function in stochastic control and harmonic analysis (2001).
- Øksendal, Bernt Stochastic differential equations. Universitext. Springer 2003.
- Krylov, N. V. Controlled diffusion processes. Springer-Verlag, 1980.